MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2015
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

ANSWERS

A) $x =$ ______________________________

B) (____________ , ____________)

C) (____________ , ____________)

A) $y = f(x)$ defines a linear function with slope of $\frac{2}{3}$ and a y-intercept of -6.
$y = g(x)$ defines a linear function perpendicular to $y = f(x)$ with a y-intercept of $+6$.
Compute the x-intercepts of $y = h(x)$, given $h(x) = f(x) \cdot g(x)$.

B) If $f^{-1}(x) = \dfrac{1-2x}{3}$, then $8 \le f(x) \le 20$ for $a \le x \le b$. Compute the ordered pair (a,b).

C) The zeros of $y = f(x) = 3x^2 + 2x - 4$ are u and v.
The zeros of $y = g(x) = 3x^2 + bx + c$ are $2u + 3v$ and $3u + 2v$.
Compute the ordered pair (b,c).

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 1

A) $f(x)=\frac{2}{3}x-6,\ g(x)=-\frac{3}{2}x+6 \Rightarrow h(x)=\left(\frac{2}{3}x-6\right)\left(-\frac{3}{2}x+6\right)$

$h(x)=0 \Rightarrow \left(\frac{2}{3}x-6\right)=0$ or $\left(-\frac{3}{2}x+6\right)=0 \Rightarrow x=\underline{\mathbf{9,\ 4}}$

B) Interchanging x and y and resolving for y, we have

$y=f^{-1}(x)=\frac{1-2x}{3} \Leftrightarrow x=\frac{1-2y}{3} \Leftrightarrow 3x+2y=1 \Leftrightarrow y=f(x)=\frac{1-3x}{2}$ Now $8 \le f(x) \le 20$

$\Leftrightarrow 8 \le \frac{1-3x}{2} \le 20 \Leftrightarrow 16 \le 1-3x \le 40 \Leftrightarrow 15 \le -3x \le 39 \Leftrightarrow -5 \ge x \ge -13$

$\Rightarrow (a,b)=\underline{(\mathbf{-13,-5})}$ The order was important, since it was required that $a \le b$!

C) If the zeros of $y=f(x) \ =3x^2+2x-4$ are u and v, then $\begin{cases} (1)\ \ u+v=-\frac{2}{3} \\ (2)\ \ \ \ uv=-\frac{4}{3} \end{cases}$.

The sum of the zeros of $y=g(x)$ is $(2u+3v)+(3u+2v)=5(u+v)=5\cdot-\frac{2}{3}=-\frac{10}{3}$.

The product of the zeros of $y=g(x)$ is $(2u+3v)(3u+2v)=6u^2+13uv+6v^2=6\left(u^2+v^2\right)+13uv$.

Squaring (1), we have $(u+v)^2=\left(u^2+v^2\right)+2uv=\frac{4}{9} \Rightarrow \left(u^2+v^2\right)=\frac{4}{9}-2uv$.

Substituting, $(2u+3v)(3u+2v)=6\left(\frac{4}{9}-2uv\right)+13uv=\frac{8}{3}+uv=\frac{4}{3}$.

A quadratic equation with roots $2u+3v$ and $3u+2v$ is $x^2+\frac{10}{3}x+\frac{4}{3}=0$.

Therefore, $y=g(x)=3x^2+10x+4 \Rightarrow (b,c)=\underline{(\mathbf{10,4})}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2015
ROUND 2 ARITHMETIC / NUMBER THEORY

ANSWERS

A) ______________________ $_{(base\ 5)}$

B) ______________________

C) ______________________

A) Given: $a_{(base\ 5)} = 1011010_{(base\ 2)} + 1111111_{(base\ 2)}$

Compute $a_{(base\ 5)}$.

B) The sum of three consecutive positive integers *a*, *b* and *c*, where *a* < *b* < *c*, is divisible by both 6 and 15. Compute the <u>sum</u> of the three <u>smallest</u> possible values of *a*.

C) Let $A = 125 \cdot (45)^x$ and $B = 18 \cdot (24)^x$, where *x* is a positive integer.

Let $n(K)$ denote the number of divisors of *K*.

Determine <u>all</u> possible values of *x* for which $\dfrac{n(A)}{n(B)} = \dfrac{3}{4}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2015 SOLUTION KEY

Round 2

A) Doing the addition is base 2, we have $\begin{array}{r} 1011010 \\ +\underline{1111111} \\ 11011001 \end{array}$.

Converting this result to base 10, we have $2^7+2^6+2^4+2^3+2^0=128+64+16+8+1=217$.

Converting to base 5, we have
$$217-\mathbf{1}\left(5^3\right)=217-125=92$$
$$92-\mathbf{3}\left(5^2\right)=92-75=17$$
$$17-\mathbf{3}\left(5^1\right)=17-15=\mathbf{2}$$

Thus, $a_5=\underline{\mathbf{1332}}$

The same conversion can be obtained by repeatedly dividing by 5 and keeping track of the quotients and remainders until the quotient becomes 0. Reading the remainders from the bottom up gives us the above answer.

$$\begin{array}{rrl} 5\,| & 217 & \\ & 43 & \mathbf{2} \\ & 8 & \mathbf{3}\ \Uparrow \\ & 1 & \mathbf{3}\ \Uparrow \\ & 0 & \mathbf{1}\ \Uparrow \end{array}$$

B) Numbers divisible by both 6 and 15 are multiples of 30.
Let n – 1, n and n + 1 denote the three consecutive integers.
Then $3n=30k \Rightarrow n=10k$
$k=1,2,3 \Rightarrow n=10,20,30 \Rightarrow (9,10,11),(19,20,21),(29,30,31)$
Therefore, the required sum is $9+19+29=\underline{\mathbf{57}}$.

C) $A=125(45)^x=3^{2x}\cdot 5^{x+3} \Rightarrow n(A)=(2x+1)(x+4)$
$B=18(24)^x=2^{3x+1}\cdot 3^{x+2} \Rightarrow n(B)=(3x+2)(x+3)$
Thus, $\dfrac{(2x+1)(x+4)}{(3x+2)(x+3)}=\dfrac{3}{4} \Rightarrow 8x^2+36x+16=9x^2+33x+18 \Rightarrow x^2-3x+2=0$
$\Rightarrow (x-1)(x-2)=0 \Rightarrow x=\underline{\mathbf{1,2}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2015
ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

ANSWERS

A) ______________________________

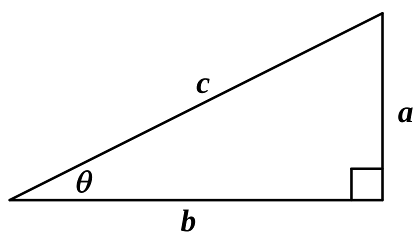
B) ______________________________

C) ______________________________

A) Compute: $\cos\left(6\cdot\left(Sin^{-1}\left(-\frac{1}{2}\right)+Tan^{-1}\left(-\sqrt{3}\right)\right)\right)$

B) Solve over $0 \le x < 2\pi$: $\sqrt{1-\cos^2(2x)} = \tan x$

C) Given: $\theta = Cos^{-1}(k) = Tan^{-1}(k)$, where $k > 0$
Compute the unique value of $\sin\theta$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2015 SOLUTION KEY

Round 3

A) $\cos\left(6\cdot\left(Sin^{-1}\left(-\frac{1}{2}\right)+Tan^{-1}\left(-\sqrt{3}\right)\right)\right)=\cos 6\cdot\left(-\frac{\pi}{6}-\frac{\pi}{3}\right)=\cos 6\cdot-\frac{\pi}{2}=\cos\left(-3\pi\right)=\cos\pi=\underline{\mathbf{-1}}$

B) Given: $\sqrt{1-\cos^2\left(2x\right)}=\tan x$

Note that the left hand side of the equation <u>always</u> returns a nonnegative value.

Using the identities, $\begin{cases}\sin^2\theta+\cos^2\theta=1\\ \tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}\end{cases}$, and squaring both sides, we have

$1-\cos^2\left(2x\right)=\tan^2 x$

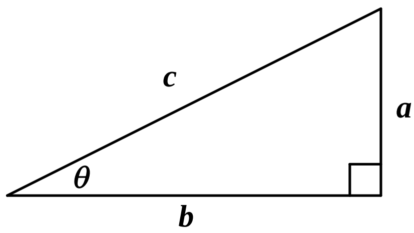
$\Rightarrow 1-\cos^2\left(2x\right)=\frac{\left(1-\cos 2x\right)^2}{\sin^2\left(2x\right)}=\frac{\left(1-\cos 2x\right)^2}{1-\cos^2\left(2x\right)}=\frac{\left(1-\cos 2x\right)^2}{\left(1-\cos 2x\right)\left(1+\cos 2x\right)}=\frac{1-\cos 2x}{1+\cos 2x}$

$\Rightarrow\left(1-\cos 2x\right)\left(1+\cos 2x\right)=\frac{1-\cos 2x}{1+\cos 2x}\Rightarrow 1-\cos 2x=0$ or $\left(1+\cos 2x\right)^2=1$

$\Rightarrow$ $\cos 2x=1\Rightarrow\underline{\mathbf{0,\ \pi}}$ or $1+\cos 2x=\pm 1\Rightarrow\cos 2x=0,\ \cancel{-2}$

$\cos 2x=0\Rightarrow 2x=\frac{\pi}{2}+n\pi\Rightarrow x=\frac{\pi}{4}+\frac{n\pi}{2}=\frac{\pi\left(2n+1\right)}{4}\Rightarrow\underline{\boldsymbol{\frac{\pi}{4}}},\ \cancel{\frac{3\pi}{4}},\ \boldsymbol{\frac{5\pi}{4}},\ \cancel{\frac{7\pi}{4}}$

C) $\theta=Cos^{-1}\left(k\right)=Tan^{-1}\left(k\right)\Rightarrow\sin\theta=\tan\theta=k\ \Rightarrow\frac{b}{c}=\frac{a}{b}\Rightarrow b^2=ac$



$a^2+b^2=c^2\Rightarrow a^2+ac=c^2$

$\Rightarrow a^2+ac-c^2=0\Leftrightarrow\left(\frac{a}{c}\right)^2+\left(\frac{a}{c}\right)-1=0$, provided $c\neq 0$

$\Leftrightarrow\sin^2\theta+\sin\theta-1=0$

Applying the quadratic formula, we have $\sin\theta=\frac{-1\pm\sqrt{5}}{2}$.

Since we know that θ is in the <u>first</u> quadrant, $\sin\theta=\underline{\mathbf{\frac{\sqrt{5}-1}{2}}}$ (only).

$\frac{-1-\sqrt{5}}{2}<-1$ is extraneous.

FYI: The approximate value of θ is $38.17270763°$.

For this value of θ, $\begin{cases}\cos\theta=0.7861513777\\ \tan\theta=0.7861513778\end{cases}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2015
ROUND 4 ALG 1: WORD PROBLEMS

ANSWERS

A) $ ____________________________

B) ______________________ minutes

C) _________________________ cups

A) The Kohl's department store is famous for giving discounts of merchandise which has already been discounted. Compute the original price of an article, if the customer pays $63 after Kohl's has discounted the article 40%, and then discounted an additional 30%.

B) One day on a cross-country road trip with my family, we started at 10:30AM. I drove at an average of 70 mph for 2 hours, stopped *k* minutes for lunch, drove for another $3\frac{1}{4}$ hours at an average speed of 60mph and then stopped for the day.
My average speed for the entire day was 50 mph. Compute *k*.

C) Heavy cream is 41% butterfat, while whole milk contains only 5% butterfat. In order to make a delicious pint of ice cream, a recipe calls for 3 cups of a mixture which is 19% butterfat. Compute the number of cups of heavy cream which must be used to produce the correct butterfat percentage.

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 4

A) The customer must pay 70% of the first discounted price which is 60% of the original price.

$.7(.6x)=63 \Rightarrow .42x=63 \Rightarrow x=\frac{63}{.42}=\frac{6300}{42}=\frac{900}{6}=\underline{\mathbf{150}}$.

B) Applying $R\cdot T=D$ and $R=\frac{D}{T}$, we have $\frac{\text{miles}}{\text{hours}}=\frac{70(2)+60(3.25)}{2+3.25+\frac{k}{60}}=50 \Leftrightarrow \frac{335}{5.25+\frac{k}{60}}=50$

$\Leftrightarrow \frac{134\cancel{0}}{21+\frac{k}{15}}=5\cancel{0} \Leftrightarrow 134=5\cdot 21+\frac{k}{3} \Leftrightarrow 29=\frac{k}{3} \Leftrightarrow k=\underline{\mathbf{87}}$ minutes.

C) Let (x, y) denote cups of (41% butterfat, 5% butterfat) required by the recipe. Then:

$\begin{cases} x+y=3 \\ .41x+.05y=.19(3) \end{cases} \Rightarrow y=3-x \Rightarrow 41x+5(3-x)=57 \Rightarrow 36x=42 \Rightarrow x=\underline{\mathbf{\frac{7}{6}}}$ cups.

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 5 PLANE GEOMETRY: CIRCLES

ANSWERS

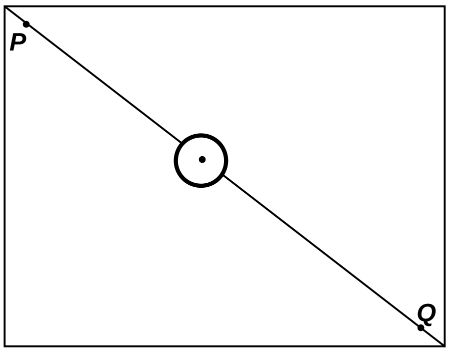
A) _______________ : _______________

B) ___________________________ units

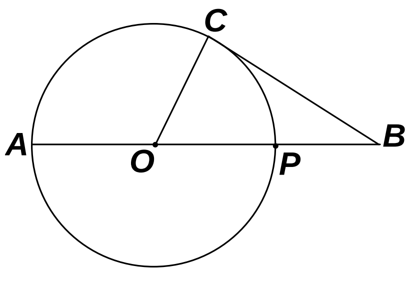
C) ___________________________ units

A) Circle 1 has a diameter of $2\sqrt{3}$. Circle 2 has a radius of $\frac{\sqrt{15}}{2}$.
Compute the ratio of the area of the larger circle to the area the smaller circle.

B) A bead with a 2 inch diameter slides along a metal rod which connects opposite corners of a rectangular 6 inch by 8 inch wooden frame. When the bead touches the side of the frame it reverses direction. Let P and Q, respectively, be the locations of the center of the bead when the bead comes closest to the opposite corners of the frame. Compute PQ.



C) The length of the secant $\overline{AB}$ is 20.
The length of tangent $\overline{BC}$ is 10.
Let $\overline{PQ}$ be a chord in circle O parallel to $\overline{BC}$. Compute PQ.



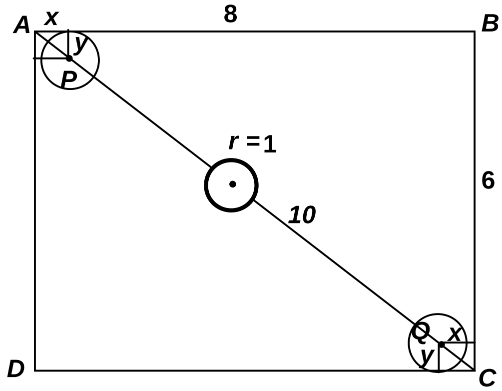
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2015 SOLUTION KEY

Round 5

A) The area of circle 1 is $\pi\left(\sqrt{3}\right)^2 = 3\pi$

The area of circle 2 is $\pi\left(\dfrac{\sqrt{15}}{2}\right)^2 = \dfrac{15}{4}\pi = 3.75\pi$

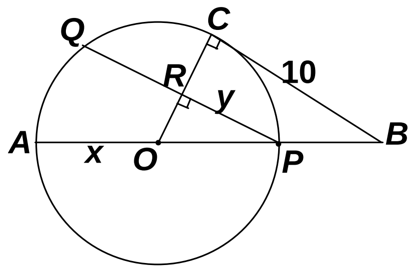
Thus, the required ratio is $\dfrac{15/4}{3} = \underline{\mathbf{\dfrac{5}{4}}}$.



B) By similar triangles, $\dfrac{y}{x} = \dfrac{6}{8}$ and $r = 1 \Rightarrow y = 1, x = \dfrac{4}{3}$,

(diag) $d = AP = CQ = \dfrac{5}{3}$

Therefore, $PQ = 10 - 2\left(\dfrac{5}{3}\right) = \underline{\mathbf{\dfrac{20}{3}}}$.



C) Let x denote $OA = OP = OC$, radii of circle O. Then:

In ΔBOC, $x^2 + 10^2 = (20 - x)^2 \Rightarrow$

$100 = 400 - 40x \Rightarrow x = 7.5$

$AB = 20 \Rightarrow PB = 5$

$\Delta ROP \sim \Delta COB \Rightarrow \dfrac{7.5}{7.5+5} = \dfrac{y}{10} \Rightarrow \dfrac{15}{25} = \dfrac{3}{5} = \dfrac{y}{10}$

$\Rightarrow y = 6 \Rightarrow PQ = \underline{\mathbf{12}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2015
ROUND 6 ALG 2: SEQUENCES AND SERIES

ANSWERS

A) $f(x)$ = ________________________

B) (________ , ________ , ________)

C) (____________ , ____________)

A) $f(x) = c_0 + c_1 x + c_2 x^2$

Find $f(x)$ under the following conditions: $f(0) = 4$, $f(1) = 3$ and $f(2) = 2$

B) The series $S = \sum_{n=a}^{n=b}(3n-5)$ generates a sum of 3 negative numbers and 2 positive numbers.

Compute the ordered triple (a, b, S).

C) The simplified ratio of the sum of the terms of the infinite geometric sequence $3, -\frac{9}{4}, \frac{27}{16}, \ldots$ to the sum of the first four terms of the sequence is $A : B$. Compute the ordered pair (A, B).

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 6

A) $f(0) = 4 = c_0$
$f(1) = 3 = 4 + c_1 + c_2$
$f(2) = 2 = 4 + 2c_1 + 4c_2$
Thus, $c_0 = 4$, $c_1 = -1$ and $c_2 = 0$ and $f(x) =$ **$\underline{4 - x}$**.

B) This series consists of terms of an arithmetic progression with a common difference of 3, starting with $3a - 5$. Thus, the sequence is $3a-5,\ 3a-2,\ 3a+1, 3a+4$, etc.
Trying $a = 1$, the sequence would begin with $-2, 1, \ldots$. Rejected, only 1 negative.
Adjusting, let $a = -1$ and the sequence is $-8, -5, -2, 1, 4$ - Bingo!
The series generates 5 terms and the sum is $(-8) + (-5) + (-2) + 1 + 4 = -10$.
Thus, $(a, b, S) =$ **$\underline{(-1, 3, -10)}$**.

C) For the geometric sequence $3, -\frac{9}{4}, \frac{27}{16}, \ldots,\ a = 3$ and $r = -\frac{3}{4}$.
Since $|r| < 1$, the sum of the series converges to $\frac{a}{1-r}$.
The sum of all the terms is $\frac{3}{1+\frac{3}{4}} = \frac{12}{4+3} = \frac{12}{7}$

The sum of n terms of any geometric series is given by $\frac{a - ar^n}{1-r}$. We have $\boxed{a = 3, r = -\frac{3}{4}}$.
In this case, the sum of the first 4 terms is $\frac{a\left(1-r^4\right)}{1-r}$.
Rather than "simply" computing $3 - \frac{9}{4} + \frac{27}{16} - \frac{81}{64}$ or plugging in specific values, let's simplify the formula. $\frac{a\left(1-r^4\right)}{1-r} = \frac{a\left(1-r^2\right)\left(1+r^2\right)}{1-r} = \frac{a\cancel{(1-r)}(1+r)\left(1+r^2\right)}{\cancel{1-r}} = a(1+r)\left(1+r^2\right) \Rightarrow$
$3\left(\frac{1}{4}\right)\left(\frac{25}{16}\right) = \frac{75}{64}$. The required ratio is $\frac{12}{7} \div \frac{75}{64} = \frac{\cancel{12}^{4}(64)}{7(\cancel{75}^{25})} = \frac{256}{175} \Rightarrow (A, B) =$ **$\underline{(256, 175)}$**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2015
ROUND 7 TEAM QUESTIONS

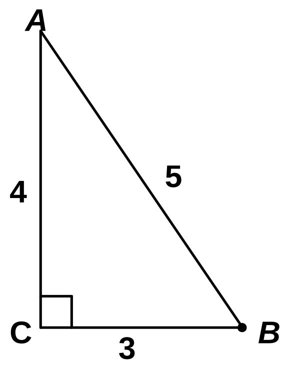
ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) ____________________________

C) ____________________________ F) ____________________________

A) Given: $f(2x) = 2ax^2 + 6bx + c$, where *a*, *b* and *c* are integers, $c > 0$ and $a \geq -\frac{3b}{4}$.
If $f(1) = 15$ and $f(8) = 36$, find the minimum value of $a + b + c$.



B) Consider 3-digit numbers of the form *A*8*B* and *B*8*A*, where $A > B$.
Find all possible ordered pairs (*A*, *B*) for which both of these 3-digit numbers are prime.

C) ΔABC is a 3-4-5 triangle. It is rotated clockwise about point *B* until C' (the image of *C*) lies between *A* and *B*. Compute $(A'C)^2$.

D) In baseball, three computations give a good indication of a player's offensive production.
They are slugging percentage (*SLG*), on-base percentage (*OBP*) and batting average (*BA*):

$$SLG = \frac{(1B) + 2(2B) + 3(3B) + 4(HR)}{AB} \qquad OBP = \frac{H + BB + HBP}{AB + BB + HBP + SF} \qquad BA = \frac{H}{AB}$$

The 4 possible hits (*H*) in baseball are 1*B* (single), 2*B* (double), 3*B* (triple) and *HR* (homerun).
AB - at-bats BB - base on balls (walks) HBP - hit by pitch SF – sacrifice fly

My last year in the majors was my best: 104 singles, 18 doubles, 2 triples and 6 homeruns in 400 at bats. My *OBP* was the average of my batting average and my slugging percentage. Thankfully, I was not a favorite target of opposing pitchers and my BB : HBP ratio was 10 : 1. How many times was I hit by a pitch, if I had fewer than 100 sacrifice flies.
Give all possible answers.

MASSACHUSETTS MATHEMATICS LEAGUE
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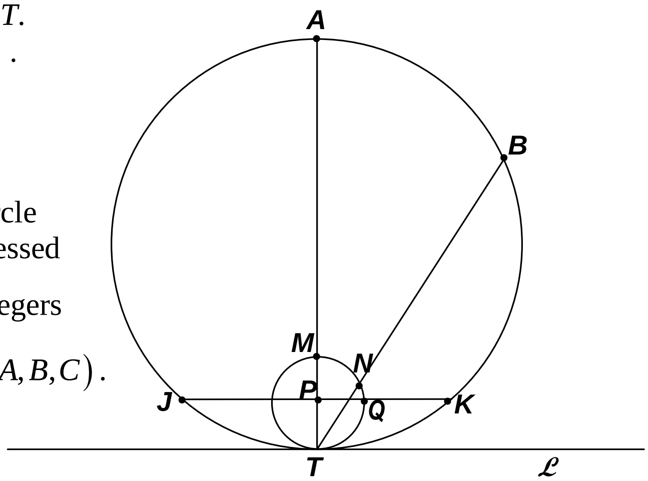
ROUND 7 TEAM QUESTIONS

E) Two circles are tangent to line $\mathcal{L}$ at point T.
$m\angle ATB = 36°$, $JK = 2$ and $PT = 3 \cdot PM$.
The length of minor arc $\overset{\frown}{AB}$ is $\dfrac{4\pi}{5}$.
$\overline{AT} \perp \mathcal{L}$, $\overline{JK} \perp \overline{AT}$.
The area of the region inside the larger circle and outside the smaller circle can be expressed in simplest form as $\dfrac{A}{B}\left(1 + C\sqrt{3}\right)\pi$, for integers A, B and C. Compute the ordered triple $\left(A, B, C\right)$.



F) x and y are the first and second terms, respectively, of an arithmetic sequence (AS).
x and y are also the first and second terms, respectively, of a geometric sequence (GS).
If the third term of the GS is –27 and the third term of the AS is 21, compute <u>all</u> possible values of the 5th term of the AS divided by the 4th term of the GS.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2015 SOLUTION KEY

Team Round

A) If $f(2x) = 2ax^2 + 6bx + c$, we can evaluate $f(1)$ and $f(8)$ by letting $x = \frac{1}{2}$ and $x = 4$ respectively.

Thus, $\begin{cases} f(1) = \frac{a}{2} + 3b + c = 15 \\ f(8) = 32a + 24b + c = 36 \end{cases}$. Subtracting $f(8)$ from $64\left(f(1)\right)$, we have $168b + 63c = 924$.

Since $168b + 63c = 924 \Leftrightarrow 84(2)b + 63c = 84(11)$, the right hand side of the equation is divisible by 84 and, therefore the left hand side must also be divisible by 84. This forces $63c$ to be a multiple of 84. Since the gcd(63, 84) = 21, c must be a multiple of 4.
Therefore, let $c = 4k$ and $168b + 63c = 924$ becomes

$168b + 252k = 924 \Leftrightarrow 2b + 3k = 11 \Leftrightarrow b = \frac{11-3k}{2}$

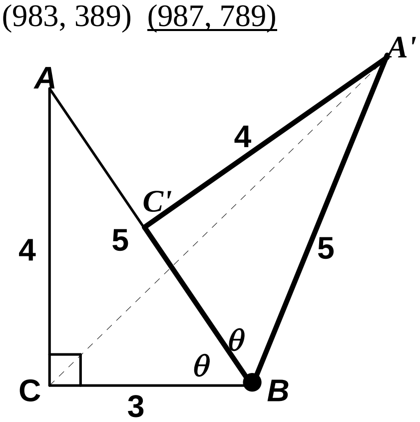
Since $2b$ is always even, $3k$ must be odd which forces k to be odd.

From $f(1)$, we have $a = 30 - 6b - 2c = 30 - 6\left(\frac{11-3k}{2}\right) - 2(4k) = 30 - 33 + 9k - 8k = k - 3$

Thus, minimizing $a + b + c$ is equivalent to minimizing $k - 3 + \frac{11-3k}{2} + 4k = \frac{5+7k}{2}$ for odd positive values of k. However, we must also verify that $a \ge -\frac{3b}{4}$.

$k = 1 \Rightarrow (a, b, c) = (-2, 4, 4)$ and $-2 \ge -\frac{3 \cdot 4}{4} = -3$. Thus, the minimum value of $a + b + c$ is **6**.

B) To be checked: (381, 183), (781, 187), (783, 387), (981, 189), (983, 389) (987, 789)
The underlined pairs fail because each is divisible by 3.
The second pair fails since 781 is divisible by 11.
Only the 5th pair needs be exhaustively checked:
389 must be checked for divisibility by primes up to $\sqrt{389} < 20$
$\Rightarrow$ 7, 13, 17, 19 (all fail - it's prime)
983 must be checked for divisibility by primes up to $\sqrt{983} < 32$
$\Rightarrow$ 7, 13, 17, 19, 23, 29, 31 (all fail - it's prime)
Thus, the only ordered pair is **(9, 3)**.



C) Using the Law of Cosines, we have

$\left(A'C\right)^2 = 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cos 2\theta = 34 - 30\cos 2\theta$. In ΔABC, $\cos\theta = \frac{3}{5} \Rightarrow \theta = Cos^{-1}\left(\frac{3}{5}\right)$.

Using the double-angle identity, $\cos 2\theta = 2\cos^2\theta - 1$, $\cos\left(2Cos^{-1}\left(\frac{3}{5}\right)\right) = 2\left(\frac{3}{5}\right)^2 - 1 = -\frac{7}{25}$

Substituting, $\left(A'C\right)^2 = 34 + 30 \cdot \frac{7}{25} = 34 + \frac{42}{5} = \mathbf{\frac{212}{5}}$ or **42.4**.

Team Round - continued

D) 104 singles, 18 doubles, 2 triples and 6 homeruns in 400 at bats

My batting average was $BA = \frac{104+18+2+6}{400} = \frac{130}{400} = \frac{13}{40} = 0.325$

My slugging percentage was $SLG = \frac{104+2(18)+3(2)+4(6)}{400} = \frac{170}{400} = \frac{17}{40} = 0.425$

Thus, my on base percentage was $\frac{15}{40}$ or 0.375. Let *x* = *HBP* and *y* = *SF*. Then:

$BB + HBP = 11x$ and we have $\frac{130+11x}{400+11x+y} = 0.375 = \frac{3}{8} \Leftrightarrow 1040 + 88x = 1200 + 33x + 3y$

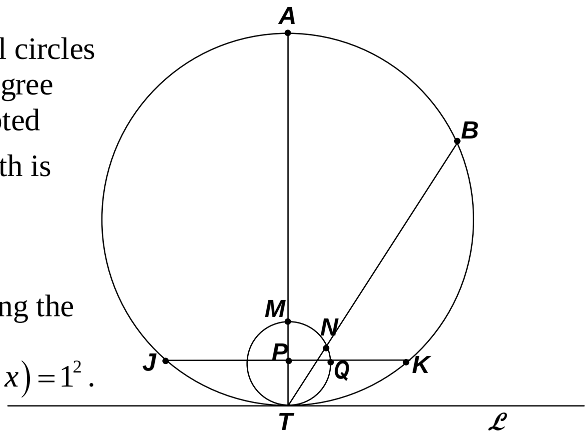
$\Leftrightarrow \Leftrightarrow 55x - 3y = 160$ (a linear function w/slope $\frac{55}{3}$) $\Leftrightarrow$ $y = \frac{5(11x-32)}{3}$

Since 3 is not a factor of 5, it must be a factor of $11x - 32$. $x = 4 \Rightarrow y = \frac{5(12)}{3} = 20$

Increasing *x* by 3 and *y* by 55, we get additional pairs: $(7,75)$, $(10,130)$,...

However, since I had fewer than 100 sac flies, only **4** and **7** are acceptable *x*-values.

E) Let *R* and *r* denote the radii of the large and small circles respectively. As an inscribed angle $\angle ATB$, its degree measure is half the degree measure of its intercepted arc. Therefore, minor arc $\overset{\frown}{AB}$ is 72°, i.e. its length is $\frac{1}{5}$ of the circumference of the circle and



$C = 5\left(\frac{4\pi}{5}\right) = 4\pi \Rightarrow R = 2$. Let $PT = x$. Applying the product-chord theorem in the larger circle, $x(4-x) = 1^2$.

$x^2 - 4x + 1 = 0 \Rightarrow x = \frac{4 \pm 2\sqrt{3}}{2} \Rightarrow PT = 2 - \sqrt{3}$

(the other root is extraneous)

$PT = 3 \cdot PM \Rightarrow TM = \frac{4}{3}\left(2-\sqrt{3}\right) \Rightarrow r = \frac{2}{3}\left(2-\sqrt{3}\right)$

Therefore, the required area is

$$4\pi - \pi\left(\frac{2}{3}\right)^2\left(2-\sqrt{3}\right)^2 = \pi\left(4 - \frac{4}{9}\left(7-4\sqrt{3}\right)\right) = \pi\left(4 - \frac{28}{9} + \frac{16}{9}\sqrt{3}\right)$$

$$= \pi\left(\frac{8}{9} + \frac{16}{9}\sqrt{3}\right) = \frac{8}{9}\left(1+2\sqrt{3}\right)\pi \Rightarrow (A,B,C) = \underline{(\mathbf{8,9,2})}.$$

Team Round - continued

F) GS: $x, y, -27, \ldots$ AS: $x, y, 21, \ldots$

In the GS, the common ratio is $\frac{y}{x} = \frac{-27}{y} \Rightarrow y^2 = -27x$.

In the AS, the common difference is $d = y - x = 21 - y \Rightarrow x = 2y - 21$.

Substituting, $y^2 = -27(2y-21) \Leftrightarrow y^2 + 54y - 27 \cdot 21 = 0$ or

$y^2 + 54y - 9 \cdot 63 = 0 \Leftrightarrow (y-9)(y+63) = 0$

$\Rightarrow\ y = 9,\ x = -3\ \ (r = -3)$ or $y = -63,\ x = -147\ \ \left(r = \frac{3}{7}\right)$

Thus, there are two possible pairs of sequences.

$$\begin{cases} \text{G.S. } -3, 9, -27, \boxed{81}, -243, \ldots \\ \text{A.S. } -3, 9, 21, 33, \boxed{45}, \ldots \end{cases} \Rightarrow \frac{45}{81} = \underline{\mathbf{\frac{5}{9}}}$$

$$\begin{cases} -147, -63, -27, -\frac{81}{7}, \ldots \\ -147, -63, 21, 105, 189, \ldots \end{cases} \Rightarrow 189\left(-\frac{7}{81}\right) = \underline{\mathbf{-\frac{49}{3}}}$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2015 ANSWERS

Round 1 Alg 2: Algebraic Functions

A) 4, 9 B) $(-13,-5)$ C) (10, 4)

Round 2 Arithmetic/ Number Theory

A) 1332 B) 57 C) 1, 2 (order irrelevant)

Round 3 Trig Identities and/or Inverse Functions

A) –1 B) $0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}$ C) $\frac{\sqrt{5}-1}{2}$

Round 4 Alg 1: Word Problems

A) $150 B) 87 C) $\frac{7}{6}$

Round 5 Geometry: Circles

A) $\frac{5}{4}$ B) $\frac{20}{3}$ C) 12

Round 6 Alg 2: Sequences and Series

A) $f(x) = 4 - x$ B) (–1, 3, –10) C) (256, 175)

Team Round

A) 6

B) (9, 3)

C) $\frac{212}{5}$ (or 42.4)

D) 4 and 7

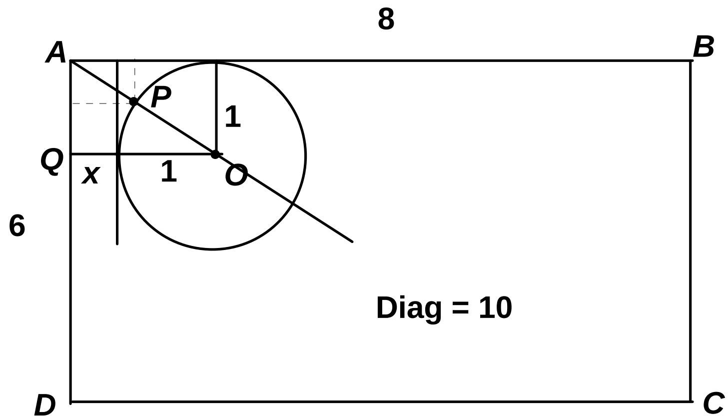
E) $(8,9,2)$

F) $\frac{5}{9}, -\frac{49}{3}$ or equivalent
(Both answers are required, but order is irrelevant.)

Appeal from Hamilton Wenham in Round 5 Question B

The original wording of the question was as follows:
A bead with a 2 inch diameter slides along a metal rod which connects opposite corners of a rectangular 6 inch by 8 inch wooden frame. When the bead touches the side of the frame it reverses direction. Let P and Q be the points along the wire where the bead comes closest to the opposite corners of the frame. Compute PQ.

Assuming P was not the center of the bead.
Zoom in on the upper left corner of the frame when the bead touches the top edge of the frame.



$$\Delta AOQ \sim \Delta ACD \Rightarrow \frac{AQ}{OQ} = \frac{AD}{CD} \Leftrightarrow \frac{1}{x+1} = \frac{6}{8} = \frac{3}{4} \Rightarrow 3x+3=4 \Rightarrow x = \frac{1}{3}$$

Using the Pythagorean Theorem on ΔAQO,

$$\left(\frac{4}{3}\right)^2 + 1^2 = AO^2 \Rightarrow AO^2 = \frac{25}{9} \Rightarrow AO = \frac{5}{3} \Rightarrow AP = \frac{5}{3} - 1 = \frac{2}{3}$$

Therefore, $PQ = 10 - 2\left(\frac{2}{3}\right) = \underline{\boldsymbol{\frac{26}{3}}}$

Alternate interpretation was accepted.