MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007
ROUND 1 ALG 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS

ANSWERS

A) $a =$ ____________ $b =$ ___________

B) ________________________________

C) ________________________________

A) Determine the values of a and b such that the solution (x, y) of the system $\begin{cases} ax + by = 5 \\ ax - by = 15 \end{cases}$ is (5, 15).

B) Determine the numerical value of $\begin{vmatrix} 3 & a \\ b & 4 \end{vmatrix}$ if the system of equations represented by

$\begin{bmatrix} 3 & a \\ b & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$ intersect at the point $P(5, 1)$.

C) The identity $\dfrac{A}{x} + \dfrac{B}{2x-1} + \dfrac{C}{x+3} \equiv \dfrac{-21}{2x^3 + 5x^2 - 3x}$ is true for exactly one ordered triple (A, B, C). Determine the value of $A + B + C$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007
ROUND 2 ALG 1: EXPONENTS AND RADICALS

ANSWERS

A) ________________________________

B) $x =$ ___________ , $y =$ ___________

C) $a =$ ______ , $b =$ ______ , $c =$ ______

A) The variables a, b, c and d have distinct values of 1, -2, 3 and -4, but not necessarily in that order. Determine the maximum possible value of the expression $a^b - c^d$.

B) Solve for x and y, if $\frac{4^{2x}}{2^{2y}} = \frac{8^{8x}}{64^y}$ and $\left(\frac{1}{3}\right)^{y-x} = 81$.

C) $\sqrt{48 - 24\sqrt{3}}$ in a simplified form can be written as $a + b\sqrt{c}$ where a, b and c are integers and c is square-free, i.e. contains no factors which are perfect squares (other than 1). Find a, b and c.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007
ROUND 3 ALG 2 POLYNOMIAL FUNCTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Clearly, $x = 1$ is a zero of the polynomial $p(x) = x^3 - 2x + 1$.
Determine the exact value of the larger of the two remaining zeros.

B) Given: $y = p(x)$ is a cubic polynomial (i.e. 3rd degree), $p(-1) = p(2) = p(1/2) = 0$ and $p(0) = 2$
Find the remainder when $p(x)$ is divided by $(x - 1)$.

C) The width of an **open** box (i.e. **without a top**) is two less than its length and its height is 7 less than 6 times its length. Numerically, the volume of the box is 5 less than 6 times its length. Find all possible surface areas of the exterior of such a box.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007
ROUND 4 ALG 1: ANYTHING

ANSWERS

A) water: ______ oz milk: ______ oz

B) ________________________________

C) ________________________________

A) The recipe calls for 1% milk and only a half gallon of 4% milk is available. How many ounces of water and how many ounces of the available 4% milk must be mixed to satisfy the recipe's requirement of 3 cups of 1% milk?
NOTE: 1 gallon (gal) = 128 ounces(oz) = 4 quarts (qt) = 8 pints (pt) = 16 cups (c)

B) Determine all values of a for which the distance between the x- and y-intercepts of the line $\frac{x}{3a}+\frac{y}{4a}=1$ is 35.

C) Let $N = (4^x + 4^x)(3^x + 3^x + 3^x)$.
Determine how many positive integer factors N has in terms of x, if x denotes a positive integer.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007
ROUND 5 PLANE GEOMETRY: ANYTHING

ANSWERS

A) ________________________________

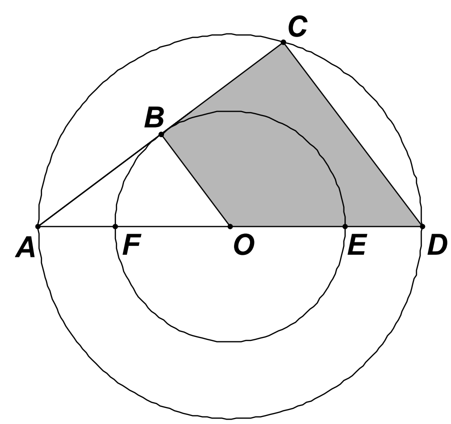
B) ________________________________

C) ________________________________

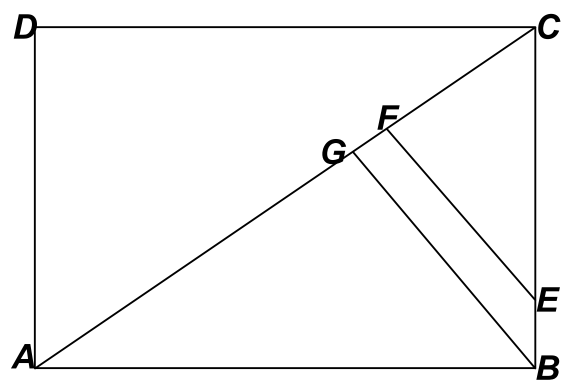
A) *ABCD* is a rectangle with sides of lengths 12 and 35.
<u>All</u> possible line segments connecting pairs of vertices are drawn.
What is the total length of all these line segments?

B) Given: concentric circles with center at point O
chord $\overline{AC}$ is tangent to the inner circle at point B
$AB = 8$, $AF = 4$

Determine the area of quadrilateral *BCDO*.



C) Given: Rectangle *ABCD*, $AB = 160$, $BC = 120$, $BE = 20$, $\overline{EF} \perp \overline{AC}$, $\overline{BG} \parallel \overline{EF}$
Find the area of *BEFG*.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007
ROUND 6 ALG 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Uncle Sam has 5 red, 4 white and 6 blue flags. If three flags are selected without replacement, what is the probability that the three flags selected are the <u>same</u> color? Give your answer as a simplified fraction.

B) One of the terms in the expansion of $\left(x^{2/3}+\dfrac{1}{2x^{3/2}}\right)^{16}$ is kx^2, where k is a constant.

Determine the value of k.

C) What is the probability that a permutation of the nonzero digits 1 … 9 will begin with a non-prime OR end with a prime?

Three such permutations are <u>**4**</u>73218569, 73218694<u>**5**</u> and <u>**4**</u>3218569<u>**7**</u>
Note: The digit 1 is <u>not</u> prime.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) ____________________________

C) (____________ , ____________) F) ____________________________

***** NO CALCULATORS ON THIS ROUND ****

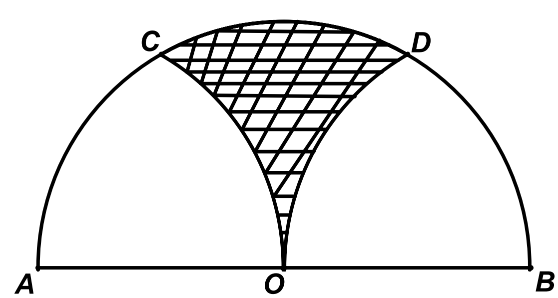
A) If $a \neq b$, GCF(a, b) = 1 and $|b - 87| < 14$, there are exactly <u>two</u> ordered pairs (a, b) for which $92_a = 29_b$. Note: a and b denote the bases of two-digit numbers. If the specific ordered pairs are (a_1, b_1) and (a_2, b_2) and $a_1 < a_2$, then what is the value of the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$?

B) Let x = the minimum positive integer for which $(a + 1)^2 + a > 1200$. Let y = the minimum integer value of a for which $|\sqrt{a+1} - \sqrt{a}| < 0.1$. Determine the sum of the prime factors of the product $x(x + y)$.

C) If A and B are integers and $(-1 + i)$ is a root of $p(x) = x^4 + Ax^2 + Bx - 6 = 0$, find the ordered pair (A, B).

D) Let A and B denote positive integers and $B \neq 1$. Find <u>all</u> possible ordered pairs (A, B) for which the ratio $(164 - 2A) : (3A - 6)$ is equal to the ratio B : 1.

E) In semicircle O, arcs $\overset{\frown}{OC}$ and $\overset{\frown}{OD}$ centered at A and B are drawn passing through O. $AO = 6$ Find the <u>exact</u> area of the shaded region?



F) What is the smallest <u>integer</u> coefficient in the expansion of $(4t + \frac{v}{2})^8$?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007 ANSWERS

Round 1 Alg 2: Simultaneous Equations and Determinants

A) $a = 2$, $b = -1/3$ B) -4 C) -6

Round 2 Alg 1: Exponents and Radicals

A) 65 B) $x = -1$, $y = -5$ C) $a = 6$, $b = -2$, $c = 3$

Round 3 Alg2: Polynomial Functions

A) $\frac{\sqrt{5}-1}{2}$ B) -2 C) 49.25

Round 4 Alg 1: Anything

A) milk: 6 water: 18 B) ±7 C) $2(x + 1)(x + 2)$
$= 2x^2 + 6x + 4$ (or equivalent)

Round 5 Plane Geometry: Anything

A) 168 B) 72 C) 1056

Round 6 Alg 2: Probability and the Binomial Theorem

A) $\frac{34}{455}$ B) $\frac{455}{4}$ C) $\frac{13}{18}$

Team Round

A) 14 B) 78 C) (-5, -10)

D) (22, 2), (10, 6), (7, 10), (4, 26)

E) $18\sqrt{3} - 6\pi$ F) 7

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007 SOLUTION KEY

Round 1

A) Adding and substituting $x = 5$, $2ax = 20$ → $\underline{\boldsymbol{a = 2}}$
Subtracting and substituting $y = 15$, $2by = -10$ → $\underline{\boldsymbol{b = -1/3}}$

B) The matrix equation is equivalent to: $\begin{cases} 3x + ay = 7 \\ bx + 4y = -6 \end{cases}$. Substituting, $\begin{cases} 15 + a = 7 \\ 5b + 4 = -6 \end{cases}$

→ $(a, b) = (-8, -2)$
Evaluating the determinant, $12 - (-8)(-2) = \underline{\mathbf{-4}}$

C) Think of -21 as $0x^2 + 0x - 21$

As a single fraction, the left hand side is $\dfrac{A(2x^2+5x-3)+B(x^2+3x)+C(2x^2-x)}{2x^3+5x^2-3x}$

Re-arranging terms, $\dfrac{(2A+B+2C)x^2+(5A+3B-C)x-3A}{2x^3+5x^2-3x}$

Thus, $\begin{cases} 2A+B+2C=0 \\ 5A+3B-C=0 \\ -3A=-21 \end{cases}$ → $A = 7$ and $\begin{cases} B+2C=-14 \\ 3B-C=-35 \end{cases}$ → $(B, C) = (-12, -1)$

and the required sum is $7 + (-12) + (-1) = \underline{\mathbf{-6}}$.

Round 2

A) Trial and error → $(a, b, c, d) = (1, -2, -4, 3)$ → $(1)^{-2} - (-4)^3 = 1 + 64 = \underline{\mathbf{65}}$

B) $x - y = 4$ and $2^{4x-2y} = 2^{24x-6y}$ → $2^{20x-4y} = 1$ → $20x - 4y = 0$ or $5x - y = 0$
Solving simultaneously, $4x = -4$ → $\underline{\boldsymbol{x = -1}}$. Substituting back, $\underline{\boldsymbol{y = -5}}$.

C) The radicand must represent a perfect square, call it $(a+b\sqrt{3})^2 = a^2 + 3b^2 + 2ab\sqrt{3}$
For integer values of a and b, $a^2 + 3b^2$ must represent an integer and $2ab\sqrt{3}$ a multiple of $\sqrt{3}$.
Thus, $a^2 + 3b^2 = 48$ and $ab = -12$. Clearly, a and b have opposite signs and checking out factors of 12 in the first equation produces either $(6, -2)$ → $\boxed{\mathbf{6 - 2\sqrt{3}}}$ which is positive or $(-6, 2)$ → $-6+2\sqrt{3}$ which is a negative value and must be rejected. Thus, $\underline{\boldsymbol{a = 6, b = -2, c = 3}}$

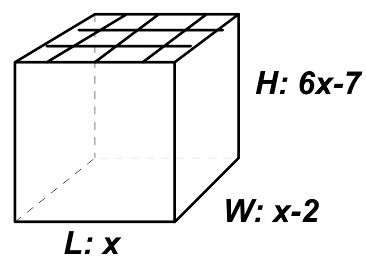
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007 SOLUTION KEY

Round 3

A) Using synthetic substitution, p(x) factors to $(x-1)(x^2+x-1)$ Using the quadratic formula on the second factor, the remaining roots are $\frac{-1\pm\sqrt{5}}{2}$ and the larger is $\underline{\boldsymbol{\frac{\sqrt{5}-1}{2}}}$.

B) $p(x) = a(x+1)(x-2)(x-½)$ and $p(0) = a(1)(-2)(-1/2) = 2 \rightarrow a = 2$
Thus, $p(x) = (x+1)(x-2)(2x-1)$.
Substituting $x = 1$ gives us the remainder of division by $(x-1)$: $2(-1)(1) =$ **<u>-2</u>**

C) $V = x(x-2)(6x-7) = 6x-5$
$\rightarrow 6x^3 - 19x^2 + 14x = 6x - 5$
$\rightarrow 6x^3 - 19x^2 + 8x + 5 = 0$



Synthetic substitution shows that $x = 1$ is a root and the quotient is $6x^2 - 13x - 5 = (3x+1)(2x-5)$. Checking the roots of 1, 5/2 and -1/3, only $x = 5/2$ gives three positive dimensions.
The dimensions of the box are 5/2 x 1/2 x 8 and

the surface area of the <u>open</u> box is

top		left/right faces		front/back faces		
1(1/2 · 5/2)	+	2(1/2 · 8)	+	2(5/2 · 8)	=	5/4 + 8 + 40 = **<u>49.25</u>**.

Round 4

A) Since 1 c = 8 oz, we need 24 oz of 1% milk.
We must determine the ratio of water and 4% milk that produces 1% milk.
To convert the entire half gallon (64 oz) of 4% milk to 1% milk by adding x oz of water:
$.04(64) + 0.00x = .01(64+x) \rightarrow 256 = 64 + x \rightarrow x = 192$
4% : water = 64 : 192 → 1 : 3 ratio
Let n denote the # ounces of 4% milk. Then $n + 3n = 24 \rightarrow n = 6$
→ milk: **<u>6 oz</u>** water: **18 oz**

B) The x- and y-intercepts of $\frac{x}{3a}+\frac{y}{4a}=1$ are (3a, 0) and (0, 4a).
The distance between these points is $\sqrt{25a^2} = 5|a| = 35 \rightarrow a =$ <u>±7</u>

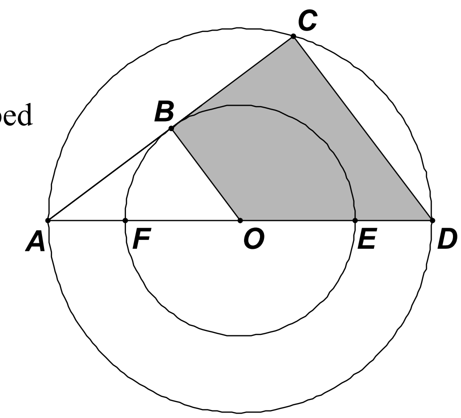
C) $(4^x + 4^x) = 2(4^x) = 2(2^{2x}) = 2^{2x+1}$
$(3^x + 3^x + 3^x) = 3(3^x) = 3^{x+1}$
2^{2x+1} has $(2x+2)$ factors, namely $1 = 2^0, 2 = 2^1, 4 = 2^2, 8 = 2^3, \ldots. 2^{2x+1}$
Similarly, 3^{x+1} has $(x+2)$ factors, namely 1, 3, 9, 27, … 3^{x+1}
The factors of N are found by multiplying any factor in the first list by any factor in the second list. Since any pair of numbers selected from these lists are relatively prime, there will be $(2x+2)(x+2)$ products with no duplicates. Thus, N has $\underline{\mathbf{2(x+1)(x+2) = 2x^2 + 6x + 4}}$ factors (or equivalent)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007 SOLUTION KEY

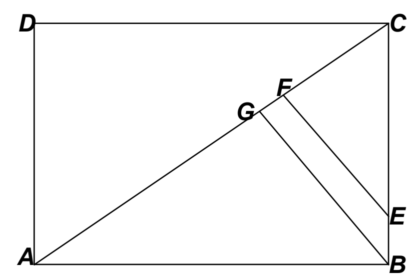
Round 5

A) There are 6 possible line segments, 4 sides and 2 diagonals.
The diagonals have length 37. $2(12 + 35 + 37) =$ **<u>168</u>**

B) Let the radius $OF = r$. Using the secant-tangent theorem, $AB^2 = AF(AD) \rightarrow 8^2 = 4(4 + 2r) \rightarrow r = 6$. Since $\angle ACD$ is inscribed in a semi-circle, it must be a right angle. Thus, $\Delta ABO \sim \Delta ACD$. Since the corresponding sides are in a 2 : 1 ratio, $CD = 12$. Quadrilateral $BCDO$ is a trapezoid and its area is $\frac{1}{2}\cdot 8\cdot(6+12) =$ **<u>72</u>**



C) $AC = 200$, $CE = 120 - 20 = 100$, Δs ABC, ABG and CFE are all 3-4-5 Δs.
Thus, $CF = 60$, $FE = 80$, $BG = 96$, $AG = 128$
Area $BEFG$ = Area ABC – Area ABG – Area CFE =
½(160(120) - ½(118)(96) - ½(60)(80)
= 9600 – 6144 – 2400 = **<u>1056</u>**



Round 6

A) $$\text{P(same color)} = \frac{5\cdot4\cdot3+6\cdot5\cdot4+4\cdot3\cdot2}{15\cdot14\cdot13} = \frac{60+120+24}{15\cdot14\cdot13} = \frac{204}{15\cdot14\cdot13} = \underline{\mathbf{\frac{34}{455}}}$$

B) The general term is $$\binom{16}{k}(x^{2/3})^{16-k}\cdot\left(\frac{1}{2x^{3/2}}\right)^k = \binom{16}{k}2^{-k}x^{\frac{32-2k}{3}-\frac{3k}{2}} = \binom{16}{k}2^{-k}x^{\frac{64-13k}{6}}$$

Thus, $\frac{64-13k}{6} = 2 \rightarrow k = 4$ and the coefficient $k = \binom{16}{4}2^{-4} = \frac{16\cdot15\cdot14\cdot13}{1\cdot2\cdot3\cdot4\cdot2^4} = \frac{5\cdot7\cdot13}{4} = \underline{\mathbf{\frac{455}{4}}}$

C) There are 5 choices for the leftmost digit, namely 1, 4, 6, 8 and 9. There are 4 choices for the rightmost digit, namely 2, 3, 5 and 7.
Thus, there are 5(8!) arrangements that begin with a non-prime and 4(8!) arrangements that end in a prime. But we must subtract the arrangements which satisfy both conditions since these have been counted twice.
5(8!) + 4(8!) – **5(4)(7!)** $\rightarrow$ 7!(40 +32 – 20) = 7!(52). This is out of the total of 9! possible arrangements. So we have $\frac{52(7!)}{9!} = \frac{4(13)(7!)}{9(8)(7!)} = \underline{\mathbf{\frac{13}{18}}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2007 SOLUTION KEY

Team Round

A) $9a + 2 = 2b + 9 \rightarrow b = (9a - 7)/2 \rightarrow (1, 1)$ which is rejected. Slope $m = 9/2$ and
$|b - 87| < 14 \rightarrow -14 < b - 87 < +14 \rightarrow 73 < b < 101$
$\rightarrow$ value of b must be between 74 and 100 inclusive $\rightarrow$ min ordered pair to be tested is
$(1 + 9(2), 1 + 9(9)) = (19, 82)$ ok
Next (21, 91) rejected (GCF = 7), (23, 100) ok
$\begin{vmatrix} 19 & 82 \\ 23 & 100 \end{vmatrix} = 19(100) - 23(82) = 1900 - 1886 =$ **<u>14</u>**

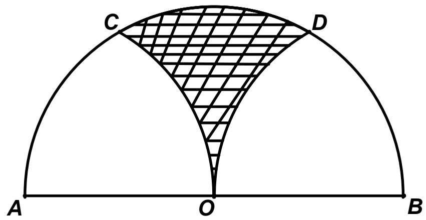
B) $34^2 + 33 = 1189$ and $35^2 + 34 = 1259 \rightarrow x = 34$

Testing $y = 24$. Is $5 - \sqrt{24} < .1$? $5 - \sqrt{24} < .1 \rightarrow 4.9 < \sqrt{24} \rightarrow 24.01 < 24$
Oops, 24 fails, but just barely. Thus, $y = 25$.
$(x, y) = (34, 25) \rightarrow$ required product = 34(59) $\rightarrow$ prime factors = 2, 17, 59 $\rightarrow$ **<u>78</u>**

C) Since the coefficients are real, complex roots must occur in conjugate pairs. Thus,
$(-1 - i)$ is also a root. Using the sum and product of the roots relation to the coefficients,
$x^2 + 2x + 2$ must be a factor of $p(x)$. Since the cubic term of $p(x)$ is missing (i.e. $0x^3$) and the constant term is -6, the other factor of $p(x)$ must be $x^2 - 2x - 3$.
Multiplying, $p(x) = x^4 - 5x^2 - 10x - 6 \rightarrow (A, B) =$ **<u>(-5, -10)</u>**

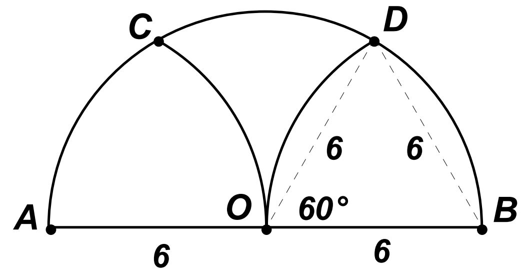
D) Solving for A in terms of $B \rightarrow A = 2 + \dfrac{160}{3B+2}$
$160 = 2^5 5^1 \rightarrow$ 160 has 12 factors: 1, 2, 4, 5, 8, 10, 16, 20, 32, 40, 80 and 160.
Equating $3B + 2$ to each of these values produces an integer value of $B = \cancel{1}$, 2, 6, 10 and 26
which correspond to $A = \cancel{34}$, 22, 10, 7 and 4 $\rightarrow$ the ordered pairs **<u>(22, 2), (10, 6), (7, 10), (4, 26)</u>**

Team Round - continued



E) Area sector $OBD = \frac{1}{6}\pi \cdot 6^2 = 6\pi$

Area (Equilateral ΔOBD) = $\frac{6^2}{4}\sqrt{3} = 9\sqrt{3}$



Area (curvilinear region *OBD*) =
Area(sector *OBD* + sector *BOD* - ΔOBD)
$= 6\pi + 6\pi - 9\sqrt{3} = 12\pi - 9\sqrt{3}$
= Area (curvilinear region *OAC)*

Area (shaded region *OCD*) = Area(semi-circle) – 2 · Area(curvilinear region *OBD*)
$= 18\pi - 2(12\pi - 9\sqrt{3}) =$ $\underline{\mathbf{18\sqrt{3} - 6\pi}}$

F) The coefficients are given by the expression $\binom{8}{k}(4)^{8-k}(\frac{1}{2})^k = \binom{8}{k}2^{16-3k}$ for $k = 0$ to 8

Combinatorial Factor (Pascal'sTriangle)	**1**	**8**	**28**	**56**	**70**	**56**	**28**	**8**	**1**
Power of 2	**2^16**	**2^13**	**2^10**	**2^7**	**2^4**	**2^1**	**2^-2**	**2^-5**	**2^-8**

Smallest: 7

Addendum

The original version of question C in round 3 did not specify “the exterior of” such a box and some enterprising students appealed that there is both an inside and outside to an open top box and, lacking and information about the thickness of the walls, the interior and exterior surfaces areas would be the same. These students appealed that an answer of 98.5 should also be accepted. Their appeal was accepted.