MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2005
ROUND 1 TRIG: RT ANGLE, LAWS SINES & COSINES

ANSWERS

A)______________

B)______________

C)______________

A) In triangle ABC with hypotenuse $\overline{AB}$, $\sin(\angle A)=0.28$. Find $\tan(\angle B)$, expressing your answer as a fraction $\frac{a}{b}$ with *a, b* relatively prime.

B) In acute ΔDEF $\sin(\angle D) = \sin(\angle F) + \frac{1}{3}$ while $EF = ED + \frac{2}{3}$

If $\sin(\angle F) = \frac{5}{9}$, find the <u>exact</u> value of EF in simplified form

C) In acute ΔABC $\cot(\angle A) = 0.75$ while $\tan(\angle B) = 2.40$ If the perimeter of ΔABC is 420, find the triangle's area.

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2005
ROUND 2 ELEMENTARY NUMBER THEORY

ANSWERS

A)_______________

B)_______________

C)_______________

A) How many different positive integers are factors of 160?

B) The product of 123_4 with 567_8 is $1ABC0_9$ a five-digit base nine number. Determine the ordered triplet of digits (A, B, C).

C) The digits of a two-digit base ten positive integer are reversed, resulting in a 108% increase in value. What was the original number?

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2004
ROUND 3 ANALYTIC GEOM OF LINE

ANSWERS

A)_____________

B)_____________

C)_____________

Definition: A lattice point is one whose coordinates are each integers.

A) A line passes through P(– 8, 3) with slope $\frac{-5}{2}$. Moving from P to the right along the line, what are the coordinates of the next lattice point on the line?

B) A line whose equation is $7x - 3y = c$ passes through the lattice points (9, 16) and (a, b) where $a > 100$. Find the minimum possible value of the sum $a + b$.

C) A triangular region is bounded by the lines $y = 0$, $3x - 2y = 0$, and $3x + 4y = 108$. Find the number of lattice points strictly in the interior of the triangle (that is, do not count points on the boundary.)

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2005
ROUND 4 ALG 2: LOGS & EXPONENTIAL FUNCTIONS

******* NO CALCULATORS ON THIS ROUND ******

ANSWERS

A)______________

B)______________

C) ______________

A) Evaluate: $\log_5 625 + \log 0.001 - \log_9 3 + \ln(e)$

B) Solve for x: $\log_5 x + \log_5(x - 20) = 3$

C) Solve for x given x>1 and $x^{\log_2(1/x)} = \dfrac{4}{x^3}$

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2005
ROUND 5 ALG 1: RATIO PROPORTION VARIATION

ANSWERS

A)______________

B)______________

C)______________**pm**

A) The power in an electric circuit values jointly as the resistance and as the square of the current. If the current is halved, what change must be made to the resistance if the power is to remain the same?

B) Given $\dfrac{a + 3b + 5c}{a} = 18$ and $\dfrac{7c}{4b} = \dfrac{35}{12}$ find $a : b : c$ as ratios of relatively prime integers

C) Fifteen hens could lay 25 eggs in 3 days. How many hens should the farmer start with if he wants 600 eggs in 10 days but will sell half his hens after 5 days?

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2005
ROUND 6 GEOMETRY: POLYGONS
ANSWERS

A)______________

B)______________

C)______________

A) ABCDEFGHI is a regular polygon. $\overline{AB}$ and $\overline{CD}$ are extended to meet at P. Find m$\angle$BPC.

B) If three consecutive sides of a convex quadrilateral have corresponding lengths of 5, 7, and 6 and the fourth side also has integral length, how many different possible lengths are there for that fourth side?

C) A rhombus has a perimeter of 60 and diagonals whose lengths are in the ratio 3:4. The longer diagonal of the rhombus is also the larger diagonal of a regular hexagon. The smaller diagonal of the rhombus is the altitude of an equilateral triangle. Find the ratio of the perimeter of the equilateral triangle to the perimeter of the regular hexagon..

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2005
ROUND 7: TEAM QUESTIONS

ANSWERS

A)______________ **D)**______________

B)______________ **E)**______ ________

C)______________ **F)**______________

A) A right prism has a square base. Consider the set of all angles formed by a diagonal of a face and a diagonal of the prism. If one such angle has a tangent of 2, find the sum of the tangents of all the angles in the set.

B) Twin primes are primes that differ by 2. There are 12 pairs of twin primes between 1 and 180. Find the largest and smallest value of N if N – 1 and N + 1 are twin primes less than 180 and N has exactly eight factors.

C) The point P(a, b) is the first quadrant lattice point on $17x - 23y - 2 = 0$ closest to the origin. Find the area of the triangle with vertices P, (4,0) and (26, 1).

D) Solve for A in terms of B and C if A, B, and C > 1 and

$$\log_C AB - \frac{1}{\log_A C} = \log_C A + \log_{\sqrt{12}} 12$$

E) Al, Bob, and Carl each have different amounts of money. They share it as follows: Al gives Bob and Carl each 1/3 of his money. Bob then gives Al and Carl each 1/3 of his new total. Carl then gives Al and Bob each 1/3 of his new total. The result is that Al ends up with \$45 more than Bob and Bob ends with \$45 more than Carl. If after the first sharing Carl had \$78, how much money did Al end up with?

F) A, B, and C are consecutive vertices of a regular polygon with less than 25 sides. A rotation centered at B maps A onto C and C onto P. If A, B, and P are not collinear find the difference between the maximum and minimum possible values of m$\angle$ABP.

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2005 ANSWERS

Round 1:	A) $\frac{24}{7}$	B) $\frac{16}{9}$	C) 8,400
Round 2	A) 12	B) (4, 8, 0)	C) 25
Round 3	A) (–6, –2)	B) 335	C) 301
Round 4	A) 1.5	B) 25	C) 2 or 4
Round 5	A) quadrupled (or x 4)	B) 2:3:5	C) 144
Round 6	A) 100	B) 17	C) $\sqrt{3}:2$
Team Round	A) 64/3	B) 30 and 138	C) 115.5
	D) (A=) $\frac{B}{C^2}$	E) \$131	F) 114

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2005 BRIEF SOLUTIONS

Round One:

A. BC/AB = 0.28 =7/25 so one rt Δ has AC = 24 by Pythagoras. Tan(∠B)=AC/BC

B. Law of Sines: $\frac{\sin(\angle D)}{EF} = \frac{\sin(\angle F)}{ED}$ so $\frac{5/9 + 1/3}{x + 2/3} = \frac{5/9}{x}$ thus
$8/9\,x = 5/9\,x + 10/27$ so ED = x = 10/9 and EF = 16/9

C. Draw altitude CD. From ∠A AD = 3x, CD = 4x. From ∠B BD = 5y, CD=12y so 12y=9x and AD = 9y. Pythagoras gives BC = 13y and AC = 15y while AB = 14y. Perimeter gives y =10 thus area is 0.5(14y)(12y) = 8400

Round Two:

A. $160 = 2^5\,5^1$ so there are (5+1)(1+1) = 12 factors.

B. 123_4 in base ten is 27; 567_8 is 375 so product is 10,125 which in base 9 is 14800_9.

C. Increase is (10u + t) – (10t +u) = 9(u – t). % increase is $\frac{9(u-t)}{10t+u} = \frac{108}{100} = \frac{27}{25}$ so
225u – 225t = 270t + 27u so 198u = 495t or 2u=5t thus u=5, t=2.

Round Three:

A. – 8 + 2 = 6; 3 – 5 = – 2.

B. Substitute (9, 16) to get c = 15 thus y = 7/3 x – 5. To get lattice pt x is a multiple of 3 so a = 102, b = 7/3 (102) – 5 = 233 sum is 335

C. Vertices are (0,0) (12, 18) and (36, 0) Consider x= 1 to 12; interior lattice point counts along vertical lines are 1,2, 4, 5, 7, 8, 10,11, 13, 14, 15, 17 sum 108. Counting from x= 35 back to x = 13 gets 0, 1, 2, 2, 3, 4, 5, 5, 6, 7, 8, 8, 9, 10, 11, 11, 12, 13, 14 14, 15 16, 17 sum 193 total is 301. OR Picks Thm: Area = (#Interor pts) + (#Boundary Pts)/2 –1 so 324 = I + (48/2) – 1 so I = 301

Round Four:

A. 4 + (– 3) – ½ + 1 = 3/2 .

B. $\log_5(x^2 - 20x) = \log_5(125)$ so $x^2 - 20x - 125 = 0$ so (x – 25)(x + 5) =0, exclude –5 as not in domain of $\log_5$ function.

C. $\log_x(\frac{4}{x^3}) = \log_2(\frac{1}{x})$ so $\log_x(4) - 3 = -\log_2(x)$ and if $a = \log_x 2$ we solve
$2a - 3 = \frac{-1}{a}$ to get a= ½ or 1 so x = 4 or 2.

Round Five:

A. Pow = c (res) $(\text{cur})^2$ = c (4 res) $(\text{curr} / 2)^2$

B. 18a = a + 3b +5c so 17a = 3b + 5c while 84 c= 140 b so c = 5/3 b. Thus 17a= 34/3 b thus a/b = 2/3.

C. The rate is 5/9 egg per hen_day. If we start with 2x hens we have 2x(5) + x(5)=15x hen days yielding 25x/3 eggs. So x=72; start with 144 hens.

Round Six:

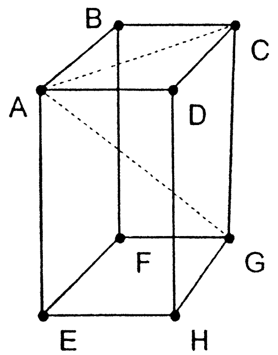
A. Exterior angles of 9-gon are each 40 so m∠BPC = 100.

B. Sketch circles of radius 5 and 6 at the endpoints of a segment of length 7 to see possibilities from 1 through 17.

C. Rhombus side 15, half diagonals 9 and 12 full diags 18, 24. Hexagon perimeter 3(24)=72, equil triangle $3(12\sqrt{3}) = 36\sqrt{3}$. Ratio simplifies to $\sqrt{3} : 2$

Team Round:

A. Each vertex (example A) is the vertex of three such angles ($\angle$GAC, $\angle$GAF, and $\angle$GAH) Since AH > AD = GH tan($\angle$GAH) = GH/AH < 1 and the angle with



tangent 2 must be $\angle$GAC. Let AD = x, then AC = $x\sqrt{2}$ so tan($\angle$GAC) gives CG = $2x\sqrt{2}$ thus hypotenuse of ΔAGH is AG=$x\sqrt{10}$ so AH =3 and the other two angles at A have tangents of 1/3. Sum at A is 8/3 so total from all vertices is 8(8/3) = 64/3

B. Brute force: 3,5 N = 4 = 2^2 = 3 factors. 5,7 N = 6 = 2(3) = 4 factors. 11,13 N = 12 = 2^2 3= 6 factors. 17,19 N = 18 = $(2)3^2$ = 6 factors. 29, 31 N = 30 = 2(3)5= 8 factors- **smallest found**. Working from top 149, 151 N = 150 = $2(3)5^2$ = 12 factors. 137, 139 N = 138 = 2(3)23 = 8 factors- **largest found.** Alternatively, if N has <u>eight</u> factors (one of which must be 2) its factorization must be p^7 (try 2^7 =128 but 129 not prime) or p^3q (if p=2 we can show one of 8q-1, 8q, 8q+1 divisible by 3 so q=3 but N=24 doesn't work. So q= 2, if p=3 no good and otherwise one of $2p^3 \pm 1$ a multiple of 3) or pqr (so we systematically try products of 2 with 2 other primes. to locate N= 2(3)5 and eventually 2(3)23)

C. $x = \dfrac{23y+2}{17} = \dfrac{17y+2(3y+1)}{17}$ so 3y+1 is a multiple of 17 (and 1 more than a multiple of 3) so 3y+1 =34 y=11 x=15. Enclose Δ in rectangle with sides y=0, y=11, x=4, x=26 area is 22(11)- 3 rt Δs = 242 – 11 – 55 – 121/2 = 115.5

D. $\dfrac{\log AB}{\log C} - \dfrac{\log A}{\log C} = \dfrac{\log A}{\log C} + 2$ so $\dfrac{\log A + \log B - \log A - \log A}{\log C} = 2$ so

$\log B - \log A = 2 \log C$ so $\log(\frac{B}{A}) = \log(C^2)$ thus $AC^2 = B$

E. Start with (27A, 9B, 3C) $\Rightarrow$(9A, 9A+9B, 9A+3C) $\Rightarrow$ (12A+3B, 3A+3B, 12A+3B+3C) $\Rightarrow$(16A+4B+C, 7A+4B+C, 4A+B+C) so if Al has 45 more 9A=45 A=5; Bob has 45 more 3A+3B=45 B=10; 9A+3C=78 C=11. 16A+4B+C=131.

F. Maximum is with pentagon m$\angle$ABP=144 . Thereafter two interior angles exceeds 180 so $\angle$ABP decreases in size. If n<25 minimum is 24-gon of 30.