MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 – DECEMBER 2009
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

******* NO CALCULATORS IN THIS ROUND *******

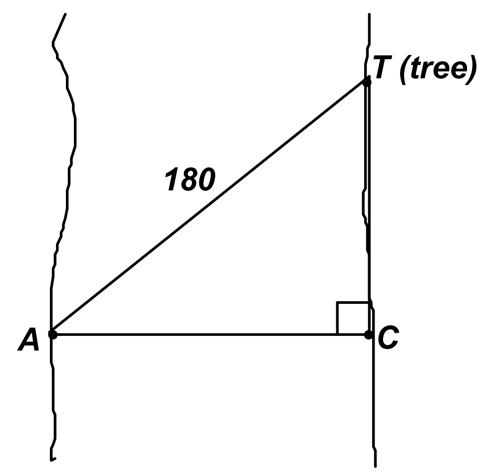
ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) From point A on a river bank, the distance to a tree on the opposite bank is 180 meters.
If $\tan\left(\angle TAC\right) = \frac{\sqrt{5}}{2}$, compute AC, the width of the river.



B) In acute ΔABC, $m\angle C = 30°$, $AB = 4$ and $AC = n$, where n is an integer.
Determine how many values of n are possible?

C) In ΔABC, $m\angle A = 30°$, $a = 10$, $b = 15$ and $\angle B$ is as large as possible.
Determine the exact value of $\sin C$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009
ROUND 2 ARITHMETIC/NUMBER THEORY

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) ______________________________ %

B) ________________________________

C) (__________ , __________)

A) A is multiplied by 4, divided by 5 and then multiplied by 6. What is the percent increase in the value of A after these three operations?

B) Find the sum of all positive integer divisors of 2009.

C) Find the ordered pair (x, y) so that $9x43y5$ is the smallest number divisible by 33.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) (____ , ____)   (____ , ____)

B) ______________________________

C) ___________________________ = 0

A) The line $4x - 3y - 11 = 0$ passes through the center of $(x-2)^2 + (y+1)^2 = 25$
Determine the coordinates of the two points of intersection.

B) Given $A(-2.9, 5.9)$, $B(0.3, k)$ and $AB = 3.2\sqrt{5}$
Determine all possible values of k.

C) Three vertices of parallelogram $PQRS$ are $P(2, 1)$ $Q(6,11)$ and $S(12, 9)$.
Determine the equation of $\overleftrightarrow{PR}$, in $ax + by + c = 0$ form, where a, b and c are integers, $a > 0$ and $\text{GCF}(a, b, c) = 1$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) _______________________________

B) _______________________________

C) $x =$ __________________________

A) Compute $\log_9\left(1+\frac{\sqrt{6}}{3}\right)+\log_9\left(1-\frac{\sqrt{6}}{3}\right)$

B) Compute all real values of x for which $\log_4 x^3 - 2\log_{16} x + \log_{64} x^6 = 3$
Any radicals in your answer must have the smallest possible index.

C) Solve for x over the reals. $\frac{2^x - 2^{-x}}{2} = -1.875$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) ______________________________

B) ______________________ buckets

C) ______________________________

A) For what value of k does the ratio $\sqrt{2009} : \sqrt{2009k}$ equal $\sqrt{2} : \sqrt{7}$?

B) June can eat a bucket of blueberries in 3 hours. Her mom picks enough by herself to fill a bucket in 5 hours, while her dad can fill a bucket by himself in 7 hours. Suppose June's parents are picking blueberries into a bucket, while June is eating from the same bucket. After a sufficient amount of time, they have accumulated one bucket full of blueberries. How many buckets of blueberries did mom and dad actually pick?

C) A round trip between A and B consists of a trip from A to B at a constant rate of R_1 and a return trip from B to A at a constant rate of R_2.
The average speed (R) on a round trip varies directly as the distance between A and B and inversely as the sum of the elapsed times traveling back and forth.

$R = 48$ when $(R_1, R_2) = (40, 60)$ and $D = 30$.
Compute R when $(R_1, R_2) = (8, 12)$ and $D = 2008$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

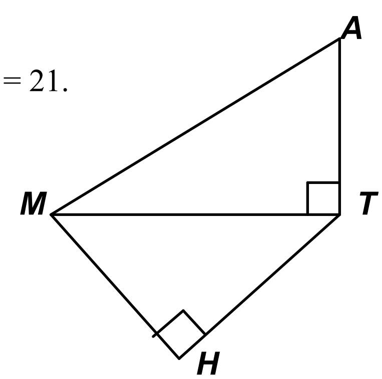
A) ______________________________ °

B) ______________________________

C) ______________________________

A) The angles of ΔPQR measure $(7x - 3)°$, $(6x + 7)°$ and $(95 - 4x)°$. The largest angle is $\angle P$ and the smallest angle is $\angle R$. Determine the $m\angle Q$.

B) Given the quadrilateral $MATH$, where $\overline{MT} \perp \overline{AT}$, $\overline{MH} \perp \overline{TH}$ and $MA = 21$. Compute the sum of the squares of the sides of the quadrilateral.



C) The interior angles of regular polygon P measure $x°$.

The interior angles of regular polygon Q measure $\left(x + \frac{1}{2}\right)^{\circ}$.

If Q has 8 more sides than P, compute the number of diagonals in Q.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009
ROUND 7 TEAM QUESTIONS

******* CALCULATORS ARE PERMITTED IN THIS ROUND *******

ANSWERS

A) ___________________________ D) ___________________________

B) ___________________________ E) ___________________________

C) ___________________________ F) ___________________________

A) There are two possible triangles with sides $x - 1$, $x + 3$ and $2x - 3$ and an angle with measure of 120°. Compute the <u>smaller</u> of the two possible perimeters.

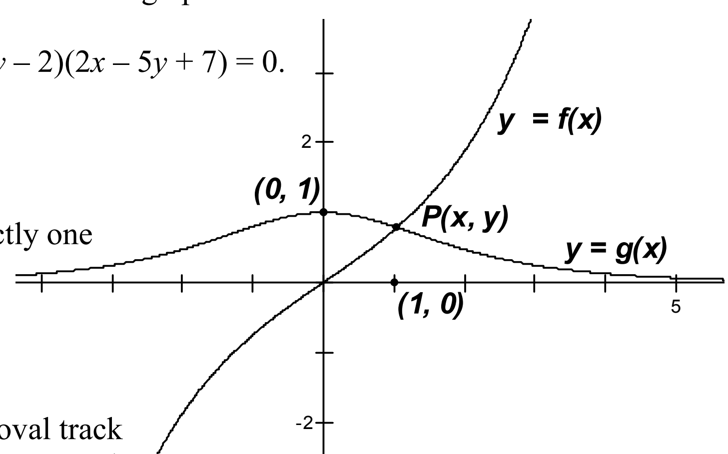
B) Consider the following list of Pythagorean triples:

Row 1:	9	40	41
Row 2:	11	60	61
Row 3:	13	84	85

The first numbers in each row form an increasing arithmetic progression.
Compute the sum of the squares of the numbers in the 11^{th} row.

C) How many points are determined by the intersection of the graphs of

$$(x - y + 2)(3x + y - 4) = 0 \text{ and } (x + y - 2)(2x - 5y + 7) = 0.$$



D) Let $f(x) = \dfrac{2^x - 2^{-x}}{2}$ and. $g(x) = \dfrac{2}{2^x + 2^{-x}}$

The graphs of $y = f(x)$ and $y = g(x)$ indicate exactly one point of intersection at $P(x, y)$.
The <u>exact</u> x-coordinate of point P can be expressed in the form $\log_2 N$. Compute N.

E) Abbott and Costello are running laps around an oval track in opposite directions. Their starting points are staggered by half a lap. Assuming both runners move at a constant speed and Abbott completes 5 laps in the time it takes Costello to complete 4, how many laps has Abbott completed when the runners have passed each other 100 times?

F) ΔABC has sides a, b and c with <u>integer</u> lengths and $a \le b$.
The median m to side c also has integer length.
Determine all possible values of m if the perimeter of ΔABC is 24 and $c = 8$ or 10.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009 ANSWERS

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

A) 120 meters   B) 1   C) $\dfrac{3\sqrt{3}-\sqrt{7}}{8}$

Round 2 Arithmetic/Elementary Number Theory

A) 380%   B) 2394   C) (0, 6) [904365]

Round 3 Coordinate Geometry of Lines and Circles

A) (5, 3), (−1, −5)   B) −0.5, 12.3   C) $9x - 7y - 11 = 0$ [$R(16, 19)$]

Round 4 Alg 2: Log and Exponential Functions

A) $-\dfrac{1}{2}$ B) $2\sqrt{2}$ C) −2

Round 5 Alg 1: Ratio, Proportion or Variation

A) $\dfrac{7}{2}$ B) 36   C) 9.6

Round 6 Plane Geometry: Polygons (no areas)

A) 60   B) 882   C) 3080

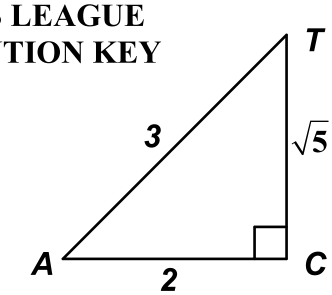
Team Round

A) 15   D) $\sqrt{2+\sqrt{5}}$

B) 354482   E) 56

C) 4   F) 5 or 7

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009 SOLUTION KEY



Round 1

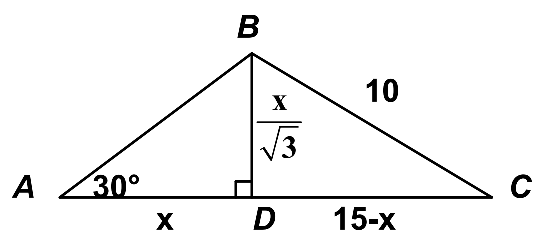
A) $\dfrac{AC}{180} = \cos(\angle A)$ ➔ $AC = 180\left(\dfrac{2}{3}\right) = \underline{\mathbf{120}}$

B) $\dfrac{\sin 30}{4} = \dfrac{\sin B}{n}$

➔ $\sin B = \dfrac{n}{8}$ ➔ $0 < \dfrac{n}{8} < 1$ ➔ $n = 1, 2, \ldots, 7$. This condition is <u>necessary</u> to guarantee the existence of a non-right ΔABC, but not <u>sufficient</u> to guarantee that the triangle is acute.
$m\angle C = 30°$ ➔ $m\angle A + m\angle B = 150°$.
Both A and B must be acute; hence $m\angle B < 90°$ ➔ $150 - m\angle A < 90$ ➔ $m\angle A > 60$
Applying the same reasoning to $\angle A$, we have $60 < m\angle A, m\angle B < 90$.
Since the sine is a strictly increasing function over this interval, we have
$\sin 60° < \sin B < \sin 90°$ ➔ $\dfrac{\sqrt{3}}{2} < \dfrac{n}{8} < 1$ ➔ $4\sqrt{3} < n < 8$. $4\sqrt{3} \approx 4(1.7) \approx 6.8$ and only $n = 7$ satisfies this requirement. Therefore, there is only **<u>one</u>** value of n for which ΔABC is acute.

C) Using the Law of Sines, $\dfrac{\sin 30°}{10} = \dfrac{\sin B}{15}$ ➔ $\sin B = 15/20 = ¾$ (and B is obtuse)

Method 1:
Using the Pythagorean Theorem on ΔBDC,



$\dfrac{x^2}{3} + (15 - x)^2 = 100$ ➔ $x^2 + 675 - 90x + 3x^2 = 300$

➔ $4x^2 - x + 375 = 0$ ➔ $x = \dfrac{90 \pm \sqrt{8100 - 16(375)}}{8}$

$= \dfrac{90 \pm 10\sqrt{21}}{8}$ and since $x < 15$, $x = \dfrac{5(9 - \sqrt{21})}{4}$

and $\sin C = \dfrac{x}{10\sqrt{3}} = \dfrac{5(9-\sqrt{21})}{4 \cdot 10\sqrt{3}} = \dfrac{9 - \sqrt{21}}{8\sqrt{3}} = \dfrac{9\sqrt{3} - 3\sqrt{7}}{8 \cdot 3} = \underline{\mathbf{\dfrac{3\sqrt{3} - \sqrt{7}}{8}}}$

Method 2: far easier w/this approach, but students in this round are not expected to know the expansion of $\sin(A - B)$

$\sin B = ¾$ and $\angle B$ obtuse ➔ $\cos B = -\dfrac{\sqrt{7}}{4}$

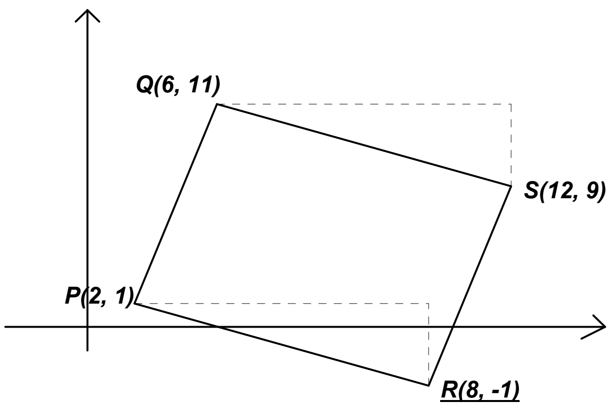
$A + B + C = 180$ ➔ $C = 150 - B$

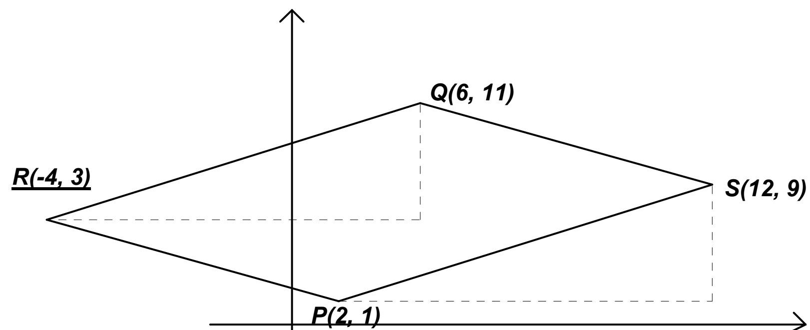
Thus, $\sin C = \sin(150 - B) = \sin 150 \cos B - \sin B \cos 150 = \dfrac{1}{2} \cdot \dfrac{-\sqrt{7}}{4} - \dfrac{3}{4} \cdot \dfrac{-\sqrt{3}}{2} = \underline{\mathbf{\dfrac{3\sqrt{3} - \sqrt{7}}{8}}}$

Round 1 –continued

C) - **continued**

Given three points $P(2, 1)$, $Q(6, 11)$ and $S(12, 9)$, there are in fact three points which could be the fourth vertex of a parallelogram. Besides (16,19) above, (8, −1) and (−4,3) are possible candidates.





In the latter two cases, the quadrilaterals would be *PQSR* and *PRQS* respectively (or a cyclical permutation thereof).
The requested quadrilateral was *PQRS* which implies *P* and *R* must be opposite vertices.
In these other two quadrilaterals, *P* and *R* are consecutive vertices.
Therefore, the only possible position for point *R* is (16, 19) and the above solution is unique.

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Round 2

A) $A \rightarrow \frac{4\cdot 6}{5}A = \frac{24}{5}A$ Thus, the actual increase is $\frac{24}{5}A - A = \frac{19}{5}A$ and $\frac{19}{5} = \frac{p}{100}$ (a p% increase)
$5p = 1900 \rightarrow p =$ **<u>380</u>**

B) $2009 = 7(287) = 7^2 \cdot 41^1 \rightarrow$ 2009 has $(2 + 1)(1 + 1) = 6$ positive integer divisors, namely
$1 + 7 + 41 + 49 + 287 + 2009 =$ **<u>2394</u>**

C) The number must be divisible by 3 and 11, so the sum of the digits must be divisible by 3, and $(9 + 4 + y) - (x + 3 + 5)$ must be divisible by 11. This means that $x + y$ could $= 0, 3, 6,$ etc. since $9 + 4 + 3 + 5 = 21$. Also $5 + y - x$ could $= 0, 11,$ etc. If $5 + y - x = 0$ and $y + x = 0$ or 3, we get negative values. So try $5 + y - x = 11$ and $y + x = 6$.
Solving, we get $y = 6$ and $x = 0 \rightarrow 904365 \rightarrow (x, y) =$ **<u>(0, 6)</u>**

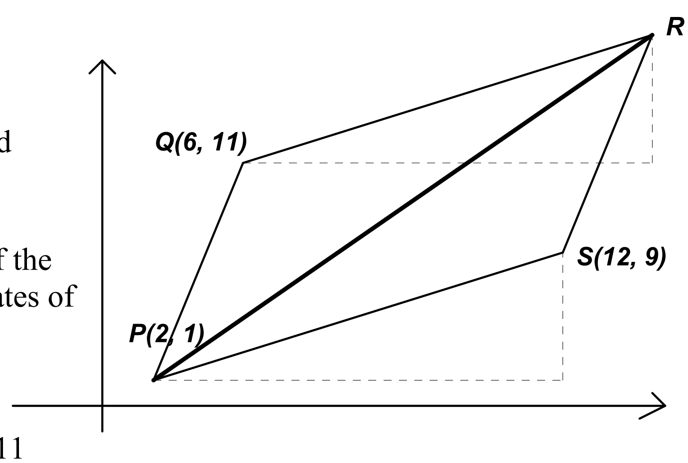
Round 3

A) Solving the linear equation for x (or y) and substituting in the quadratic equation would be messy. Instead, note that the center of the circle is $(2, -1)$ and the slope of the line is $\frac{4}{3}$.
Thus, the required points of intersection are:

$P(2 + 3, -1 + 4) =$ **<u>(5, 3)</u>** and $Q(2 - 3, -1 - 4) =$ **<u>(–1, –5)</u>**

B) $AB^2 = (k-5.9)^2 + 3.2^2 = (3.2\sqrt{5})^2 = 3.2^2(5) \rightarrow (k-5.9)^2 = 3.2^2(5-1) = 3.2^2(2^2)$
$\rightarrow k - 5.9 = \pm 6.4 \rightarrow k = 5.9 \pm 6.4 \rightarrow$ **<u>–0.5, 12.3</u>**

C) The slope of $\overline{PS}$ is 8/10.
R is then located by translating Q 10 units right and 8 units up. $R(16, 19)$
Thus, the slope of $\overleftrightarrow{PR}$ is $18/14 = 9/7$
and the equation is $9x - 7y + k = 0$ and the value of the constant is determined by substituting the coordinates of either P or R.



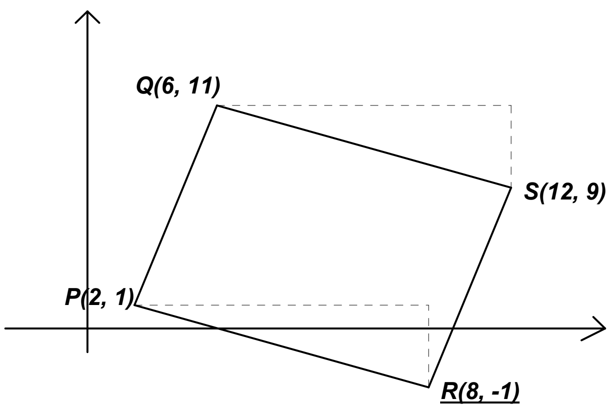
$P \rightarrow 18 - 7 + k = 0 \rightarrow k = -11$ or
$R \rightarrow 9(16) - 7(19) + k = 0 \rightarrow k = -144 + 123 = -11$
Thus, the required equation is **<u>$9x - 7y - 11 = 0$</u>**

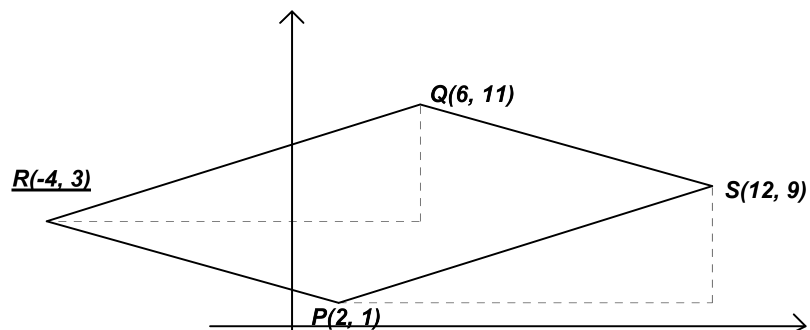
or once the coordinates of R have been determined use the 2-point form of a straight line.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009 SOLUTION KEY

Round 3
C) - **continued**

Given three points $P(2, 1)$, $Q(6, 11)$ and $S(12, 9)$ there are in fact three points which could be the fourth vertex of a parallelogram. Besides (16,19) above, (8, −1) and (−4,3) are possible candidates.





In the latter two cases, the quadrilaterals would be *PQSR* and *PRQS* respectively (or a cyclical permutation thereof).
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Therefore, the only possible position for point *R* is (16, 19) and the above solution is unique.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009 SOLUTION KEY

Round 4

A) $\log_9\left(1+\frac{\sqrt{6}}{3}\right)+\log_9\left(1-\frac{\sqrt{6}}{3}\right) = \log_9\left(\left(1+\frac{\sqrt{6}}{3}\right)\cdot\left(1-\frac{\sqrt{6}}{3}\right)\right)=\log_9\left(1-\frac{6}{9}\right)=\log_9\left(\frac{1}{3}\right)=N$

→ $9^N = 3^{2N} = \frac{1}{3} = 3^{-1}$ → $N =$ $\underline{\mathbf{-\frac{1}{2}}}$

B) $\log_4 x^3 - 2\log_{16} x + \log_{64} x^6 = 3$ → $\log_4 x^3 - \log_{16} x^2 + \log_{64} x^6 = 3$

→ $\log_4 x^3 - \log_4 x + \log_4 x^2 = 3$ → $\log_4 \frac{x^3 \cdot x^2}{x} = \log_4 x^4 = 3$ → $x^4 = 4^3 = 64$

→ $x =$ $\underline{\mathbf{+2\sqrt{2}}}$ only

C) $\frac{2^x - 2^{-x}}{2} = -1.875$ → $2^x - 2^{-x} = -3.75 = -\frac{15}{4}$ Let $A = 2^x$. Then:

$A - \frac{1}{A} = -\frac{15}{4}$ → $4A^2 + 15A - 4 = (4A - 1)(A + 4) = 0$ → $A = 2^x = \begin{cases} \frac{1}{4} = 2^{-2} \\ -4 \text{ (rejected)} \end{cases}$ → $x =$ $\underline{\mathbf{-2}}$

Round 5

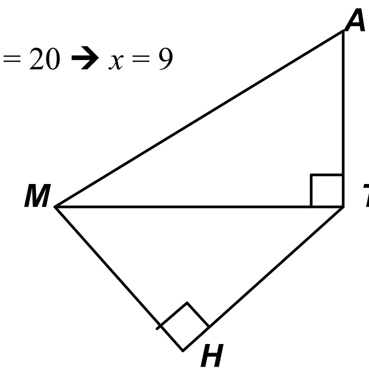
A) $\frac{\sqrt{2009}}{\sqrt{2009k}} = \frac{1}{\sqrt{k}} = \frac{\sqrt{2}}{\sqrt{7}}$ → $\sqrt{k} = \frac{\sqrt{7}}{\sqrt{2}}$ → $k =$ $\underline{\mathbf{\frac{7}{2}}}$

B) Since June is eating (not picking) her rate is −1/3.
Let T denote the time spent picking (and eating). Then:
$\frac{T}{5}+\frac{T}{7}-\frac{T}{3}=1$ → $21T + 15T - 35T = 105$ → $T = 105$
Buckets picked by mom and dad = $\frac{105}{5}+\frac{105}{7} = 21 + 15 =$ **<u>36</u>**

C) $48 = \dfrac{kD}{\frac{D}{40}+\frac{D}{60}} = \dfrac{k}{\frac{1}{40}+\frac{1}{60}} = \dfrac{k(40)(60)}{40+60} = 24k$ → $k = 2$
The distance cancels out, so we can ignore D.
Thus, $R = \dfrac{2}{\frac{1}{8}+\frac{1}{12}} = \dfrac{2(8)(12)}{20} = \dfrac{96}{10} =$ **<u>9.6</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009 SOLUTION KEY

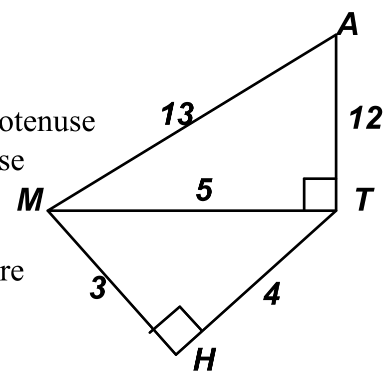
Round 6



A) $(7x - 3) + (6x + 7) + (95 - 4x) = 180$ ➔ $9x + 99 = 180$ ➔ $x + 11 = 20$ ➔ $x = 9$
Thus, the angle measures are 60°, 61° and 59°. m$\angle Q$ = **<u>60</u>**

B)
$$MA^2 + AT^2 + \left(TH^2 + MH^2\right) =$$
$$MA^2 + \left(AT^2 + MT^2\right) =$$
$$MA^2 + MA^2 = 2MA^2 = 21^2 + 21^2 = 441 + 441 = \underline{\mathbf{882}}$$

Alternate solution (motivated by Pope John XIII mathletes):



Simplify, Simplify, Simplify! Working with 21 as the length of the hypotenuse does not allow both of the other legs to have integer lengths. Instead, let's use $AM = 13$ and look for a pattern. (13 was picked because it allows us to use two common Pythagorean triples, namely 3 – 4 – 5 and 5 – 12 – 13.)
We ignore the fact that the diagram suggests that $MT > AT$, since diagrams are not necessarily drawn to scale.
Note that $MA^2 + AT^2 + TH^2 + MH^2 =$
$$3^2 + 4^2 + 12^2 + 13^2 = (9 + 16 + 144) + 169 = 2(169) = 2(13)^2 = 338$$
This suggests a pattern: if $AM = x$, then $MA^2 + AT^2 + TH^2 + MH^2 = 2x^2$.
Thus, 21 ➔ $2(21)^2$ = **<u>882</u>**.
The first argument actually proves this contention using the Pythagorean Theorem twice.

C) Let P have n sides and Q have $(n + 8)$ sides. The measure of the exterior angle of Q is 0.5° **<u>less than</u>** the measure of the exterior angle of P. Thus, $\frac{360}{n} - \frac{360}{n+8} = \frac{1}{2}$ ➔
$$360\left(\frac{8}{n(n+8)}\right) = \frac{1}{2} \text{ ➔ } n(n+8) = 360(16)$$
Rather than trying to factor a quadratic trinomial or forcing the quadratic formula, let's look for a factorization of 360(16) where the factors differ by 8. Redistributing factors of 2, 4 and 5 we have $180 \cdot 32$ ➔ 148 $\quad 90 \cdot 64$ ➔ 26 $\quad 72 \cdot 80$ ➔ 8 $\quad$ Bingo!
Thus, Q has 80 sides and $\frac{80(77)}{2} = 40(77) = \underline{\mathbf{3080}}$ diagonals.

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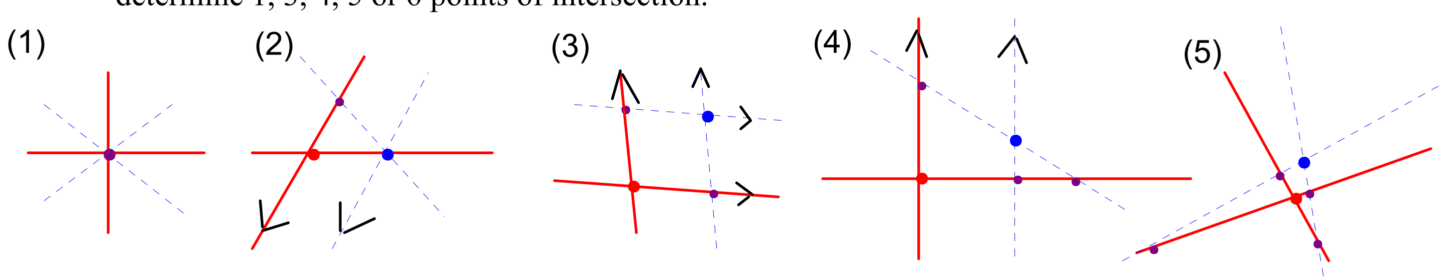
Team Round

A) Since the side opposite the 120° must be the longest side, the side opposite 120° can't be $x - 1$.
Using the law of cosines:
Case 1: $(x + 3)^2 = (x - 1)^2 + (2x - 3)^2 - 2(x - 1)(2x - 3)(-1/2)$
$\rightarrow x^2 + 6x + 9 = 5x^2 - 14x + 10 + 2x^2 - 5x + 3$
$\rightarrow 6x^2 - 25x + 4 = (6x - 1)(x - 4) = 0 \rightarrow 4$ only
Per $= (4 - 1) + (2 \cdot 4 - 3) + (4 - 3) = 3 + 5 + 7 = \underline{\mathbf{15}}$

Case 2: $(2x - 3)^2 = (x - 1)^2 + (x + 3)^2 - 2(x - 1)(x + 3)(-1/2)$
$\rightarrow 4x^2 - 12x + 9 = 2x^2 + 4x + 10 + x^2 + 2x - 3$
$\rightarrow x^2 - 18x + 2 = 0 \rightarrow (x - 9)^2 = -2 + 81 = 79 \rightarrow x = 9 + \sqrt{79}$
Per $= (9 + \sqrt{79} - 1) + (9 + \sqrt{79} + 3) + \left(2(9 + \sqrt{79}) - 3\right)$
$= (8 + 12 + 15) + (1 + 1 + 2)\sqrt{79} = \underline{\mathbf{35 + 4\sqrt{79}}}$ (but this is a larger perimeter)

B) The numbers in the n^{th} row are: $2n + 7$, m and $(m + 1)$
$n = 11 \rightarrow 29^2 + m^2 = (m + 1)^2 \rightarrow 841 = 2m + 1 \rightarrow m = 420$
We must evaluate $29^2 + 420^2 + 421^2$, but we know from the Pythagorean Theorem that the sum of the first two terms equals the third.
Thus, the required sum is $2(421)^2 = 2(177241) = \underline{\mathbf{354482}}$

C) Each graph consists of two intersecting lines. In general, two pair of intersecting lines can determine 1, 3, 4, 5 or 6 points of intersection.



$(x - y + 2)(3x + y - 4) = 0 \rightarrow$ lines with slopes of +1 and −3 and intersecting at $P\left(\frac{1}{2}, \frac{5}{2}\right)$

$(x + y - 2)(2x - 5y + 7) = 0 \rightarrow$ lines with slopes of −1 and 2/5 and intersecting at $Q\left(\frac{3}{7}, \frac{11}{7}\right)$

Thus, since none of the given lines are parallel and at least 2 points were determined, the maximum number of points were determined, i.e. 6 - diagram (5) above. However, each graph consists of a pair of intersecting lines and the intersection of the graphs consists of only points determined by a red line and a blue line, $6 - 2 = \underline{\mathbf{4}}$.
An aside:
Call the lines A, B, C and D. Two lines determine at most one point of intersection.
Thus, four lines determine at most ${}_4C_2 = 6$ possible points of intersection. Clearly, four parallel lines would produce no points of intersection. The diagrams above indicate the five other possible cases. Try as you may, exactly two points of intersection is impossible.

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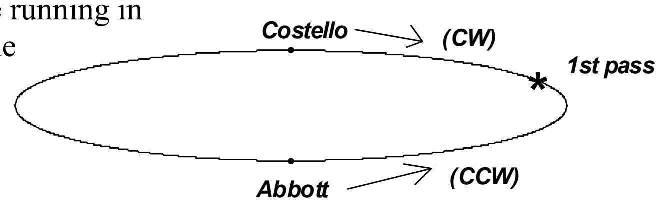
D) $f(x) = g(x) \Rightarrow \dfrac{2^x - 2^{-x}}{2} = \dfrac{2}{2^x + 2^{-x}} \Rightarrow \left(2^x\right)^2 - \left(2^{-x}\right)^2 = 4 \Rightarrow 4^x - 4^{-x} = 4$

Let $Y = 4^x$. Then: $Y - \dfrac{1}{Y} = 4 \Rightarrow Y^2 - 4Y - 1 = 0 \Rightarrow Y = \dfrac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$

($2 - \sqrt{5}$ is extraneous.) Thus, $4^x = 2 + \sqrt{5} \Rightarrow 2^{2x} = 2 + \sqrt{5} \Rightarrow 2x = \log_2\left(2 + \sqrt{5}\right)$

$\Rightarrow x = \dfrac{1}{2}\log_2\left(2+\sqrt{5}\right) = \log_2\sqrt{2+\sqrt{5}} \Rightarrow N = \underline{\boldsymbol{\sqrt{2+\sqrt{5}}}}$

E) Since distance equals rate times time, the runners cover distances of $5t$ and $4t$ laps respectively in t units of time. Since they are running in opposite directions, they pass for the first time when $5t + 4t = \dfrac{1}{2}$, for the second time when $5t + 4t = 1 + \dfrac{1}{2} = \dfrac{3}{2}$, etc.



In general, they pass for the m^{th} time when $5t + 4t = m - \dfrac{1}{2}$ or when $t = \dfrac{2m-1}{18}$.

For Abbott to complete n laps takes $t = \dfrac{n}{5}$ units of time. Therefore, the number of times the runners pass each other is the <u>largest</u> integer m for which $\dfrac{2m-1}{18} \le \dfrac{n}{5}$ or $m \le \dfrac{18n+5}{10}$

If k denotes the number of laps Abbott has completed when the runners have passed each other for the 100^{th} time, then $100 \le \dfrac{18k+5}{10} \Rightarrow 18k \ge 995 \Rightarrow k \ge 55^{+} \Rightarrow k = \underline{\mathbf{56}}$

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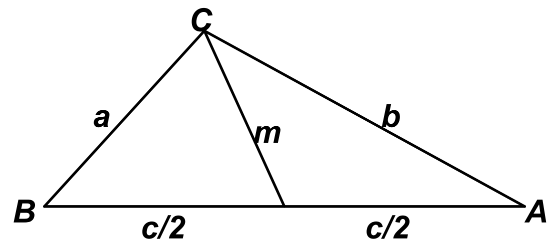

F) Using Stewart's Theorem, $a^2\dfrac{c}{2}+b^2\dfrac{c}{2}=m^2c+c\cdot\dfrac{c}{2}\cdot\dfrac{c}{2}$

$c \neq 0$ ➔ $\dfrac{a^2+b^2}{2}=m^2+\dfrac{c^2}{4}$ ➔ $2(a^2+b^2)-c^2=4m^2$

➔ $m=\dfrac{1}{2}\sqrt{2(a^2+b^2)-c^2}$

<u>Case 1</u>: $c = 8$ and $a + b = 16$

$m=\dfrac{1}{2}\sqrt{2(a^2+(16-a)^2)-8^2} = \dfrac{1}{2}\sqrt{4a^2-64a+448} = \sqrt{a^2-16a+112} = \sqrt{(a-8)^2+48}$

Since $c = 8$, $a + b = 16$, $a < b$, the triangle inequality implies we must only try $a = 5\ldots8$.

5 ➔ $\sqrt{57}$, 6 ➔ $\sqrt{52}$, 8 ➔ $\sqrt{48}$, $a = 7$ ➔ $m = \underline{\mathbf{7}}$ (and $b = 9$)

(Check: $7^2\dfrac{8}{2}+9^2\dfrac{8}{2}=7^2 8+8\cdot\dfrac{8}{2}\cdot\dfrac{8}{2}$ = 520)

<u>Case 2</u>: $c = 10$ and $a + b = 14$

$m=\dfrac{1}{2}\sqrt{2(a^2+(14-a)^2)-10^2} = \dfrac{1}{2}\sqrt{4a^2-56a+292} = \sqrt{a^2-14a+73} = \sqrt{(a-7)^2+24}$

We must try $a = 1 \ldots 7$.

$a = 1$ ➔ $\sqrt{60}$, 3 ➔ $\sqrt{40}$, 4 ➔ $\sqrt{33}$, 5 ➔ $\sqrt{28}$, 7 ➔ $\sqrt{24}$

$a = 2$, $m = 7$, $b = 12$, $c = 10$ (rejected – Triangle Inequality fails)

$a = 6$, $m = \underline{\mathbf{5}}$, $b = 8$, $c = 10$

(Check: $6^2\dfrac{10}{2}+8^2\dfrac{10}{2}=5^2 10+10\cdot\dfrac{10}{2}\cdot\dfrac{10}{2}$ = 500)

The following code snippet found two other triangles with a smaller perimeter.

```
FOR a = 1 TO 9
  FOR b = a TO 10
    FOR c = 1 TO 10

      IF ItsaTriangle(a, b, c) THEN
        m = 2 * (a ^ 2 + b ^ 2) - c ^ 2
        IF m > 0 THEN m = .5 * SQR(m)

        IF m = INT(m) THEN
          IF ItsaTriangle(a, m, c / 2) AND ItsaTriangle(m, b, c / 2) THEN
              PRINT a; b; c; "M = "; m

FUNCTION ItsaTriangle% (p, q, r)
  ItsaTriangle% = ((p + q > r) AND (p + r > q) AND (q + r > p))
```

```
➔      5       5       6       M = 4
       5       5       8       M = 3
       6       8      10       M = 5
       7       9       8       M = 7
```

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F) - continued

Note that each of the first three solutions involves at least one right triangle.
Figure 4 is the smallest solution that does <u>not</u> involve a right triangle.

Fig. 1 (Per = 16)

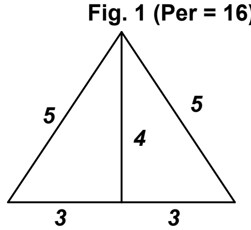

Fig. 2 (Per = 18)

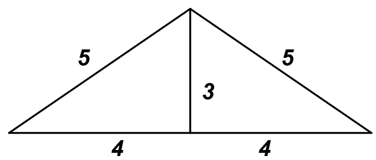

Fig. 3 (Per = 24, m = 5)

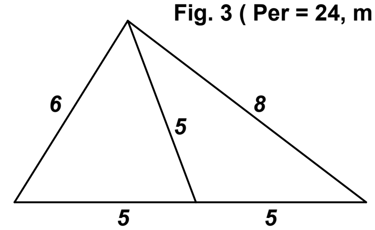

Fig. 4 (Per = 24, m = 7)