MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2003
ROUND 1: COMPLEX NUMBERS

ANSWERS

A)_____

B)_____

C)_____

A) Calculate ab when $\dfrac{5}{4 + 3i}$ is in simplified a + bi form.

$(a \times b)$

B) Calculate the absolute value of : $\dfrac{-2\sqrt{3} + 6i}{-\sqrt{2} - i\sqrt{2}}$ in simplified radical form.

C) Find the imaginary roots of the equation $x^4 + x^3 - 6x^2 - 14x - 12 = 0$

ANSWERS

A)_____

B)_____

C)_____

A) Sharnell could earn a weekly salary of $200 plus 15% commission of sales at the Radio Barn, or she could earn $300 plus 10% commission on sales at Woofen Etc store. Compute the dollar amount of sales Sharnell would have to sell to generate the same salary at each business.

B) Solve for x: $\dfrac{3}{x-1} - \dfrac{14}{x^2} = \dfrac{6}{x^3 - x^2}$.

C) If the graph of $\{(x, y): y = ab^x\}$ contains the points (0, 3) and (3, 24); compute the value of a + b.

ANSWERS

A)_____

B)_____

C)_____

A) The shorter diagonal of a rhombus is equal to the diagonal of a square, while the longer diagonal is equal to twice the side of the same square. Calculate in simplified radical form the ratio of the area of the rhombus to the area of the square.

B) A regular hexagon and a square share a common side. Calculate the ratio in simple radical form of the area of the hexagon to the area of the square.

C) In regular hexagon ABCDEF, the area of triangle ABD is 9. Calculate the area of the hexagon.

ANSWERS

A)_____

B)_____

C)_____

A) Factor completely: $\left(x^2 - 5x + 2\right)^2 - 4$

B) Solve for x: $\left(x - 5\right)^3 + 5 = x$

C) Find both sets of three consecutive odd integers such that four times the square of the third diminished by three times the square of the first is 89 less than two times the square of the second.

ANSWERS

A)_____

B)_____

C)_____

A) Evaluate and leave the result in simplest radical form.

$\sin 30^0 + \cos 30^0 + \cot 225^0 + \tan 135^0 + \csc 120^0 + \sec 300^0$

B) Solve for $0^0 \le x < 180^0$: $\cos(x)\cot^2(x) = \cos(x)$

C) The area of a regular octagon is $288\sqrt{2}$. The circumference of its circumscribed circle is $k\pi$. What is the value of k?
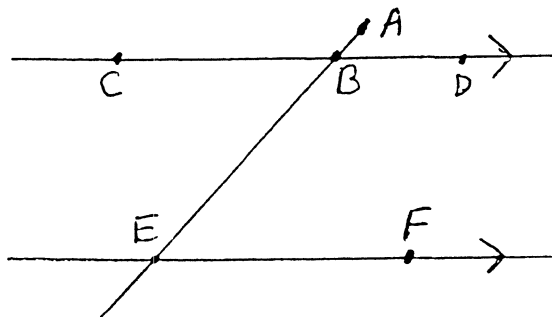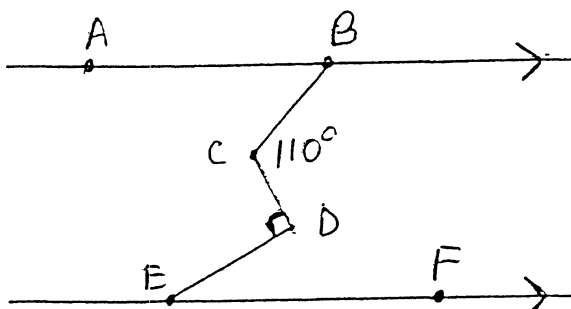
ANSWERS

A)_____

B)_____

C)_____

A) Given two parallel lines cut by a transversal. If $\angle ABC = 6x + 6y - 36$, $\angle CBE = 4x$, and $\angle BEF = 3y$; calculate the degree measure of $\angle DBE$.

B) What is the smaller angle formed by the hands of a clock at 4:36?

C) Given two parallel lines. If $\angle ABC = 3x - 5$, $\angle BCD = 110^0$, $\angle CDE = 90^0$, and $\angle DEF = x + 5$; calculate the value of x.

ANSWERS

A)_____   D)_____

B)_____   E)_____

C)_____   F)_____

A) Express the value of a + b in simple radical form when $9i^{14}(1 + i)^3(\sqrt{2} - i)^{-2}$ is in standard a + bi form.

B) Cameron is thinking of a positive integer with the property that it is the average of a prime and a perfect cube where the cube is larger than the prime. How many integers less than 100, and which are primes satisfy Cameron's property?

C) A rectangle with integral sides is divided into four smaller rectangles by drawing a pair of perpendicular lines in its interior. If the areas of the four smaller rectangles are x, 2, 3, and 6; list all possible values of x.

D) The top and base of a fish tank are congruent rectangles whose lengths are 10 inches more than their widths. If the sum of the height and width of the tank is 50 inches, and the combined areas of the top and base is 400 square inches less than the total area of the four sides, what are the dimensions of the tank?

E) The pilot of an airplane calculates the angle of depression of an airport to be 30^0. The angle of depression to the airport from a second airplane 1000 feet directly above the first is 45^0. In simple radical form, what is the altitude of the first airplane?

F) The sides of a heptagon are extended to form a seven pointed star. What is the sum of the angles formed at the points of the star?

ANSWERS

A) $-12/25$

B) $2\sqrt{3}$

C) $-1 \pm i$

A) Simplify and leave in a + bi form. $\dfrac{5}{4+3i}$, Calculate ab,

$$\frac{5}{4+3i}\cdot\frac{4-3i}{4-3i} = \frac{20-15i}{25} = \frac{4}{5} - \frac{3}{5}i,$$

$$ab = -12/25$$

B) Calculate the absolute value of: $\dfrac{-2\sqrt{3}+6i}{-\sqrt{2}-i\sqrt{2}}$ in simple radical form.

$$\frac{\left|-2\sqrt{3}+6i\right|}{\left|-\sqrt{2}-i\sqrt{2}\right|} = \frac{\sqrt{12+36}}{\sqrt{2+2}} = \frac{\sqrt{48}}{\sqrt{4}} = \sqrt{12} = 2\sqrt{3}$$

C) Find the imaginary roots of the equation $x^4 + x^3 - 6x^2 - 14x - 12 = 0$

$$
\begin{array}{c|ccccc}
 & 1 & 1 & -6 & -14 & -12 \\
3 & 1 & 4 & 6 & 4 & 0 \\
-2 & 1 & 2 & 2 & 0 &
\end{array}
$$

$$x^2 + 2x + 2 = 0$$
$$x^2 + 2x + 1 = -2 + 1$$
$$(x+1)^2 = -1$$
$$x = -1 \pm i$$

ANSWERS

A) $\cancel{\$} 2000$

B) $2/3, 4$

C) 5

A) Sharnell could earn a weekly salary of $200 plus 15% commission of sales at the Radio Barn, or she could earn $300 plus 10% commission on sales at Woofen Etc store. Compute the dollar amount of sales Sharnell would have to sell to generate the same salary at each business.

$$200 + .15x = 300 + .10x$$

$$.05x = 100$$

$$x = 2000$$

B) Solve for x: $\dfrac{3}{x-1} - \dfrac{14}{x^2} = \dfrac{6}{x^3 - x^2}$.

$$3x^2 - 14(x-1) = 6 \qquad (3x-2)(x-4) = c$$

$$3x^2 - 14x + 14 = 6 \qquad x = 2/3, 4$$

$$3x^2 - 14x + 8 = 0$$

C) If the graph of $\{(x, y): y = ab^x\}$ contains the points (0, 3) and (3, 24); compute the value of a + b.

$$ab^0 = 3, \quad a = 3$$

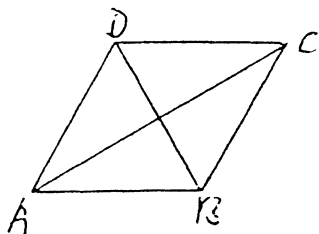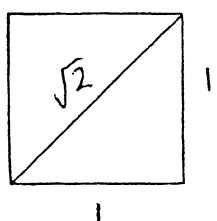$$3b^3 = 24, \quad b^3 = 8, \quad b = 2 \qquad a + b = 5$$

ANSWERS

A) $\sqrt{2} : 1$

B) $3\sqrt{3}/2$

C) 27

**A)** The shorter diagonal of a rhombus is equal to the diagonal of a square, while the longer diagonal is equal to twice the side of the same square. Calculate in simplified radical form the ratio of the area of the rhombus to the area of the square.
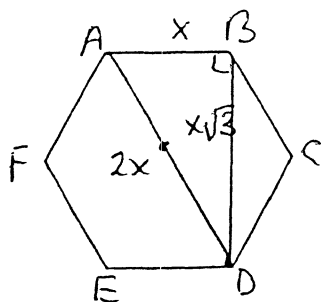
$BD = \sqrt{2}$ Area rhombus $= \frac{1}{2} \cdot 2\sqrt{2} = \sqrt{2}$

$AC = 2$ Area square $= 1$

ANS $\sqrt{2} : 1$

B) A regular hexagon and a square share a common side. Calculate the ratio in simple radical form of the area of the hexagon to the area of the square.
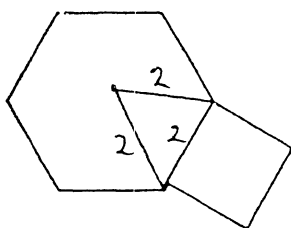
$$\frac{A_H}{A_S} = \frac{6 \cdot \frac{4\sqrt{3}}{4}}{4} = \frac{6\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$$

C) In regular hexagon ABCDEF, the area of triangle ABD is 9. Calculate the area of the hexagon.

$\frac{1}{2}x^2\sqrt{3} = 9$, $x^2\sqrt{3} = 18$, $x^2 = 6\sqrt{3}$

$A_H = 6 \cdot \frac{x^2\sqrt{3}}{4} = \frac{6 \cdot 6\sqrt{3}\sqrt{3}}{4} = \frac{36 \cdot 3}{4} = 27$

MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2003
ROUND 4: ALGEBRA I FACTORING

ANSWERS

A) $x(x-1)(x-4)(x-5)$

B) $4, 5, 6$

C) $29, 31, 33$ or $-5, -3, -1$

A) Factor completely: $\left(x^2 - 5x + 2\right)^2 - 4$

$\left(x^2 - 5x + 2 + 2\right)\left(x^2 - 5x + 2 - 2\right) = \left(x^2 - 5x + 4\right)\left(x^2 - 5x\right)$

$(x - 1)(x - 4)(x - 5)x$

B) Solve for x: $(x - 5)^3 + 5 = x$

$(x - 5)^3 - (x - 5) = 0 \qquad (x - 5)\left[(x - 5)^2 - 1\right] = 0$

$(x - 5)(x - 5 + 1)(x - 5 - 1) = 0$

$(x - 5)(x - 4)(x - 6) = 0$

$x = 4, 5, 6$

C) Find three consecutive odd integers such that four times the square of the third diminished by three times the square of the first is 89 less than two times the square of the second.

$x, x+2, x+4 \qquad 4(x+4)^2 - 3x^2 = 2(x+2)^2 - 89$

$4x^2 + 32x + 64 - 3x^2 = 2x^2 + 8x + 8 - 89$

$x^2 + 32x + 64 = 2x^2 + 8x - 81$

$x^2 - 24x - 145 = 0 \qquad (x+5)(x-29) = 0$

$-5, -3, -1$ or $29, 31, 33$

ANSWERS

A) $(15 + 7\sqrt{3})/6$

B) $45°, 90°, 135°$

C) 24

A) Evaluate and leave the result in simplest radical form.

$$\sin 30° + \cos 30° + \cot 225° + \tan 135° + \csc 120° + \sec 300°$$

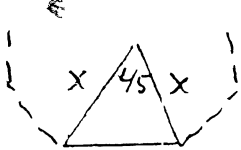$$\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 - 1 + \frac{2}{\sqrt{3}} + 2$$

$$\frac{\sqrt{3}}{2\sqrt{3}} + \frac{3}{2\sqrt{3}} + \frac{4}{2\sqrt{3}} + \frac{4\sqrt{3}}{2\sqrt{3}} = \frac{7 + 5\sqrt{3}}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{15 + 7\sqrt{3}}{6}$$

B) Solve for $0° \le x < 180°$: $\quad \cos(x)\cot^2(x) = \cos(x)$

$$\cos x = 0 \qquad \cot^2 x = 1, \quad \tan x = \pm 1$$

$$x = 90°, \qquad x = 45°, 135°$$

C) The area of a regular octagon is $288\sqrt{2}$. The circumference of its circumscribed circle is $k\pi$. What is the value of k?

$$8 \cdot \frac{1}{2} x^2 \sin 45° = 288\sqrt{2} \qquad C = 2\pi x = 24\pi$$

$$4x^2 \frac{\sqrt{2}}{2} = 288\sqrt{2} \qquad k = 24$$

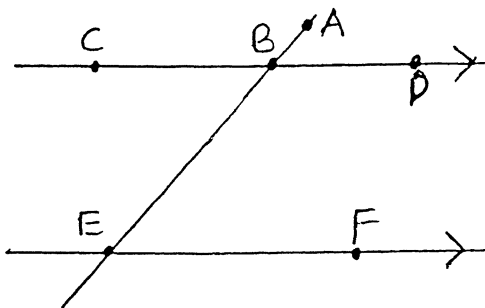$$2x^2 = 288, \quad x^2 = 144$$

$$x = 12$$

ANSWERS

A) $132°$

B) $78°$

C) 15

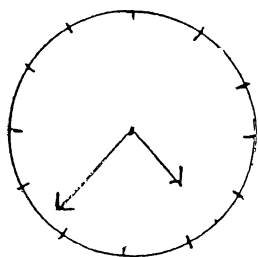**A)** Given two parallel lines cut by a transversal. If $\angle ABC = 6x + 6y - 36$, $\angle CBE = 4x$, and $\angle BEF = 3y$; calculate the degree measure of $\angle DBE$.
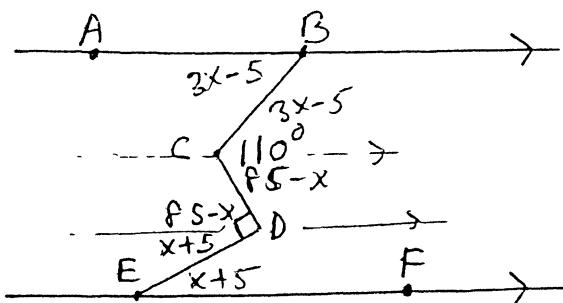
$6x + 6y - 36 + 4x = 180$

$10x + 6y = 216$

$5x + 3y = 108$

$4x - 3y = 0$

$9x = 108$

$x = 12$

$4x = 48°$

$\angle DBE = 180° - 48°$

$= 132°$

B) What is the smaller angle formed by the hands of a clock at 4:36?

min hand $\dfrac{36}{60} \cdot \dfrac{360°}{1} = 216°$

hour hand $120° + \dfrac{3}{5} \cdot \dfrac{30}{1} = 138°$

$216° - 138° = 78°$

C) Given two parallel lines. If $\angle ABC = 3x - 5$, $\angle BCD = 110°$, $\angle CDE = 90°$, and $\angle DEF = x + 5$; calculate the value of x.

$3x - 5 + 85 - x = 110°$

$2x + 80° = 110°$

$2x = 30°$

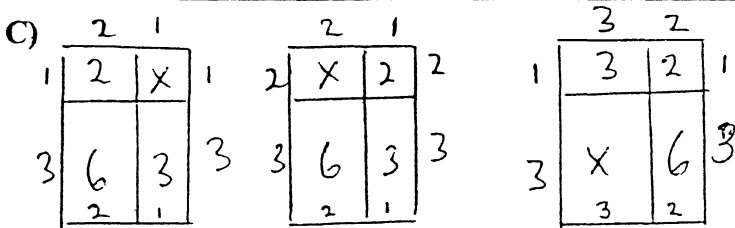$x = 15$

ANSWERS

A) $8\sqrt{2}$ D) $30 \times 40 \times 20$

B) 7 E) $500 + 500\sqrt{3}$

C) $1, 4, 9$ F) $540°$

A) $9i^4 = -9$, $(1+i)^3 = -2+2i$, $(\sqrt{2}-i)^2 = 1 - 2\sqrt{2}i$.

$\dfrac{-9(-2+2i)}{1-2\sqrt{2}i}$, $\dfrac{1+2\sqrt{2}i}{1+2\sqrt{2}i} = \dfrac{-9}{9}(-2-4\sqrt{2}+2i-4\sqrt{2}i) =$

$2+4\sqrt{2}-2i+4\sqrt{2}i$, $a+b = 8\sqrt{2}$

B) 2 ⑤ 8   19 ㉓ 27   53 ㊽ 125

7 ⑰ 27   17 ㉑ 125   ANS: 7

11 ⑲ 27   41 ㊸ 125

C)

D) The top and base of a fish tank are rectangles whose length is 10 inches more than the width. If the sum of the height and width of the tank is 50 inches, and the combined areas of the top and base is 400 square inches less than the total area of the four sides, what are the dimensions of the tank?

$w = x$, $\ell = x+10$, $h = 50-x$ $2x(x+10) = 2(x+10)(50-x)+2x(50-x)-400$

Simplifies to $3x^2 - 80x - 300 = 0$ $(3x+10)(x-30) = 0$

$w = x = 30$, $\ell = 40$, $h = 20$

E) The pilot of an airplane calculates the angle of depression of an airport to be $30°$. The angle of depression to the airport from a second airplane 1000 feet directly above the first is $45°$. In simple radical form, what is the altitude of the first airplane?
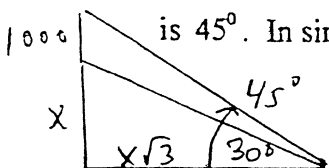
$x\sqrt{3} = x + 1000$, $x = \dfrac{1000}{\sqrt{3}-1} = 500(\sqrt{3}+1)$

$= 500 + 500\sqrt{3}$

F) The sides of a heptagon are extended to form a seven pointed star. What is the sum of the angles formed at the points of the star?

Use reg. hept. Star $\angle = 180° - 2 \cdot 51\frac{3}{7} = 77\frac{1}{7}$. Sum $= 7 \cdot 77\frac{1}{7} = 540°$