MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 – OCTOBER 2012
ROUND 1 VOLUME & SURFACES

ANSWERS

A) ______________________ units

B) ______________________ units^3

C) (____________ , ____________)

A) The total surface area of a cube is numerically equal to one-third of its volume.
Compute all possible positive lengths of an edge of this cube.

B) Compute the volume of a rectangular prism that has three faces with areas 54, 36 and 24 square units.

C) A sphere is inscribed in a cube.
Let P denote the volume of the sphere.
Let Q denote the volume of the region inside the cube and outside the sphere.
Let R denote the exact value of $\frac{P}{Q}$.
The denominator of this fraction cannot be rationalized.
Let L denote the larger volume (either P or Q).
Specify the ordered pair (R, L).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012
ROUND 2 PYTHAGOREAN RELATIONS IN RECTILINEAR FIGURES

ANSWERS

A) ________________________ units

B) ________________________ units^2

C) (____________ , ____________)

A) If the legs of a right triangle were 8 and 9, the hypotenuse would have a length of $\sqrt{145}$.
Since 145 is not a perfect square, this length is irrational.
However, if the short leg is reduced by the same integer amount as that by which the long leg is increased, the hypotenuse has an integer length. This is true for exactly one integer value x.
For this value of x, what is the length of the hypotenuse?

B) In square $ABCD$, E and F are midpoints of $\overline{BC}$ and $\overline{CD}$, respectively, and the area of ΔAEF is R square units. Find the area of square $ABCD$ as a simplified expression in terms of R.

C) Mitt and Barack start from the same point. Mitt travels north on a plane for 3 hours.
Barack leaves one hour later and travels west on a bus for 2 hours.
At the end of this time interval Mitt and Barack are 500 miles apart.
If the plane covers a distance of 340 miles more than the bus, compute the rates of the plane (P) and the bus (B) in miles per hour. Express your answer as an ordered pair (P, B).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) (________ , ________ , ________)

A) Solve for *x* in terms of *a*, *b*, *c*, and *d*. $a+b-c\div d\cdot x=1$
Express your answer as a single simplified fraction using a minimum number of minus signs.

B) Find all possible positive values of *n* for which $\dfrac{1+2+3+...+n}{n}=2012$.

C) A collection of exclusively dimes and quarters is worth \$49.75 .
There is at least one of each type of coin.
The total number of coins in the collection can take on *n* different values.
The minimum possible number of coins is *m* and the maximum possible number of coins is *M*.
Determine the ordered triple, (*n*, *m*, *M*).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

ANSWERS

A) 1st: ___________ 2nd: ____________

B) ______________________________

C) (____________ , _____________)

A) Dick and Joe try to guess how many jelly beans are in two large jars at the local grocery store.
Dick and Joe love jelly beans, but are <u>bad</u> at guessing.
Dick's guess is 60% less than the correct number in the first jar.
Joe's guess is 20% more than the correct number in the second jar.
The two jars together actually contain 1200 jelly beans.
Unbelievably, Dick's guess for jar #1 and Joe's guess for jar #2 actually add up to 1200.
How many jelly beans are there in each jar?

B) Determine <u>all</u> values of x for which $Q = \dfrac{\frac{3}{x}}{\frac{1}{x+1} + \frac{2}{x-1}}$ is undefined.

C) Consider both terms on the left-hand side of the equation to be <u>mixed numbers</u> (not products), where A and B are positive integers with no common factor (other than 1).

$$2\frac{A}{B} + 4\frac{A+1}{B} = 7.375$$

Compute the ordered pair (A, B), where $A + B$ is a minimum.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 – OCTOBER 2012
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) If $y = 10 - 3x$ and $1 \le 2x - 3 \le 7$, compute the minimum possible value of y.

B) Solve for x. $$\frac{2}{x+6} \ge \frac{5}{x-3}$$

C) Solve for x over the reals: $\left|x^2 - 3x - 1\right| \le 3$ and $\left|2x\right| \ge 5$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012
ROUND 6 ALG 1: EVALUATIONS

ANSWERS

A) ______________________________ ft^3

B) $ ____ . ____ ___

C) ______________________________

A) A box is 8 inches x 9 inches x 15 inches. Compute the volume of this box in ft^3.

B) Turkey Hill ice cream costs $1.49/quart and $5.29/gallon. No other sizes are available.
If a gallon and 3 quarts will serve 15 people equal-sized portions, how much will the minimum ice cream required to serve 100 people the same size portions cost?

Note: 4 quarts = 1 gallon

C) Given: $a \nabla b = \dfrac{2ab}{a+b} - \sqrt{ab}$ Compute $\dfrac{1}{2} \nabla \dfrac{8}{9}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012
ROUND 7 TEAM QUESTIONS

ANSWERS

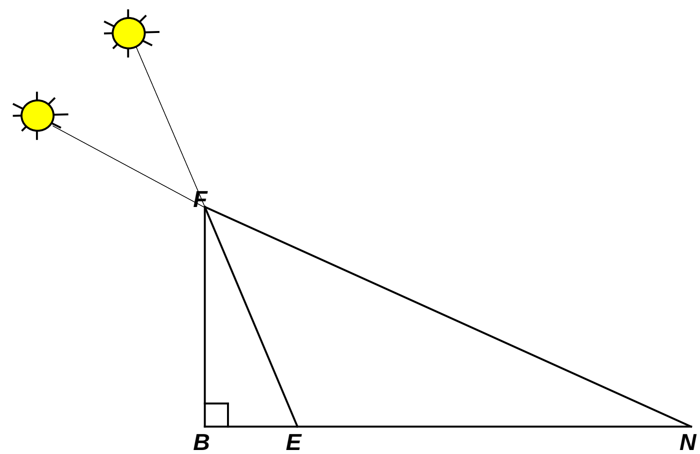
A) ______________________________ D) (__________ , ___________)

B) _________________________ft. E) ______________________________

C) (____ , ____ , ____ , ____) F) (_______ , ________ , _______)

A) A rectangular block of ice cream has dimensions n inches , $(n - 1)$ inches and $(n - 2)$ inches, for $n \geq 4$. It is completely covered with chocolate (like a Klondike Bar). Suppose it is then cut up into one inch cubes by making cuts parallel to the faces. Let A be the number of cubes with 1 or 3 faces covered with chocolate. Let B be the number of cubes with 0 or 2 faces covered with chocolate. Compute all values of n for which $A = B$.

B) At 9:00 AM, a 12-foot flagpole casts a shadow 68 feet longer than it will at 11:30 AM. If the perimeter of ΔFEN is 153, compute the length (in feet) of the shadow at 11:30.



C) Solve the following system of equations

$$\begin{cases} 64A+16B+4C+D=204 \\ 27A+9B+3C+D=104 \\ 8A+4B+2C+D=46 \\ A+B+C+D=18 \end{cases}$$

Express your answer as the ordered 4-tuple (A, B, C, D).

D) Ted Williams was the last Major League Baseball player to bat over 0.400 for an entire season. Through the next to the last day of the season he had 179 hits in 448 trips to the plate for an average of 0.39995. This would have rounded to 0.400 and he could have sat out the last day and backed into the history books. He chose to play on the last day in both games of a doubleheader. If he had gone hitless in x plate appearances, his average would have dropped below 0.393. Let x be the minimum number of plate appearances for which this would happen. He actually got h hits in x plate appearances and finished the season with an average of 0.4057 which put him in the record books at 0.406. Determine the ordered pair (h, x).

E) Solve for x: $\sqrt{|x+4|}+\sqrt{|x-1|}=\sqrt{|x-4|}$

F) The sum $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{1999}{2000!}$ may be expressed in simplified form as $A-\frac{B}{C!}$, where A, B and C are positive integers. Compute the ordered triple (A, B, C), where the sum $A + B + C$ is a minimum.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012 ANSWERS

Round 1 Geometry Volumes and Surfaces

A) 18 B) 216 C) $\left(\frac{\pi}{6-\pi}, P\right)$

Round 2 Pythagorean Relations

A) 13 B) $\frac{8R}{3}$ C) (160, 70)

Round 3 Linear Equations

A) $\frac{d(a+b-1)}{c}$ or $\frac{ad+bd-d}{c}$ B) 4023 C) (99, 202, 496)

Round 4 Fraction & Mixed numbers

A) 1st: 300 2nd: 900 B) $0, \pm 1, -\frac{1}{3}$ C) (5, 8)

(all 4 values are required)

Round 5 Absolute value & Inequalities

A) –5 B) $x \le -12$ or $-6 < x < 3$ C) $\frac{5}{2} \le x \le 4$

Round 6 Evaluations

A) $\frac{5}{8}$ (or 0.625) B) $62.66 C) $-\frac{2}{75}$

Team Round

A) 4, 5 and 6 D) (6, 8)

B) 3.5 E) $-\frac{13}{3}, \frac{-7 \pm 2\sqrt{31}}{5}$ [Do <u>not</u> accept $\frac{-7 \pm \sqrt{124}}{5}$.]

C) (2, 3, 5, 8) F) (1, 1, 2000)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012 SOLUTION KEY

Round 1

A) Let x denote the length of an edge. Then: $6x^2 = \frac{1}{3}x^3 \Rightarrow x = \underline{\mathbf{18}}$ (since $x > 0$).

B) Let the dimensions be l, w and h. Then:

$$\begin{cases} lw = 54 \\ lh = 36 \\ wh = 24 \end{cases} \Rightarrow \frac{l}{w} = \frac{36}{24} = \frac{3}{2} \Rightarrow \frac{3}{2}w^2 = 54$$

$$\Rightarrow w^2 = 36 \Rightarrow (l, w, h) = (9, 6, 4)$$

$$\Rightarrow V = 9 \cdot 6 \cdot 4 = \underline{\mathbf{216}}$$

C) Let s denote the length of the side of the cube. Then the radius of the inscribed sphere is $\frac{s}{2}$.

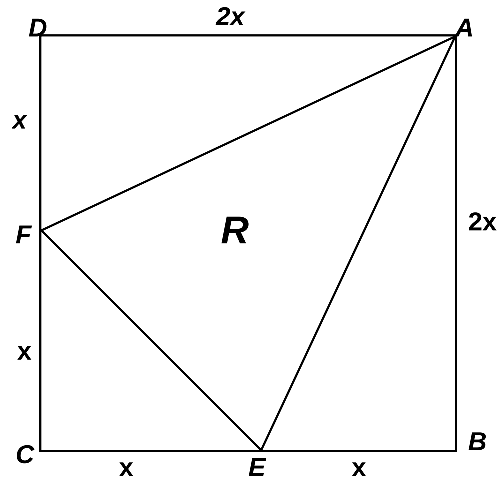
The required ratio is $\dfrac{\frac{4}{3}\pi\left(\frac{s}{2}\right)^3}{s^3 - \frac{4}{3}\pi\left(\frac{s}{2}\right)^3} = \dfrac{\frac{\pi s^3}{6}}{s^3 - \frac{\pi s^3}{6}} = \underline{\dfrac{\boldsymbol{\pi}}{\mathbf{6} - \boldsymbol{\pi}}}$

Since $\pi > 3$, $\dfrac{\pi}{6-\pi} \approx \dfrac{3^+}{3^-} > 1 \Rightarrow P > Q \Rightarrow \underline{\left(\dfrac{\boldsymbol{\pi}}{\mathbf{6}-\boldsymbol{\pi}}, \boldsymbol{P}\right)}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012 SOLUTION KEY

Round 2

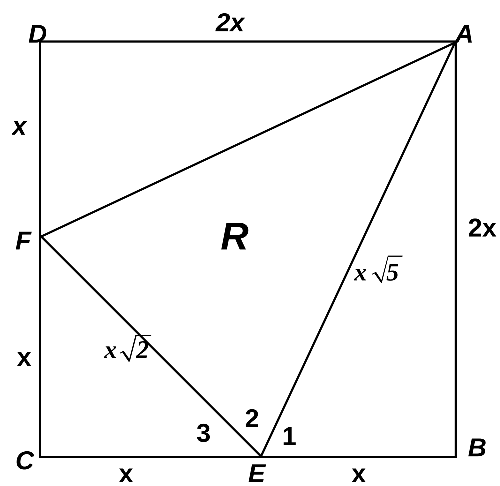
A) $(8, 9) \Rightarrow (7,10) \Rightarrow (6,11) \Rightarrow (5, 12)$ which is part of the recognizable Pythagorean Triple (5, 12, 13). The hypotenuse has length **13**.



B) $R = 4x^2 -$ area($\Delta ADF + \Delta FCE + \Delta ABE$)
$= 4x^2 - x^2 - x^2/2 - x^2 = 3x^2/2$

$$\frac{area\square}{area\Delta} = \frac{4x^2}{3x^2/2} = \frac{8}{3} \Rightarrow \text{area}(ABCD) = \underline{\mathbf{\frac{8R}{3}}}$$

Alternate solution using trig:



$\tan(\angle 1) = 2$
$m\angle 3 = 45°$
Thus, $Tan^{-1}(2) + m\angle 2 + 45 = 180$
and $m\angle 2 = 135 - Tan^{-1}(2)$
Using the expansion for $\tan(A - B)$,

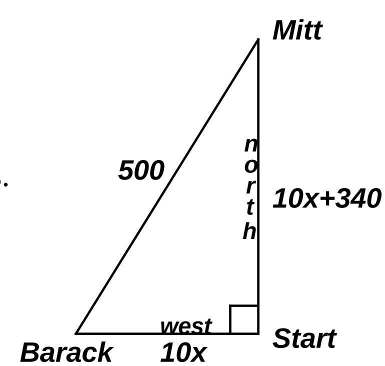
$$\tan(\angle 2) = \frac{-1-2}{1+(-1)(2)} = \frac{-3}{-1} = 3 \Rightarrow$$

$$\sin(\angle 2) = \frac{3}{\sqrt{10}}$$

$$\text{Area}(\Delta) = \frac{1}{2}ab\sin\theta = \frac{1}{2}\cdot x\sqrt{2}\cdot x\sqrt{5}\cdot\frac{3}{\sqrt{10}} = \frac{3}{2}x^2$$

$$\Rightarrow \frac{R}{area\square} = \frac{3x^2/2}{4x^2} = \frac{3}{8} \Rightarrow \text{area}(ABCD) = \underline{\mathbf{\frac{8R}{3}}}$$

C) Let $10x$ denote the distance traveled by the bus in 2 hours. Then:



$500^2 = (10x)^2 + (10x + 340)^2$
Scale the linear dimensions by a factor of 10 to avoid excessive computations.
$\Rightarrow 50^2 = (x)^2 + (x + 34)^2$
$\Rightarrow 2500 = 2x^2 + 68x + 1156 \Rightarrow 2x^2 + 68x - 1344 = 0$
$\Rightarrow x^2 - 34x - 672 = 0 \quad 672 = 2^5\cdot 3\cdot 7 \Rightarrow 14(48) \Rightarrow (x - 14)(x + 48) = 0$
$\Rightarrow x = 14$. Thus, the bus traveled 140 miles in 2 hours and the plane traveled 480 miles in 3 hours. $\Rightarrow (P, B) =$ **(160, 70)**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012 SOLUTION KEY

Round 3

A) $a+b-c \div d \cdot x = 1 \Leftrightarrow a+b-\frac{cx}{d}=1 \Leftrightarrow \frac{cx}{d}=a+b-1 \Leftrightarrow x=$ $\underline{\boldsymbol{\frac{d(a+b-1)}{c}}}$ or $\underline{\boldsymbol{\frac{ad+bd-d}{c}}}$

Equivalent expressions are acceptable (switching the order of the terms in the numerator or the order of the factors in the products in the numerator) as long as the expression contains exactly one minus sign.

B) The sum of the terms in the numerator is $\frac{n(n+1)}{2}$.

If you were unfamiliar with this result, here's how it was derived.
Let S denote the sum of the terms in the numerator of the left hand expression.
Reversing the order of the sum, of course, has no effect.

$$\begin{cases} S=1+\quad 2 \quad + \quad 3 \quad +\ldots+n \\ S=n+(n-1)+(n-2)+\ldots+1 \end{cases}$$

But notice that in each of the n columns the sum is $(n + 1)$.

Thus, adding, $2S = (n + 1)n \Rightarrow S = \frac{n(n+1)}{2}$.

This means the equation simplifies to $\frac{n+1}{2}=2012 \Rightarrow n =$ **<u>4023</u>**.

C) Let (D, Q) denote the number of dimes and quarters respectively in the collection.

$10D + 25Q = 4975 \Rightarrow 2D + 5Q = 995 \Rightarrow Q=\frac{995-2D}{5}=199-\frac{2}{5}D$

The smallest value of D is 5, resulting in $(D, Q) = (5, 197)$
For every 5 dimes added, 2 fewer quarters are required $\Rightarrow$ (10, 195), (15, 193), …, (495, 1)

Thus, the maximum number of coins required is 496, the minimum is 202 and $n = \frac{495}{5}=99$.

$\Rightarrow$ **<u>(99, 202, 496)</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012 SOLUTION KEY

Round 4

A) Let A and B denote the number of jelly beans in first and second jars respectively.

$$\begin{cases} .4A+1.2B=1200 \\ A+B=1200 \end{cases}$$

The first equation simplifies to $A + 3B = 3000$. Subtracting, B = **900** and A = **300**.

B) For x = **0, ±1**, one of the three fractions in either the numerator or denominator is undefined, but there is an additional value as well.

$$\frac{\frac{3}{x}}{\frac{1}{x+1}+\frac{2}{x-1}}=\frac{3}{x}\div\frac{x-1+2(x+1)}{(x+1)(x-1)}=\frac{3}{x}\cdot\frac{(x+1)(x-1)}{3x+1}$$

Thus, $\mathbf{-\frac{1}{3}}$ also causes division by zero.

C) $2\frac{A}{B}+4\frac{A+1}{B}=6+\frac{2A+1}{B}\Rightarrow\frac{2A+1}{B}=1.375=\frac{11}{8}\Rightarrow(A,B)=\mathbf{(5,8)}$

Other possibilities exist, but $A+B>13$. For example, cross multiplying, $8(2A+1)=11B$.
$11B$ is a multiple of 8 and, since 8 does not divide into 11, B must be a multiple of 8.
Therefore, $8(2A+1)=11(8k)$, where k is an integer.
It follows that $2A+1=11k$. For all integer values of A, the left side denotes an odd integer, so k must be odd.
$k=3\Rightarrow A=16,\ B=24$, but $k=5\Rightarrow A=27,\ B=40$
Thus, another relatively prime ordered pair (A, B) exists, but $A+B=67$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012 SOLUTION KEY

Round 5

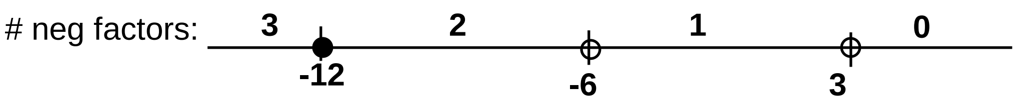
A) $1 \le 2x-3 \le 7 \Leftrightarrow 4 \le 2x \le 10 \Leftrightarrow 2 \le x \le 5$

Since $y = 10-3x$, the minimum value of y occurs for the largest possible value of x.

For $x = 5$, $y = 10 - 3(5) =$ **<u>–5</u>**.

B) $\dfrac{2}{x+6} \ge \dfrac{5}{x-3} \Rightarrow \dfrac{2}{x+6} - \dfrac{5}{x-3} \ge 0 \Leftrightarrow \dfrac{2(x-3)-5(x+6)}{(x+6)(x-3)} \ge 0 \Leftrightarrow \dfrac{-3x-36}{(x+6)(x-3)} \ge 0 \Leftrightarrow \dfrac{(x+12)}{(x+6)(x-3)} \le 0$

The critical values are –12, –6 and +3.
Since the latter two values cause division by zero, these values must be excluded from any solution.
On the number line, all factors assume negative values to the left and positive values to the right.
Moving from left to right, as each critical value is passed, there is one less negative factor.
This is summarized in the diagram below:



Since the quotient must be negative or zero, we require an <u>odd</u> number of negative factors.
From the diagram, we see that the solution set is $x \le -12$ or $-6 < x < 3$.

C) $\left|x^2-3x-1\right| \le 3 \Leftrightarrow -3 \le x^2-3x-1 \le +3 \Leftrightarrow \begin{cases} x^2-3x+2 \ge 0 \\ x^2-3x-4 \le 0 \end{cases} \Leftrightarrow \begin{cases} (x-1)(x-2) \ge 0 \\ (x-4)(x+1) \le 0 \end{cases}$

We must take the intersection between $x \le 1$ or $x \ge 2$ (2 rays) and $-1 \le x \le 4$ (a segment)
The intersection is two segments: $-1 \le x \le 1$ or $2 \le x \le 4$

$\left|2x\right| \ge 5 \Leftrightarrow x \le -\dfrac{5}{2}$ or $x \ge +\dfrac{5}{2}$

Taking the intersection between these two constraints, we have <u>$\boldsymbol{\dfrac{5}{2} \le x \le 4}$</u>.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012 SOLUTION KEY

Round 6

A) A cubic foot is equivalent to a cube 1 foot or 12 inches on side. Therefore, 1 $ft^3 = 12^3 = 1728$ in^3.

$$\frac{8\cdot 9\cdot 15}{1728}=\frac{2^3\cdot 3^3\cdot 5}{\left(2^2\cdot 3\right)^3}=\frac{5}{2^3}=\underline{\boldsymbol{\frac{5}{8}}}\ \text{ft}^3$$

or alternately converting each measure to a fractional number of feet, $\underset{(8")}{\frac{2}{\cancel{3}}}\cdot\underset{(9")}{\frac{\cancel{3}}{4}}\cdot\underset{(15")}{\frac{5}{4}}=\frac{10}{16}=\underline{\boldsymbol{\frac{5}{8}}}$

B) $\dfrac{\text{quarts}}{\text{persons served}}=\dfrac{7}{15}=\dfrac{x}{100}\Rightarrow\dfrac{7}{3}=\dfrac{x}{20}\Rightarrow x=\dfrac{140}{3}=46^{+}$

Thus, 47 quarts are required. Since the cost of 1 gallon is less than the cost of 4 quarts, we need to maximize the number of gallons purchased. We need 11 gallons and 3 quarts.
11(5.29) + 3(1.49) = **<u>$62.66</u>**

C) $\dfrac{1}{2}\nabla\dfrac{8}{9}=\dfrac{2\cdot\frac{1}{2}\cdot\frac{8}{9}}{\frac{1}{2}+\frac{8}{9}}-\sqrt{\dfrac{1}{2}\cdot\dfrac{8}{9}}=\dfrac{\frac{8}{9}}{\frac{1}{2}+\frac{8}{9}}-\dfrac{2}{3}=\dfrac{8}{9}\cdot\dfrac{18}{25}-\dfrac{2}{3}=\dfrac{16}{25}-\dfrac{2}{3}=\dfrac{48-50}{75}=\underline{\boldsymbol{-\dfrac{2}{75}}}$

Also accept $\dfrac{-2}{75},\ \dfrac{2}{-75},\ -0.02\overline{6}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012 SOLUTION KEY

Team Round

A) 3 faces: corner cubes (at the 8 vertices) $\Rightarrow$ 8

2 faces: edge cubes – excluding the corners (12 edges) $\Rightarrow$ 4 ea. @ $(n-2)$, $(n-3)$ and $(n-4)$

$\Rightarrow 12n - 36$ or $12(n-3)$

1 face: interior cubes on each face (6 faces, i.e. 3 pairs of opposite faces)

n x $(n-1)$ faces: $2[(n-2)(n-3)]$

n x $(n-2)$ faces: $2[(n-2)(n-4)]$

$(n-1)$ x $(n-2)$ faces: $2[(n-3)(n-4)]$

$\Rightarrow 6n^2 - 36n + 52$ or $6(n-3)^2 - 2$

0 faces: interior cubes only $\Rightarrow (n-2)(n-3)(n-4)$ or $n^3 - 9n^2 + 26n - 24$

We can proceed by direct evaluation or algebraically (table is included below)

We require that $6n^2 - 36n + 60 = \left(n^3 - 9n^2 + 26n - 24\right) + 12(n-3) = n^3 - 9n^2 + 38n - 60$

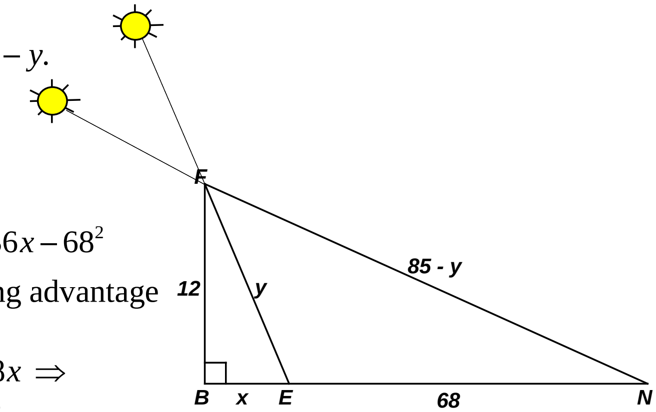
$\Leftrightarrow n^3 - 15n^2 + 74n - 120 = 0 \Leftrightarrow (n-4)(n-5)(n-6) = 0 \Rightarrow n =$ **<u>4, 5, 6</u>**.

The following table confirms our results:

n	**(3)**	**[2]**	**(1)**	**[0]**	
4	8	12	4	0	= 12
5	8	24	22	6	= 30
6	8	36	52	24	= 60
7	8	48	94	60	102 < 108
8	8	60	148	120	156 < 180
9	8	72	214	210	222 < 288
10	8	84	292	336	gap continues to widen

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012 SOLUTION KEY

Team Round



B) Let (BE, FE) = (x, y). Then FN = 153 – (y + 68) = 85 – y.

$$\begin{cases} \Delta FBE: y^2 - x^2 = 12^2 \\ \Delta FBN: 12^2 + (x+68)^2 = (85-y)^2 \end{cases}$$

Substituting for $y^2 - x^2$,

$$\left(y^2 - x^2\right) = (85-y)^2 - (x+68)^2 = 85^2 - 170y + y^2 - x^2 - 136x - 68^2$$

Re-arranging terms, pulling out common factors and taking advantage of the difference of perfect squares

$$\Rightarrow\ 0 = 85^2 - 68^2 - 170y - 136x = (153)(17) - 17\cdot 10y - 17\cdot 8x\ \Rightarrow$$

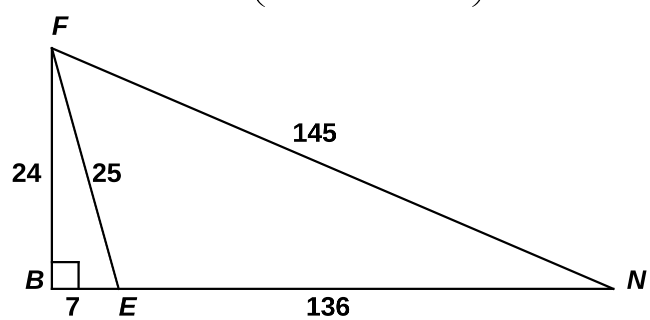
$$10y + 8x = 153 \Rightarrow\ y = \frac{153-8x}{10}\quad \text{Substituting, } \left(\frac{153-8x}{10}\right)^2 - x^2 = 12^2$$

$$\Rightarrow\ \left(153-8x\right)^2 - 100x^2 = 100\cdot 12^2 \Rightarrow 153^2 - 16\cdot 153x - 36x^2 = 100\cdot 12^2$$

Dividing through by 3^2 = 9, $51^2 - 16\cdot 17x - 4x^2 = 100\cdot 16 = 40^2\ \Rightarrow\ 4x^2 + 16\cdot 17x + \left(\overbrace{40^2 - 51^2}^{(91)(-11)=-7\cdot 11\cdot 13}\right) = 0$

$$\Rightarrow\ \left(2x-7\right)\left(2x+11\cdot 13\right) = 4x^2 + \left(\overbrace{2\cdot 11\cdot 13 - 2\cdot 7}^{2(143-7)=2\cdot 8\cdot 17}\right)x - 7\cdot 11\cdot 13 = 0$$

$\Rightarrow\ x$ = **<u>3.5</u>**.



Note the familiar 7-24-25 right triangle embedded in the diagram. Dividing the dimensions by 2 gives us the conditions and answer we derived above.

Alternate solution (Norm Swanson – Hamilton Wenham)

In right ΔFBN, $\cos N = \dfrac{x+68}{85-y}$. Using the law of Cosines in ΔFEN,

$\cos N = \dfrac{(85-y)^2 + 68^2 - y^2}{2\cdot 68\cdot (85-y)} = \dfrac{85^2 + 68^2 - 170y}{2\cdot 68\cdot (85-y)}$. Noting that each term in the trinomial in the numerator is a multiple of 17, we simplify this to $\dfrac{697-109y}{8(85-y)}$. Equating, we have $8x + 10y = 153$. Since the coefficients 8 and 10 are even and the constant 153 is odd, x and y must be of the form $\dfrac{x'}{2}$ and $\dfrac{y'}{2}$ respectively. Substituting, $4x' + 5y' = 153 = 9(17)$.

Clearly, $x' = y' = 17$ is a solution. We are looking for $P(x', y')$ which produces a Pythagorean Triple form the sides of ΔFBE doubled. $\left(FB, BE, FE\right) = \left(12, \dfrac{x'}{2}, \dfrac{y'}{2}\right) = (24, 17, 17)$ fails. Using the slope of $\dfrac{4}{-5}$, we move <u>4 up and 5 left</u> until we find the required triple $\cancel{(24,12,21)}$, <u>$\mathbf{(24,7,25)}$</u> - Bingo!

But remember we have doubled the sides of ΔFBE, so BE = $\dfrac{7}{2}$ = **<u>3.5</u>** .

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2012 SOLUTION KEY

Team Round

C) $\begin{cases} 64A+16B+4C+D=204 \\ 27A+9B+3C+D=104 \\ 8A+4B+2C+D=46 \\ A+B+C+D=18 \end{cases}$ Number these equations (#1) through (#4) respectively.

(#1) – (#2) $\Rightarrow$ (#5) $37A + 7B + C = 100$ (#5) – (#6) $\Rightarrow$ (#8) $18A + 2B = 42$ (#8) – (#9) $\Rightarrow$ $6A = 12$
(#2) – (#3) $\Rightarrow$ (#6) $19A + 5B + C = 58$ (#6) – (#7) $\Rightarrow$ (#9) $12A + 2B = 30$
(#3) – (#4) $\Rightarrow$ (#7) $7A + 3B + C = 28$
Thus, $A = 2$. Substituting in (#9), $B = 3$. Substituting in (#7), $14 + 9 + C = 28 \Rightarrow C = 5$.
Substituting in (#4), $D = 8 \Rightarrow$ **<u>(2, 3, 5, 8)</u>**.

D) If he goes hitless, his average is $\dfrac{179}{448+x} < 0.393$

To lose over 7 points on his average, I guesstimate he made at least 7 plate appearances
$\dfrac{179}{455} \approx 0.3934$. This average is too high, implying he made <u>at least</u> 8 plate appearances in the doubleheader and $x = 8$. Now we know $\dfrac{179+h}{456} \approx 0.406$. To maintain his average he needed at least 2 hits in 5 plate appearances; therefore, he must have gotten at least 4 hits to maintain a 0.400 average. To increase his average 6 points, I guesstimate he got $h = 6$ hits. $\dfrac{185}{456} = 0.4057$ and we have confirmation.
$(h, x) =$ **<u>(6, 8)</u>**.

E) The absolute value inside the radicals insures that the radicands will be positive, but since all real values of x will produce a nonnegative radicand, we must subdivide the domain into four regions and be on guard for extraneous answers. The critical values (where the radicands become zero) are –4, 1 and +4.

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x +4------------------------------0+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
x - 1-----------------------------------------------------------------------------0+++++++++++++++++++++++++++++++++
x - 4------------------------------------------------------------------------------------------------------------------0+++++++++
                                 - 4                                        + 1                                  + 4
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Squaring both sides, $|x+4|+|x-1|+2\sqrt{|x+4|}\sqrt{|x-1|}=|x-4|$

Case 1: $x \le -4$

$-x-4+1-x+2\sqrt{-x-4}\sqrt{1-x}=4-x$

$\Rightarrow\ 2\sqrt{-x-4}\sqrt{1-x}=4-x+2x+3=x+7$

$\Rightarrow\ 4\left(x^2+3x-4\right)=x^2+14x+49$

$\Rightarrow 3x^2 - 2x - 65 = (3x + 13)(x - 5) = 0 \ \Rightarrow x =$ $\underline{-\mathbf{\dfrac{13}{3}}}$ (5 is extraneous)

MASSACHUSETTS MATHEMATICS LEAGUE
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E) - continued

$|x+4|+|x-1|+2\sqrt{|x+4|}\sqrt{|x-1|}=|x-4|$

Case 2: $-4 < x \le 1$

$x+4+1-x+2\sqrt{x+4}\sqrt{1-x}=4-x$

$\Rightarrow\ 2\sqrt{x+4}\sqrt{1-x}=-1-x$

$\Rightarrow\ 4\left(-x^2-3x+4\right)=1+2x+x^2$

$\Rightarrow\ 5x^2+14x-15=0\ \Rightarrow\ \dfrac{-14\pm\sqrt{196+300}}{10}=\dfrac{-14\pm\sqrt{16(31)}}{10}=\underline{\dfrac{\mathbf{-7\pm 2\sqrt{31}}}{\mathbf{5}}}$ (Both values check!)

[$2\sqrt{31}=\sqrt{124}\approx\sqrt{121}=11\ \Rightarrow\ \approx\dfrac{4}{5},-\dfrac{18}{5}$ Both of these values fall in the required range for case 2.]

Case 3: $1 < x \le 4$

$x+4+x-1+2\sqrt{x+4}\sqrt{x-1}=4-x \Rightarrow 2\sqrt{x+4}\sqrt{x-1}=1-3x$

For any x in this domain, $1 - 3x < 0$. Any solutions would be extraneous, since the left side of the equation <u>must</u> be nonnegative.

Case 4: $x > 4$

$x+4+x-1+2\sqrt{x+4}\sqrt{x-1}=x-4\ \Rightarrow 2\sqrt{x+4}\sqrt{x-1}=-x-7$

For any x in this domain, $-x-7<0$ and any solutions would be extraneous.

F) Note that $\dfrac{n}{(n+1)!}$ can be written as $\dfrac{1}{n!}-\dfrac{1}{(n+1)!}$. $\left[\dfrac{(n+1)!-n!}{n!(n+1)!}=\dfrac{\cancel{n!}\left((n+1)-1\right)}{\cancel{n!}(n+1)!}=\dfrac{n}{(n+1)!}\right]$

Expressing each of the 1999 terms as a difference we have:

$\left(1-\dfrac{1}{2!}\right)+\left(\dfrac{1}{2!}-\dfrac{1}{3!}\right)+\left(\dfrac{1}{3!}-\dfrac{1}{4!}\right)+\ldots+\left(\dfrac{1}{1998!}-\dfrac{1}{1999!}\right)+\left(\dfrac{1}{1999!}-\dfrac{1}{2000!}\right)=1-\dfrac{1}{2000!}$ → <u>**(1, 1, 2000)**</u>

Note that the expression $1-\dfrac{2001}{2001!}$ may be equivalent, but $A+B+C>2002$.