MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2012
ROUND 1 ALG 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS

ANSWERS

A) (_______ , _______ , _______, _______)

B) (_________ , _________ , _________)

C) ___________________________________

******* NO CALCULATORS IN THIS ROUND *******

A) For a unique ordered quadruple (x, a, b, c), $\begin{bmatrix} 19-3x & 2x+b \\ x+a & \frac{x}{3}+c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the 2 x 2 identity matrix.

Compute (x, a, b, c).

B) There are 3 integers n_1, n_2 and n_3 for which $\begin{vmatrix} 1 & x & 2 \\ x & 1 & 2 \\ 1 & 2 & x \end{vmatrix} = 0$.

Compute the ordered triple (n_1, n_2, n_3), where $n_1 < n_2 < n_3$.

C) For positive integers a and b, the system of equations $\begin{cases} x+y+1=0 \\ 2x-y+a=0 \\ 3x+4y+b=0 \end{cases}$ defines a set of <u>concurrent</u> lines, i.e. they intersect at a common point. Determine (x, y), the coordinates of the point of concurrency, for which $a + b$ is a <u>maximum</u>.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2012
ROUND 2 ALG1: EXPONENTS AND RADICALS

ANSWERS

A) (______________ , ______________)

B) ________________________________

C) (________ , ________ , ________)

******* NO CALCULATORS IN THIS ROUND *******

A) For positive integers k and A, $A^2 = 6000k$.
Compute (k, A), where k is the smallest integer for which the ordered pair satisfies the equation.

B) Let $L = \dfrac{0.125}{2}$. For many integers P and Q, $2^{-P} < L$ and $3^{-Q} < L$.
Compute the <u>minimum</u> value of $P + Q$.

C) Usually radicals with different indices cannot be combined.
Given: A, B are positive integers and $A, B < 4$
Compute the ordered triple (N, C, X), where N, C and X are positive integers and C is as small as possible, for which

$$\sqrt[12]{16(27)(128)(1024)} + 3\sqrt[4]{2^A 3^B} = N\left(\sqrt[C]{X}\right)$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2012
ROUND 3 TRIGONOM ETRY: ANYTHING

ANSWERS

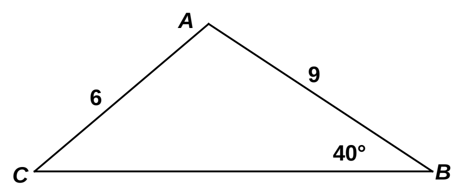
A) (______________ , _____________ °)

B) (______________ , ______________)

C) ________________________________

******* NO CALCULATORS IN THIS ROUND *******

A) In ΔABC, $AB = 9$, $AC = 6$ and $m\angle B = 40°$, as shown in the diagram at the right.
If $\sin(C) = r\cos\theta°$, compute the ordered pair $(r, \theta°)$, where $r > 0$ and θ is an <u>acute</u> angle.



B) The graph of the function defined by $y = -3\sin\left(4x - \frac{\pi}{2}\right) + 1$ attains a maximum value at the point $P(a, b)$. Compute the ordered pair (a, b), where a has the <u>smallest</u> possible positive value.

C) Given: $\left(-\sqrt{3} + i\right)^{400} = A + Bi$, where $i = \sqrt{-1}$. Compute $\left(\frac{A}{B}\right)^4$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2012
ROUND 4 ALG 1: ANYTHING

ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) _________________________ years

******* NO CALCULATORS IN THIS ROUND *******

A) Compute the lengths of the sides of a right triangle with sides $\sqrt{n}$, $\sqrt{n+4}$, $\sqrt{n+8}$
Be sure to express all radicals in simplified form.
List the side lengths from smallest to largest.

B) Despite the arbitrary constant, the slope of the line through $P(6 - a, 4)$ and $Q(3, 7 - a)$ <u>always</u> has a numerical value of m unless $a = k$. Compute the ordered pair (m, k).

C) Five years ago, Sam and David's ages were in a 5 : 2 ratio. 15 years from now Sam will be twice as old as David will be 7 years from now. In how many years (from now) will the ratio of Sam's age to David's age be 10 : 7?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2012
ROUND 5 PLANE GEOMETRY: ANYTHING

ANSWERS

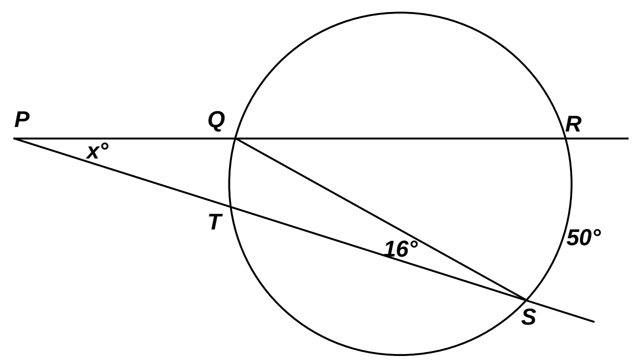
A) ______________________________

B) ______________________________

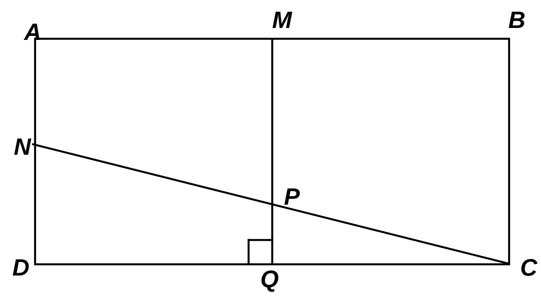
C) ______________________________

******* NO CALCULATORS IN THIS ROUND *******

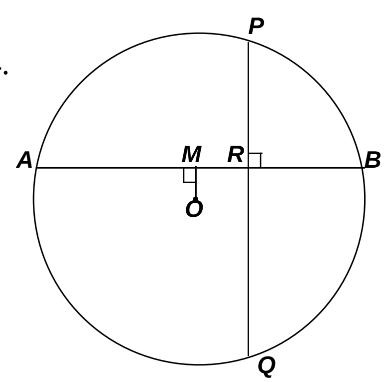
A) Compute x.



B) M and N are midpoints of $\overline{AB}$ and $\overline{AD}$ in rectangle $ABCD$.
What fraction of the area of rectangle $ABCD$ is the area of the trapezoidal region $PQDN$?



C) Chord $\overline{AB}$ is 2 units from the center of circle O whose diameter is 14.
Chord $\overline{PQ}$ is perpendicular to chord $\overline{AB}$, intersecting in common point R.
If $m\angle PBA = 60°$ and $AR : BR = 2 : 1$, compute QR.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2012
ROUND 6 ALG 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

******* NO CALCULATORS IN THIS ROUND *******

A) Two fair cubical dice are tossed. The sum of the dice is considered.
Find the ratio of the probability of getting a sum larger than nine to the probability of <u>not</u> getting a sum of three or four.

B) Alice, Ben, Carol, David and Ethan are seated on stage in 6 seats arranged in a row. One seat remains empty. In how many of these arrangements is at least one person seated between Alice and Carol, but no seats between Alice and Carol are empty?

C) The ratio of the constant term in the expansion of $\left(x^3+\frac{1}{x^2}\right)^{15}$ to the constant term in the expansion of $\left(x^4+\frac{1}{x^3}\right)^n$ is $\frac{5}{3}$.
If n is a positive integer, compute the value of n.
Note: The constant term in each of these binomial expansions is the term with no x's.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2012
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ____________________________ D) ____________________________

B) K = __________ , J = __________ E) ____________________________

C) ____________________________ F) ____________________________

***** NO CALCULATORS IN THIS ROUND *****

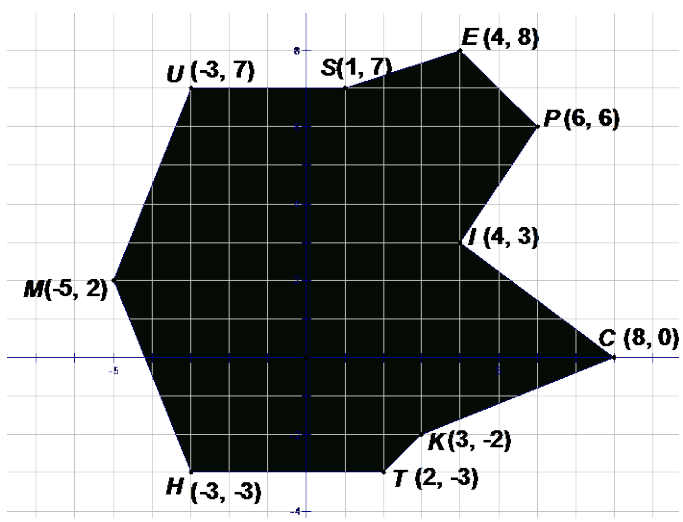
A) Compute the ordered triple (a, b, c) which solves the system $\begin{cases} x+7y+5z=12 \\ 2x+9y+4z=20 \\ 6x+Ay+3z=19 \end{cases}$ over the integers for the smallest possible positive integer value of A.

B) Suppose $K = a^{2x} + a^{-2x}$ and $J = \left(a^{x} + a^{-x}\right) + \left(a^{4x} + a^{-4x}\right)$.
a and x are real numbers ($a > 0$), but J and K are both positive integers.
Compute the ordered pair (K, J), if K is the minimum value for which $J > 2012$.

C) A circle of radius r ($0 < r < 1$) is centered at (4, 1).
Two particles A and B both starting at $(4 + r, 1)$ rotate around the circle.
A stops at point P after rotating 945° (i.e. counterclockwise).
B stops at point Q after rotating –1140° (i.e. clockwise).
Compute exactly how much closer to the x-axis one point is than the other, in terms of r.

D) Compute all real values of x for which
$\left(x^2-2x-8\right)^2 = 2(x-1)^2+17$.

E) Compute the area of polygon *EPICKTHMUS*.



F) A license plate consists of 6 distinct nonzero digits. The plate $ABC \cdot DEF$ is considered "memorable" if all six digits are either in increasing or decreasing order, but not necessarily consecutive.
[Ex: 234 · 567, 123 · 789, 875 · 421 are memorable, 125 · 489 and 125 · 976 are not.]

One startled, but alert, eyewitness to a bank robbery couldn't be specific about the digits, but was sure the plate was "memorable", while another independent witness reported that the leftmost digit was neither a 1 nor a 2. If both of these witnesses were reliable, how many plates remained for the police to cross check?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2012 ANSWERS

Round 1 Alg 2: Simultaneous Equations and Determinants

A) $(6, -6, -12, -1)$ B) $(-3, 1, 2)$ C) $(-3, 2)$

Round 2 Alg 1: Exponents and Radicals

A) $(15, 300)$ B) 8 C) $(5, 4, 24)$

Round 3 Trigonometry: Anything

A) $(1.5, 50°)$ B) $\left(\frac{\pi}{2}, 4\right)$ C) $\frac{1}{9}$

Round 4 Alg 1: Anything

A) $2, 2\sqrt{2}, 2\sqrt{3}$ B) $(-1, 3)$ C) 15 [Ages now S: 25 D: 13]

Round 5 Plane Geometry: Anything

A) 9 B) $\frac{3}{16}$ C) $\frac{4}{3}\sqrt{15}$

Round 6 Alg 2: Probability and the Binomial Theorem

A) 6 : 31 B) 240 C) 14

Team Round

A) $(3, 2, -1)$

D) $-3, -1, 3, 5$ (in any order)

B) $(47, 2214)$

E) 94

C) $\frac{r}{2}\left(\sqrt{3}-\sqrt{2}\right)$

F) 91

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2012 SOLUTION KEY

Round 1

A) $19 - 3x = 1 \Rightarrow x = 6$ which in turn implies $a = -6$, $b = -12$ and $c = -1$.
Therefore, (x, a, b, c) = **<u>(6, –6, –12, –1)</u>**.

B) $\begin{vmatrix} 1 & x & 2 \\ x & 1 & 2 \\ 1 & 2 & x \end{vmatrix} \begin{matrix} 1 & x \\ x & 1 \\ 1 & 2 \end{matrix} \Rightarrow (x + 2x + 4x) - (2 + 4 + x^3) = -x^3 + 7x - 6 = 0$

Clearly, $x = 1$ is a solution and by synthetic division
$-x^3 + 7x - 6 = -1(x^3 - 7x + 6) = -1(x-1)(x^2 + x - 6) = -1(x-1)(x-2)(x+3) = 0$
The roots are 1, 2 and –3 and $(n_1, n_2, n_3) = \underline{\mathbf{(-3,1,2)}}$.

C) Adding the first two equations, $3x + (1 + a) = 0$ or $x = \frac{-(1+a)}{3}$.
Multiplying the first equation by 4 and subtracting the third, $x = b - 4$ (***).
Equating, $\frac{-(1+a)}{3} = b - 4 \Leftrightarrow 3b + a = 11$.
Over positive integers, $(a, b) = (2, 3), (5, 2)$ and $(8, 1)$. The maximum occurs for $a = 8$ and $b = 1$.
Substituting for b in (***), $x = -3$.
Substituting back in the first equation ($x + y + 1 = 0$), we have **(x, y) = <u>(–3, 2)</u>.**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2012 SOLUTION KEY

Round 2

A) $A^2 = 6000k = 2(3)\left(10^3\right)k = 2^4 3^1 5^3 k$

If A is an integer, each prime on the right hand side of the equation must be raised to an even power. The smallest value of k which provides this luxury is $k = 3(5) = 15$.

Thus, $A^2 = 2^4 3^2 5^4 \Leftrightarrow A = 2^2 \cdot 3 \cdot 5^2 = 3(100) = 300 \Rightarrow (k, A) = \underline{\mathbf{(15, 300)}}$.

B) $$\left.\begin{array}{l} 2^{-P} = \dfrac{1}{2^P} \\ 3^{-Q} = \dfrac{1}{3^Q} \end{array}\right\} < \frac{0.125}{2} = \frac{2^{-3}}{2} = 2^{-4} = \frac{1}{16} \Rightarrow 2^P > 16 \text{ and } 3^Q > 16$$

$\Rightarrow P \geq 5$ and $Q \geq 3 \Rightarrow (P + Q)_{\min} = \underline{\mathbf{8}}$.

C) $16(27)(128)(1024) = 2^4 3^3 2^7 2^{10} = 2^{21} 3^3$

$\sqrt[12]{16(27)(128)(1024)} = \sqrt[12]{2^{21} 3^3} = 2\sqrt[12]{2^9 3^3} = 2\sqrt[4]{2^3 3^1}$

Radicals can be added only if the radicands and the indices are the same.

Thus, $(A, B) = (3,1)$ and the sum would be $5\sqrt[4]{2^3 3^1} = 5\sqrt[4]{24}$ and the required ordered triple is **<u>(5, 4, 24)</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2012 SOLUTION KEY

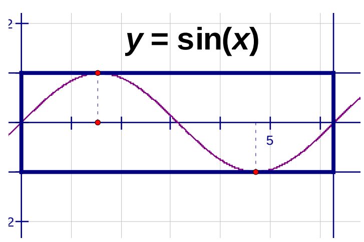
Round 3

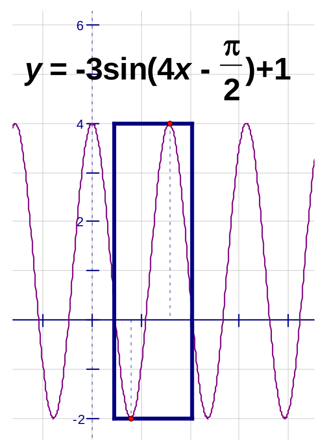
A) Using the Law of Sines, $\frac{\sin 40}{6} = \frac{\sin C}{9} \Rightarrow \sin C = \frac{3}{2}\sin 40° = 1.5\cos 50° \Rightarrow (r,\theta) = \underline{\mathbf{(1.5,\ 50°)}}$.

B) The sine function normally assumes values between a minimum of –1 (at $\frac{3\pi}{2}$) and a maximum of 1 (at $\frac{\pi}{2}$). The factor of –3 "flips" the graph over the *x*-axis and increases the fluctuation to 3 units above and below the center line (normally the *x*-axis). The +1 shifts the entire graph vertically 1 unit. So the maximum value is +4 and we need only determine for what value(s) of *x* it occurs. The maximums occur when $4x - \frac{\pi}{2} = \frac{\mathbf{3\pi}}{\mathbf{2}} + 2n\pi$ (Remember the graph is flipped.)

Thus, $x = \frac{2\pi + 2n\pi}{4} = \frac{\pi(n+1)}{2}$. For *n* = 0, we have a maximum at $\underline{\left(\boldsymbol{\frac{\pi}{2}}, \mathbf{4}\right)}$.

In each case, the rectangle encloses one period of the graph. For $y = \sin(x)$ the period is 2π and the rectangle extends left to right from 0 to 2π, and top to bottom from –1 to +1. Divide the rectangle into quarters and we have located a maximum, a zero and a minimum. For $y = -3\sin\left(4x - \frac{\pi}{2}\right) + 1$ the period is $\frac{\pi}{2}$ and the rectangle extends left to right from $\frac{\pi}{8}$ to $\frac{5\pi}{8}$, and top to bottom from –2 to +4. Likewise, dividing the rectangle into quarters we can locate the critical points. Here are the graphs of these two functions.





C) Converting to trig form, $-\sqrt{3} + i = 2cis\left(\frac{5\pi}{6}\right)$ [$r^2 = \left(\sqrt{3}\right)^2 + 1^2$, $\theta \in$ QII and $\tan\theta = -\frac{1}{\sqrt{3}}$]

$= \left(-\sqrt{3} + i\right)^{400} = 2^{400} cis\left(\frac{1000\pi}{3}\right) = 2^{400} cis\left(\frac{4\pi}{3}\right) = 2^{400}\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right] =$

$2^{400}\left(-\frac{1}{2}\right)\left[1 + i\sqrt{3}\right] = A + Bi$ Evaluating $\left(\frac{A}{B}\right)^4$, the factors of 2^{400} and $-\frac{1}{2}$ will cancel!

Thus, $\left(\frac{A}{B}\right)^4 = \left(\frac{A}{A\sqrt{3}}\right)^4 = \underline{\mathbf{\frac{1}{9}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2012 SOLUTION KEY

Round 4

A) Using the Pythagorean Theorem, $\left(\sqrt{n}\right)^2 + \left(\sqrt{n+4}\right)^2 = \left(\sqrt{n+8}\right)^2$

Since $n > 0$, $n + (n + 4) = (n + 8) \Rightarrow n = 4 \Rightarrow$ sides: 2, $\sqrt{8}$, $\sqrt{12}$ $\Rightarrow$ <u>**2, $2\sqrt{2}$, $2\sqrt{3}$**</u>

B) If the line is vertical (or $P = Q$), the slope is undefined (or indeterminate).
Equating the x - coordinates, the line is vertical when $6 – a = 3$, namely for $a = 3$.
For any other value of a, the slope is always the same, so we simply pick another value for a and substitute

$a = 0 \Rightarrow P(6, 4), Q(3, 7) \Rightarrow m = \dfrac{7-4}{3-6} = \dfrac{3}{-3} = -1$

Thus, (m, k) = <u>**(–1, 3)**</u>.

<u>**FYI**</u>: Proof that the slope is always –1 (unless $a = 3$)

$$m = \frac{4-(7-a)}{(6-a)-3} = \frac{-3+a}{3-a} = \frac{-\cancel{(3-a)}}{\cancel{3-a}} = -1$$

C) $\begin{cases} \dfrac{s-5}{d-5} = \dfrac{5}{2} \\ s+15 = 2(d+7) \end{cases} \Rightarrow s = 2d - 1$. Substituting, $\dfrac{2d-6}{d-5} = \dfrac{5}{2}$

Cross multiplying, $4d – 12 = 5d – 25 \Leftrightarrow d = 13$ and $s = 25$.

If the 10 : 7 ratio occurs in x years, $\dfrac{25+x}{13+x} = \dfrac{10}{7} \Leftrightarrow 175 + 7x = 130 + 10x \Leftrightarrow 3x = 45 \Leftrightarrow x =$ <u>**15**</u>

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2012 SOLUTION KEY

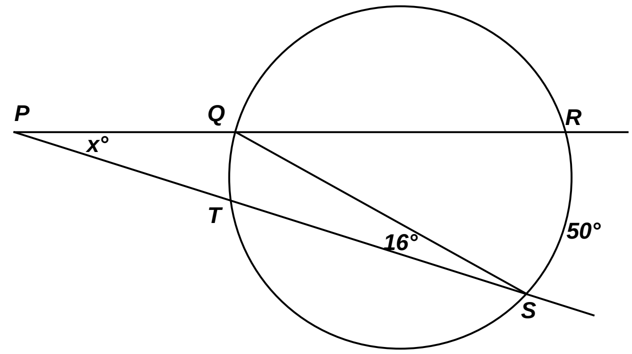
Round 5

A) Let $\angle P = x°$

As an exterior angle of ΔPQS, m$\angle RQS$ = $(x + 16)°$.

As an inscribed angle, m$\angle RQS$ = $\frac{1}{2}(50) = 25°$.

Equating, x = **<u>9</u>**°.



B) Let $AM = MB = a$ and $AN = ND = b$.

Then: $ABCD$ has area $4ab$.

Since $\Delta CPQ \sim \Delta CND$, $\frac{PQ}{ND} = \frac{CQ}{CD} = \frac{1}{2} \Rightarrow PQ = \frac{b}{2}$.

$PQDN$ has area $\frac{1}{2}(a)\left(b + \frac{b}{2}\right) = \frac{3ab}{4}$

Thus, the required fraction is $\frac{3ab/4}{4ab} = \underline{\mathbf{\frac{3}{16}}}$.

C) Let O denote the center of the circle.

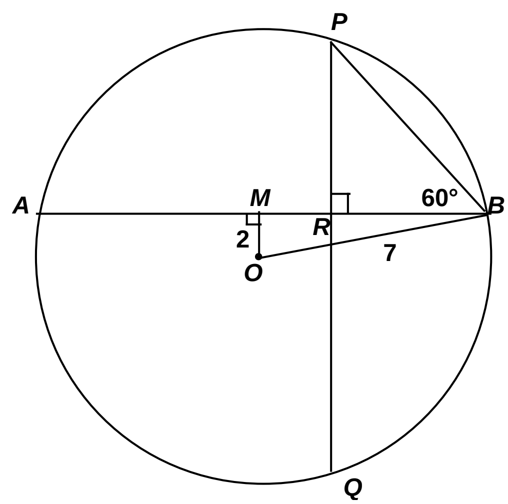
$(OM, OB) = (2, 7) \Rightarrow BM = 3\sqrt{5}$

Since a line through the center of a circle perpendicular to a chord bisects that chord, $AB = 6\sqrt{5}$.

$AR : RB = 2 : 1 \Rightarrow BR = 2\sqrt{5} \Rightarrow PR = \left(2\sqrt{5}\right)\cdot\sqrt{3} = 2\sqrt{15}$

Using the product chord theorem, $AR \cdot RB = PR \cdot RQ$.

$\left(4\sqrt{5}\right)\left(2\sqrt{5}\right) = 2\sqrt{15}\cdot RQ \Rightarrow RQ = \frac{20}{\sqrt{15}} = \underline{\mathbf{\frac{4}{3}\sqrt{15}}}$



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2012 SOLUTION KEY

Round 6

A) Sum greater than 9: (4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6) – 6 possibilities
Sum not 3 and not 4: Of the 36 possible ordered pairs, reject (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), leaving 31 possibilities. Thus, the required ratio is **6 : 31**.

B) In all cases

Seats	1	2	3	4	5	6

- Pick seats for Alice and Carol
- Seat Alice and Carol in these seats
- Fill the in-between seats
- Fill the other seats (leaving one empty)

Case 1: (1 person in-between) $4 \cdot 2 \cdot 3 \cdot 3! = 144$ [A _ C in seats 123 … 456]
Case 2: (2 persons in-between) $3 \cdot 2 \cdot (3 \cdot 2) \cdot 2! = 72$ [A _ _ C in seats 1234… 3456]
Case 3: (3 persons in-between) $2 \cdot 2 \cdot (3 \cdot 2 \cdot 1) \cdot 1 = 24$ [A _ _ _ C in seats 12345 … 23456]
Total: **240**

C) For $\left(x^3 + \frac{1}{x^2}\right)^{15}$, the $(k+1)^{\text{st}}$ term is $\binom{15}{k}\left(x^3\right)^{15-k}\left(x^{-2}\right)^k = \binom{15}{k} x^{45-5k}$ and this will be the constant term when $x = 9$, i.e. the constant term is $\binom{15}{9} = \frac{15!}{9! \cdot 6!}$.

Similarly, for $\left(x^4 + \frac{1}{x^3}\right)^n$, the $(k+1)^{\text{st}}$ term is $\binom{n}{k}\left(x^4\right)^{n-k}\left(x^{-3}\right)^k = \binom{n}{k} x^{4n-7k}$ and the constant term requires $4n - 7k = 0$, so $k = \frac{4n}{7}$. The required ratio is $\frac{\binom{15}{9}}{\binom{n}{k}} = \frac{15! \cdot k! \cdot (n-k)!}{9! \cdot 6! \cdot n!} = \frac{5}{3}$

Since 7 is prime, n must be divisible by 7.

$n = 7$ and $k = 4$ clearly will not produce a fraction which reduces to $\frac{5}{3}$.

Try $n =$ **14**. Then: $k = 8$ and $\frac{15! \cdot k! \cdot (n-k)!}{9! \cdot 6! \cdot n!} = \frac{15! \cdot 8! \cdot 6!}{9! \cdot 6! \cdot 14!} = \frac{15}{9} = \frac{5}{3}$, the required ratio.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2012 SOLUTION KEY

Team Round

A) $\begin{cases} x+7y+5z=12 \\ 2x+9y+4z=20 \\ 6x+Ay+3z=19 \end{cases}$

Solution #1: (Pretty much brute force – Tedious but nothing difficult)

- Use first two equations to get expressions for *x* and *z* in terms of *y*.
- Substitute into third equation and get an expression for *y* in terms of *A*
- Find smallest positive value for *A* which makes this expression an integer
- Substitute back to get *x*, *y* and *z*.

Multiply equation 1 by –2 and add to second equation: –5y – 6z = –4 or $z = \dfrac{4-5y}{6}$

Using equation 1, $x = 12 - 7y - 5z \Rightarrow x = 12 - 7y - 5\left(\dfrac{4-5y}{6}\right) = \dfrac{72-42y-20+25y}{6} = \dfrac{52-17y}{6}$

In equation 3, $6\left(\dfrac{52-17y}{6}\right) + Ay + 3\left(\dfrac{4-5y}{6}\right) = 19 \Leftrightarrow 104 - 34y + 2Ay + 4 - 5y = 38$

$\Leftrightarrow 70 + (2A-39)y = 0 \Leftrightarrow y = \dfrac{70}{39-2A}$

Since *A* = 2 is the smallest positive value of *A* that produces an integer value of *y*,
$y = \dfrac{70}{35} = 2$, $x = \dfrac{52-34}{6} = 3$ and $z = \dfrac{4-10}{6} = -1 \Rightarrow$ **(3, 2, –1)**.

Solution #2: Triangularization – The Big Idea
Elementary row operations (EROs) on a square matrix are the equivalent of using linear combinations on the system of coefficients of (linear) equations like the system we have in this problem. Just as linear combinations produce equivalent systems that are "easier" to solve, EROs focus on the coefficients of the equations in the system and ignore the variables. The variables are understood – from left to right the columns hold the *x*, *y*, *z*-coefficients (unless columns are interchanged)
There are three types of EROs:

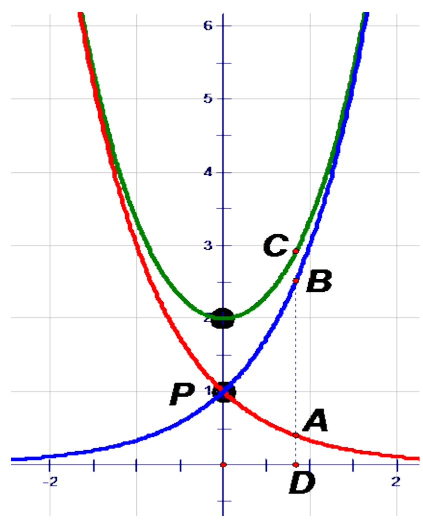
- interchanging any two rows (or columns)
- replacing a row (column) by that row (column) times and nonzero constant.
- replacing a row with a linear combination of itself and any other row, e.g. for any two rows *A* and *B* and nonzero constants *m* and *n*, row *B* may be replaced by *mA* + *nB*. Again this property also holds for columns.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2012 SOLUTION KEY

Team Round
A) - continued
The strategy will be to:

Use elementary row operations on the matrix of coefficients $\begin{bmatrix} 1 & 7 & 5 \\ 2 & 9 & 4 \\ 6 & A & 3 \end{bmatrix}$ and convert it to a triangular matrix where all the entries below the main diagonal are zero.

This square matrix will take on the form $\begin{bmatrix} 1 & 7 & 5 \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix}$. In fact, we will tack on the constants from the right side of the equation before we start the triangularization process and get a matrix of coefficients for an **equivalent** system of equations, namely $\begin{bmatrix} 1 & 7 & 5 & n_1 \\ 0 & a & b & n_2 \\ 0 & 0 & c & n_3 \end{bmatrix} \Rightarrow \begin{cases} x+7y+5z=n_1 \\ ay+bz=n_2 \\ cz=n_3 \end{cases}$ which is easily solved by **backtracking**. Substitute $z=\dfrac{n_3}{c}$ into the 2nd equation to get y. Substitute both of these values into the 1st equation to get x. This is a systematic process which is relatively easy to adapt to a computer algorithm and let the computer do the number crunching.
Details are in the addendum at the end of the solution key.



B) $\left(a^x+a^{-x}\right)^2=\left(a^{2x}+a^{-2x}\right)+2=K+2 \Rightarrow a^x+a^{-x}=+\sqrt{K+2}$

Examining the graph of $y=a^x+a^{-x}$ (for $a>0$), we see that $-\sqrt{K+2}$ is extraneous. Any vertical line selects corresponding points on the graphs of $y=a^x$ and $y=a^{-x}$. Adding these two values always produces a value greater than or equal to 1 + 1 = 2 (at P, the point of intersection).
For example, on the vertical dotted line, if $BC = AD$, then point C would represent the point on $y=a^x+a^{-x}$ for the selected x - value. Applying the same arguments to $y=a^{2x}+a^{-2x}$, we see that $K \ge 2$.

$a^{4x}+a^{-4x}=\left(a^{2x}+a^{-2x}\right)^2-2=K^2-2$

Thus, $J=\sqrt{K+2}+\left(K^2-2\right)$. Consequently, $(k + 2)$ must be a perfect square.
Examining K = 2, 7, 14, 23, …, we get $(2^2 + 0)$, $(7^2 + 1)$, $(14^2 + 2)$, $(23^2 + 3)$… .
Continuing $34^2 + 4 < 2012$, but $47^2 + 5 = (50 - 3)^2 + 5 = 2500 - 300 + 9 + 5 = 2214 > 2012$.
Thus, (K, J) = **(47, 2214)**.

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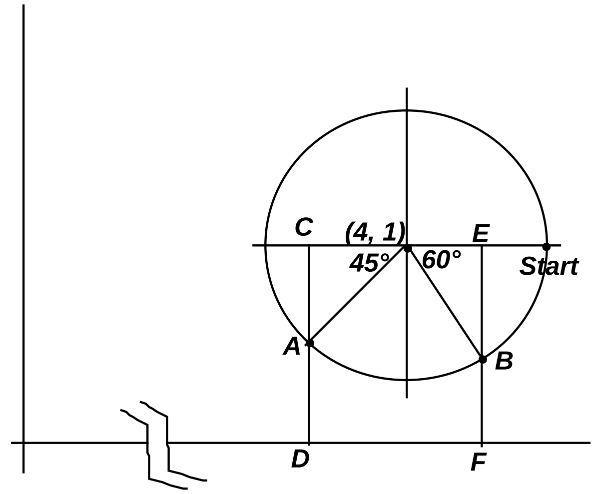
Team Round



C) $945° = 2(360°) + 225° \Rightarrow$ Q3 (45° reference angle)
$-1140° = -4(360°) + 300° \Rightarrow$ Q4 (60° reference angle)

$AD = CD - AC = 1 - \dfrac{r\sqrt{2}}{2}$

$BF = EF - EB = 1 - \dfrac{r\sqrt{3}}{2}$

$AD > BF \Rightarrow$ positive difference $= AD - BF$

$= \left(1 - \dfrac{r\sqrt{2}}{2}\right) - \left(1 - \dfrac{r\sqrt{3}}{2}\right) = \underline{\boldsymbol{\dfrac{r}{2}\left(\sqrt{3} - \sqrt{2}\right)}}$

D) $\left(x^2 - 2x - 8\right)^2 = 2(x-1)^2 + 17 \Rightarrow \left((x-1)^2 - 9\right)^2 = 2\left((x-1)^2 - 9\right) + 35$

Let $A = (x-1)^2 - 9$. Substitute, move the terms to the left side and factor.

$A^2 - 2A - 35 = (A-7)(A+5) = 0 \Rightarrow A = +7, -5$

$\Rightarrow (x-1)^2 - 9 = 7 \Rightarrow x - 1 = \pm 4 \Rightarrow x = 5, -3$ or

$\Rightarrow (x-1)^2 - 9 = -5 \Rightarrow x - 1 = \pm 2 \Rightarrow x = 3, -1$

Thus, there are 4 solutions: **$\underline{-3, -1, 3, 5}$**

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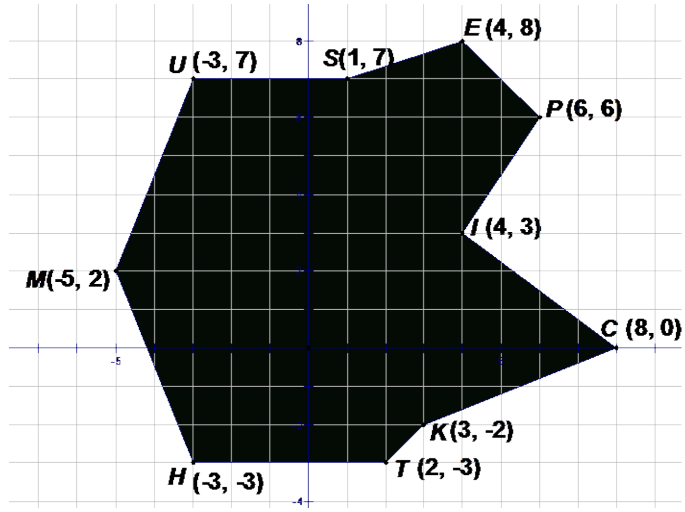
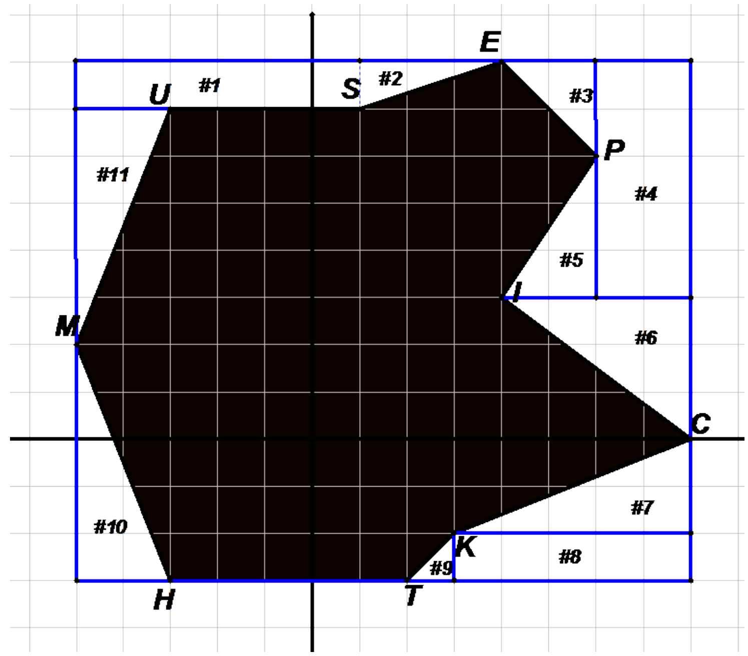
E) Using Pick's Theorem $\left(A = I + \frac{B}{2} - 1\right)$, where

I = # of interior points
B = # points on the boundary

There are 12 vertical lines through the interior of the region.
Counting interior points from left to right:
5 + 5(9) + 10 + 9 + 8 + 6 + 2 + 1 = 86
Starting at *U* and moving clockwise, besides the 10 vertex points, there are 7 points along the horizontal sides and 1 on $\overline{PE}$ for a total of 18.
Thus, the area is 86 + 18/2 – 1 = **<u>94</u>** square units.



Alternative Solution: Draw vertical lines through *M* and *C* and horizontal lines through *E* and *H*.
This creates an 11 x 13 rectangle (Area 143).
Now we must subtract off any unshaded regions inside the rectangle and outside the shaded region. This region may be subdivided into exclusively rectangles and "half-rectangles".
Starting with region 1, we have
Excess = 6 + 1.5 + 2 + 10 + 3 + 6 + 5 + 5 + .5 + 5 + 5 = 49. Thus, A = 143 – 49 = **<u>94</u>**.



MASSACHUSETTS MATHEMATICS LEAGUE
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Team Round

F) To generate a memorable license plate we must pick 6 of the 9 nonzero digits and put them in increasing or decreasing order. This can be done in $2\binom{9}{6}=2(84)=168$ ways.

Let x denote the # of "memorable" plates to be cross checked.

Left most digit:	Possible Plates	Count
4	**56** 789	***1***
3	**45** 678 / 679 / 689 / 789	
	46 789	
	56 789	***6***

Thus, $2(1 + 6 + k) = 168 \Rightarrow k + 7 = 84 \Rightarrow k = 77$, the number of strictly increasing memorable plates beginning with 1 or 2, which have not been specifically enumerated.
$\Rightarrow x = 168 - 77 =$ **91**

FYI: [Here's the enumerated list for those who must see to believe.]

2	**34** 567 / 568 / 569 / 578 / 579 / 589 / 678 / 679 / 689 / 789	
	35 678 / 679 / 689 / 789	
	36 **7**89	
	45 678 / 679 / 689 / 789	
	46 789	
	56 789	***21***
1	**23** $\Rightarrow$ 456 457 458 459 467 468 469 478 479 489	
	567 568 569 578 579 589	
	678 679 689	
	789 $\Rightarrow$ 20	
	24 $\Rightarrow$ 10 **45** $\Rightarrow$ 4	
	25 $\Rightarrow$ 4 **46** $\Rightarrow$ 1	
	26 $\Rightarrow$ 1 **56** $\Rightarrow$ 1	
	34 $\Rightarrow$ 10	
	35 $\Rightarrow$ 4	
	36 $\Rightarrow$ 1	***56***

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Addendum
Team A) continued

The triangularization process:

$$\begin{bmatrix} 1 & 7 & 5 & 12 \\ 2 & 9 & 4 & 20 \\ 6 & A & 3 & 19 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 7 & 5 & 12 \\ 0 & -5 & -6 & -4 \\ 6 & A & 3 & 19 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 7 & 5 & 12 \\ 0 & 5 & 6 & 4 \\ 6 & A & 3 & 19 \end{bmatrix}$$

row 2 replaced by row 2 $-$ 2(row 1)
Then row 2 multiplied by -1

$$\Rightarrow \begin{bmatrix} 1 & 5 & 7 & 12 \\ 0 & 6 & 5 & 4 \\ 6 & 3 & A & 19 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 7 & 12 \\ 0 & 6 & 5 & 4 \\ 0 & -27 & A-42 & -53 \end{bmatrix}$$

columns 2 and 3 interchanged
Then row 3 replaced by row 3 $-$ 6(row 1)

$$\begin{bmatrix} 1 & 5 & 7 & 12 \\ 0 & 6 & 5 & 4 \\ 0 & 0 & 2A-39 & -70 \end{bmatrix}$$

row 3 replaced by 9(row2) + 2(row3)

The equivalent system: $\begin{cases} x+5z+7y=12 \\ 6z+5y=4 \\ (2A-39)y=-70 \end{cases} \Rightarrow y = \dfrac{-70}{2A-39}$

Smallest possible positive value of A = 2, $y = \dfrac{-70}{-35} = 2$

Backtracking (substituting for y in 2nd equation), $6z + 10 = 4 \Rightarrow z = -1$
Backtracking (substituting for x and y in 1st equation), $x + 5(-1) + 7(2) = 12 \Rightarrow x = 3$
Thus, (x, y, z) = **(3, 2, –1)**.

Changes to original questions:
2C) The original problem was stated:
Usually radicals with different indices cannot be combined.
Given: *A*, *B* are integers and 0 < *A*, *B* < 4
Compute the ordered triple (*N*, *C*, *X*), where *N*, *C* and *X* are positive integers and *C* is as small as possible, for which

$$\sqrt[12]{16(27)(128)(1024)}+3\sqrt[4]{2^A3^B}=N\left(\sqrt[C]{X}\right)$$

The appeal argued that 0 < *A*, *B* < 4 was ambiguous and could be interpreted as 0 < *A* and *B* < 4. The intent was that the values of both *A* and *B* were strictly between 0 and 4 and I thought single "and" connector made it clear that there were two conditions, not three. However, the appeal was not without merit and I decided to grant the appeal.
Here is the solution submitted by Amelia Paine (Winchester HS):

$$\sqrt[12]{16(27)(128)(1024)}+3\sqrt[4]{2^A3^B}=N\left(\sqrt[C]{X}\right)$$
$$\sqrt[12]{2^{21}\cdot 3^3}+3\sqrt[4]{2^A3^B}=$$
$$2^{\frac{21}{12}}\cdot 3^{\frac{3}{12}}+3\cdot 2^{\frac{A}{4}}\cdot 3^{\frac{B}{4}}=$$
$$2^{\frac{7}{4}}\cdot 3^{\frac{1}{4}}+2^{\frac{A}{4}}\cdot 3^{\frac{B+4}{4}}$$

Let *A* = 7, *B* = –3. Then:

$$2^{\frac{7}{4}}\cdot 3^{\frac{1}{4}}+2^{\frac{7}{4}}\cdot 3^{\frac{1}{4}}=2\left(2^{\frac{7}{4}}\cdot 3^{\frac{1}{4}}\right)=2^{\frac{11}{4}}\cdot 3^{\frac{1}{4}}=\sqrt[4]{2^{11}\cdot 3^1}=4\sqrt[4]{2^3\cdot 3}=4\sqrt[4]{24}$$

Therefore, (*N*, *C*, *X*) = **<u>(4, 4, 24)</u>**.

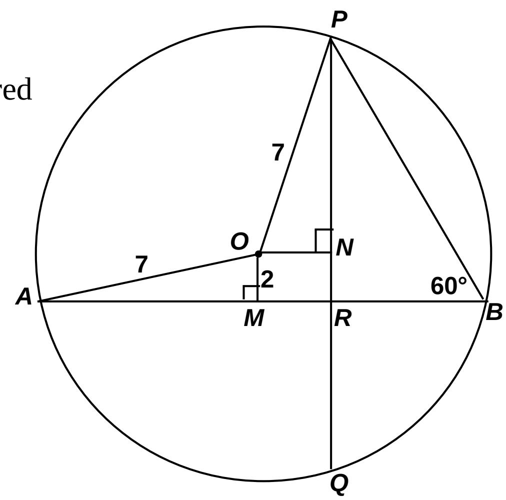
Changes to original questions:

5C) The original problem did not have a diagram.

Chord $\overline{AB}$ is 2 units from the center of a circle whose diameter is 14.
Chord $\overline{PQ}$ is perpendicular to chord $\overline{AB}$, intersecting in common point *R*.
If m$\angle PBA$ = 60° and *AR* : *BR* = 2 : 1, compute *QR*.

I thought the given conditions could produce only one diagram and I was wrong.
The following appeal made by David Fink (Acton Boxborough) was accepted.
The official answer was accepted as well as "No Answer" (due to inconsistent conditions).

Note that David's diagram is totally consistent with the given information.
Here is David's argument:



Since the distance of a chord from the center of a circle is measured along a perpendicular, *AOM* is a right triangle and

$AM = \sqrt{7^2 - 2^2} = 3\sqrt{5}$. We were given that *AR* : *BR* = 2 : 1, so

$AB = 6\sqrt{5}$, $AR = 4\sqrt{5}$, $BR = 2\sqrt{5}$ and $MR = \sqrt{5}$.

Since MRNO must be a rectangle, $NP = NQ = \sqrt{7^2 - 5} = 2\sqrt{11}$.

Therefore, $\boxed{PR = 2\sqrt{11} + 2}$.

BUT since ΔBPR is a 30 – 90 – 90 right triangle,

$\boxed{PR = BR\sqrt{3} = 2\sqrt{15}}$

What a predicament! $2\sqrt{11} + 2 = 2\sqrt{15}$

Could this be????

Dividing by 2, only if $\sqrt{11} + 1 = \sqrt{15}$

Squaring both sides, only if $12 + 2\sqrt{11} = 15 \Leftrightarrow 2\sqrt{11} = 3$

Squaring again, only if 44 = 9

I'm sure this isn't true, so we have two different lengths for the same segment, an impossibility!
Therefore, the given conditions are inconsistent.
No answer is possible.