MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 1: ALGEBRAIC FUNCTIONS

ANSWERS

A)_______________

B)_______________

C)_______________

A) If $f(x) = -2x^2 + 7x - 3$, calculate $f(3 + h) - f(3 - h)$ in terms of h.

B) If $f(x) = x + 5$ and $g(x) = x^2$, solve the equation $f(g(2 - a)) = g(f(a - 3)$ for a.

C) If $f(x) = 2x + 1$ and $g(x) = 3x - 2$, solve the equation $f^{-1}(f^{-1}(w)) = f(g^{-1}(w))$ for w.

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 2: NUMBER THEORY

ANSWERS

A)_______________

B)_______________

C)_______________

A) Given $(ABA)_9 = (BB0)_{11}$ where 0 is zero, and A and B are distinct natural numbers. Determine both possible values of A and B. Write the answers in the form (A, B).

B) Determine the units digit for the sum of $7^{2003} + 9^{2003}$.

C) How many positive even integers are divisors of $(12^3)(18^4)$?

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 3: TRIG. IDENTITIES OR INVERSES

ANSWERS

A)_______________

B)_______________

C)_______________

A) Simplify $\dfrac{(\cot\theta - \cos\theta)(1 + \sin\theta)}{\cos^3\theta}$ to the form T(θ) where T is one of the six trig functions.

B) For $0^0 \le \theta < 360^0$, solve $\dfrac{2\tan\theta}{1 + \tan^2\theta} = \dfrac{\sqrt{3}}{2}$.

C) Using principle values, express $\cos(\sec^{-1}\dfrac{3}{2} - \cos^{-1}\dfrac{1}{5})$ in simple radical form.

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 4: WORD PROBLEMS

ANSWERS

A)_______________

B)_______________

C)_______________

A) What is the larger of the two numbers whose sum is ten, if the sum of their reciprocals is 8/15.

B) An elevator went from the bottom of a tower to the top at a speed of 4 meters/second. It remained at the top for ninety seconds, and then returned to the bottom at a speed of 5 m/sec. If the total trip took 4 5 minutes, how high is the tower?

CONSECUTIVE

C) The sum of the squares of three ~~positive~~ odd integers is 967 more than the sum of the squares of the two even integers between them. Calculate the sum of the five consecutive integers.

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 5: GEOMETRY CIRCLES
NON-CALCULATOR

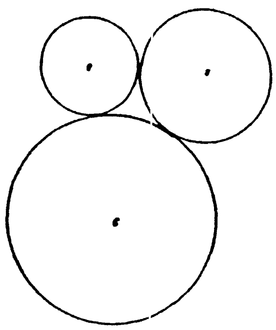
ANSWERS

A)______________

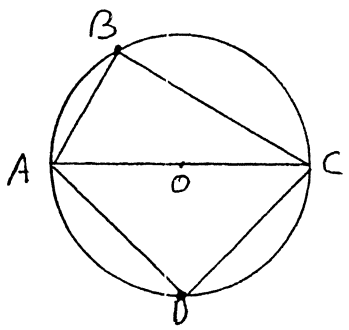
B)______________

C)______________

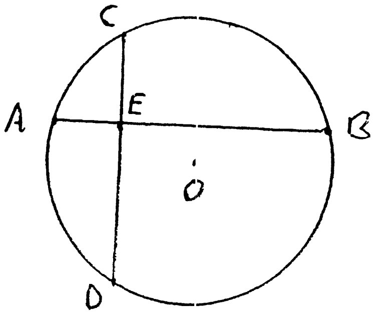
A) Three circles of areas π, 4π, *and* 9π are drawn tangent to each other. Calculate the area of the triangle formed by connecting the centers of the three circles.



B) In the figure, $\overline{AC}$ is a diameter of circle O, $\overset{\frown}{AB} = \frac{1}{2}\overset{\frown}{BC}$, D is the midpoint of $\overset{\frown}{AC}$. Find the value of BC/AD in simplified radical form.



C) In circle O, $\overline{CD} \perp \overline{AB}$, CE = 5, CD = 14, and the ratio of AE to AB is 1 to 6. The area of circle O is $k\pi$. What is the value of k?



MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 6: SEQUENCES & SERIES

ANSWERS

A)_______________

B)_______________

C)_______________

A) In an arithmetic sequence of ten terms, the tenth term is 14, and their sum is 5. Find the second term.

B) The second term of a geometric sequence is 12, and the sixth term is 1024/27. Find the first term.

C) The six terms $2x - 3$, t, $7 - 12y$, $x + 3$, $3y - 4$, $x + 12$ are in arithmetic sequence. Find the ordered triple (x, y, t).

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 7: TEAM QUESTIONS

ANSWERS

A)____________ D)____________

B)____________ E)____________

C)____________ F)____________

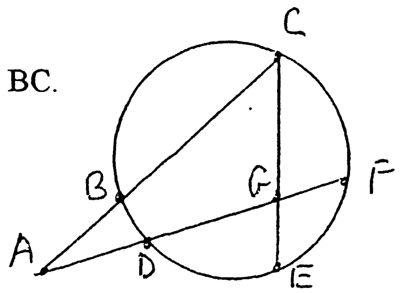
A) If $f(x) = 2x^2 - 17x + 24$ and $f(x + a) = 2x^2 - 5x - 9$, calculate the value of a.

B) Determine the 142^{nd} positive integer divisible by three or five.

C) Express $\cos^2 \frac{7\pi}{24} - \sin^2 \frac{7\pi}{24}$ in simplified radical form.

D) The hundred's digit of a three-digit number is one more than twice the units digit, and the tens digit is three less than the units digit. If the digits were reversed, the number obtained would be 396 less than the original number. Find the original number.

E) In the figure, AC = 9, GC = 6, GE = 3, and AD = DG = GF. Find BC.
(not to scale)



F) An infinite geometric series has a sum of 54 while the sum of the first three terms is 52. What is the first term?

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 1: ALGEBRAIC FUNCTIONS

ANSWERS

A) $-10h$

B) $5/8$

C) $-37/5$

A) If $f(x) = -2x^2 + 7x - 3$, calculate $f(3 + h) - f(3 - h)$ in terms of h.

$$[-2(9+6h+h^2)+7(3+h)-3]-[-2(9-6h+h^2)+7(3-h)-3]$$

$$=[-18-12h-2h^2+21+7h-3]-[-18+12h-2h^2+21-7h-3]$$

$$=(-5h-2h^2)-(5h-2h^2) = -10h$$

B) If $f(x) = x + 5$ and $g(x) = x^2$, solve the equation $f(g(2 - a)) = g(f(a - 3)$ for a.

$$f[(2-a)^2] = g[(a-3)+5], \quad (2-a)^2+5 = (a+2)^2$$

$$4-4a+a^2+5 = a^2+4a+4$$

$$5 = 8a$$

$$a = 5/8$$

C) If $f(x) = 2x + 1$ and $g(x) = 3x - 2$, solve the equation $f^{-1}(f^{-1}(w)) = f(g^{-1}(w))$ for w.

$$f^{-1}(x) = \frac{x-1}{2}, \quad g^{-1}(x) = \frac{x+2}{3}$$

$$\frac{\left(\frac{w-1}{2}\right)-1}{2} = 2\left(\frac{w+2}{3}\right)+1, \quad \frac{w-1-2}{4} = \frac{2w+4+3}{3},$$

$$\frac{w-3}{4} = \frac{2w+7}{3}, \quad 3w-9 = 8w+28$$

$$-37 = 5w, \quad w = -37/5$$

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 2: NUMBER THEORY

ANSWERS

A) (3,2), (6,4)

B) 2

C) 120

A) Given $(ABA)_9 = (BB0)_{11}$ where 0 is zero, and A and B are distinct natural numbers. Determine both possible values of A and B. Write the answers in the form (A, B).

$81A + 9B + A = 121B + 11B$

$82A = 132B - 9B = 123B$

$2A = 3B$, $A = 3$, $B = 2$ or $A = 6$, $B = 4$

B) Determine the units digit of $7^{2003} + 9^{2003}$.

7^0	1
7^1	7
7^2	9
7^3	3
7^4	1

9^0	1
9^1	9
9^2	1

$4 \overline{)2003}$ $R=3$

$2 \overline{)2003}$ $R=1$

$3 + 9 = 12$

C) How many positive even divisors does $(12^3)(18^4)$ have?

$(2^2 \cdot 3)^3 (2 \cdot 3^2)^4 = 2^6 \cdot 3^3 \cdot 2^4 \cdot 3^8 = 2^{10} \cdot 3^{11}$

divisors $= (10+1)(11+1) = 11(12) = 132$

Ans $132 -$ odd divisors $= 132 - 12 = 120$

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 3: TRIG. IDENTITIES OR INVERSES

ANSWERS

A) $\csc\theta$

B) $30°, 60°, 210°, 240°$

C) $(2+2\sqrt{30})/15$

A) Simplify $\dfrac{(\cot\theta - \cos\theta)(1+\sin\theta)}{\cos^3\theta}$ to the form T(θ) where T is one of the six trig functions.

$$\frac{\left(\frac{\cos\theta}{\sin\theta} - \cos\theta\right)(1+\sin\theta)}{\cos^3\theta} = \frac{(\cos\theta - \sin\theta\cos\theta)(1+\sin\theta)}{\sin\theta\cos^3\theta}$$

$$= \frac{\cos\theta(1-\sin^2\theta)}{\sin\theta\cos^3\theta} = \frac{1}{\sin\theta} = \csc\theta$$

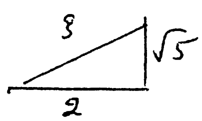
B) For $0^0 \le \theta < 360^0$, solve $\dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{\sqrt{3}}{2}$.

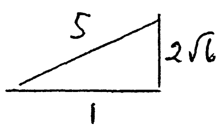
$$\frac{\frac{2\sin\theta}{\cos\theta}}{\sec^2\theta} = \frac{2\sin\theta}{\cos\theta}\cdot\frac{\cos^2\theta}{1} = 2\sin\theta\cos\theta = \sin 2\theta = \frac{\sqrt{3}}{2}$$

$2\theta = 60°; 120°, 420°, 480°$

$\theta = 30°, 60°; 210°, 240°$

C) Using principle values, express $\cos(\sec^{-1}\frac{3}{2} - \cos^{-1}\frac{1}{5})$ in simple radical form.





$\cos A\cos B + \sin A\sin B =$

$$\frac{2}{3}\cdot\frac{1}{5} + \frac{\sqrt{5}}{3}\cdot\frac{2\sqrt{6}}{5} = \frac{2+2\sqrt{30}}{15}$$

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 4: WORD PROBLEMS

ANSWERS

A) 15/2

B) 400

C) ±155

A) What is the larger of the two numbers whose sum is ten, if the sum of their reciprocals is 8/15.

$x,\ 10-x \qquad \frac{1}{x}+\frac{1}{10-x}=\frac{8}{15}$

$15(10-x)+15x=8(10x-x^2)$

$150=80x-8x^2,\quad 8x^2-80x+150=0$

$4x^2-40x+75=0 \quad (2x-5)(2x-15)=0 \quad x=\frac{5}{2},\frac{15}{2}$ ANS 15/2

B) An elevator went from the bottom of a tower to the top at a speed of 4 meters/second. It remained at the top for ninety seconds, and then returned to the bottom at a speed of 5 m/sec. If the total trip took 4.5 minutes, how high is the tower?

x = ht of elevator $\quad \frac{x}{4}+90+\frac{x}{5}=270,\quad \frac{x}{4}+\frac{x}{5}=180$

$5x+4x=180\cdot 20$

$\frac{9x}{9}=\frac{180\cdot 20}{9},\quad x=20\cdot 20=400$

C) The sum of the squares of three positive odd integers is 967 more than the sum of the squares of the two even integers between them. Calculate the sum of the five consecutive integers. $\quad x,\ x+2,\ x+4$

$x^2+(x+2)^2+(x+4)^2=967+(x+1)^2+(x+3)^2$

$x^2+4x+4+8x+16=967+2x+1+6x+9$

$x^2+12x+20=8x+977,\quad x^2+4x-957=0$

$(x+33)(x-29)=0,\quad x=29,\quad x+1=30,\ x+2=31,\ x+3=32,\ x+4=33$

OR $x=-33 \rightarrow -155$

ANS ±155

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 5: GEOMETRY CIRCLES
NON-CALCULATOR

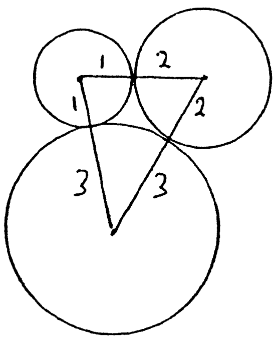
ANSWERS

A) 6

B) $\sqrt{6}/2$

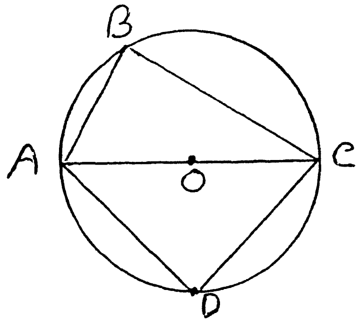
C) 85

A) Three circles of areas π, 4π, *and* 9π are drawn tangent to each other. Calculate the area of the triangle formed by connecting the centers of the three circles.



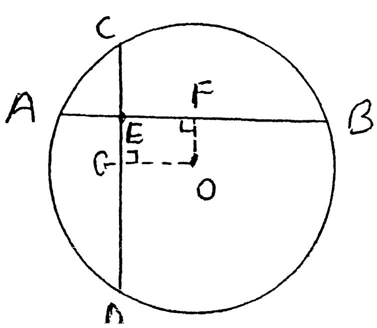
$3, 4, 5\ \triangle$, area $= \frac{1}{2} \cdot 3 \cdot 4 = 6$

B) In the figure, $\overline{AC}$ is a diameter of circle O, $\overset{\frown}{AB} = \frac{1}{2}\overset{\frown}{BC}$, D is the midpoint of $\overset{\frown}{AC}$. Find the ratio of BC to AD in simplified radical form.



Let $OA = OC = 1$, Then $AB = 1$, $BC = \sqrt{3}$, $AD = \sqrt{2}$. $BC/AD = \sqrt{3}/\sqrt{2} = \sqrt{6}/2$

C) In circle O, $\overline{CD} \perp \overline{AB}$, CE = 5, CD = 14, and the ratio of AE to AB is 1 to 6. The area of circle O is $k\pi$. What is the value of k?



$GE = 2$, $AE = x$, $EB = 5x$, $5x^2 = 45$, $x = 3$,
$EB = 15$, $EF = 6$, $FB = 9$, $OF = GE = 2$,
$OB^2 = 2^2 + 9^2 = 85 = k$

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 6: SEQUENCES & SERIES

ANSWERS

A) -10

B) ± 9

C) $\left(-\frac{15}{2}, \frac{4}{3}, -\frac{27}{2}\right)$

A) In an arithmetic sequence of ten terms, the tenth term is 14, and their sum is 5. Find the second term.

$a_1 + 9d = 14$, $5(2a_1 + 9d) = 5$ so $a_1 + 9d = 14$, $2a_1 + 9d = 1$, $a_1 = -13$

$-13 + 9d = 14$, $9d = 27$, $d = 3$.

$a_2 = a_1 + d = -13 + 3 = -10$

B) The second term of a geometric sequence is 12, and the sixth term is 1024/27. Find the first term.

$12r^4 = \frac{1024}{27}$, $r^4 = \frac{1024}{12 \cdot 27} = \frac{256}{81}$, $r = \pm\frac{4}{3}$

$a_1 = \frac{a_2}{r} = \frac{12}{\pm 4/3} = \pm \frac{12}{1} \cdot \frac{3}{4} = \pm 9$

C) The six terms $2x - 3$, t, $7 - 12y$, $x + 3$, $3y - 4$, $x + 12$ are in arithmetic sequence. Find the ordered triple (x, y, t).

$(x+12) - (x+3) = 9 = 2d$, $d = \frac{9}{2}$. $(3y-4) - (7-12y) = 9$.

$15y - 11 = 9$, $15y = 20$, $y = \frac{20}{15} = \frac{4}{3}$. $a_3 = 7 - 12y = 7 - \frac{12}{1} \cdot \frac{4}{3} =$

$7 - 16 = -9$. $t = a_2 = a_3 - d = -9 - \frac{9}{2} = -\frac{27}{2}$. $a_4 = x + 3 = a_3 + d =$

$-9 + \frac{9}{2} = -\frac{9}{2}$ so $x + 3 = -\frac{9}{2}$ and $x = -3 - \frac{9}{2} = -\frac{6-9}{2} = -\frac{15}{2}$.

ANS $(x, y, t) = \left(-\frac{15}{2}, \frac{4}{3}, -\frac{27}{2}\right)$

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 7: TEAM QUESTIONS

ANSWERS

A) 3 D) 703

B) 305 E) 3

C) $(\sqrt{2}-\sqrt{6})/4$ F) 36

A) If $f(x) = 2x^2 - 17x + 24$ and $f(x+a) = 2x^2 - 5x - 9$, calculate the value of a.

$2(x+a)^2 - 17(x+a) + 24 = 2x^2 - 5x - 9$

$2(x^2 + 2xa + a^2) - 17(x+a) + 24$ so $4a - 17 = -5$, $a = 3$

B) Determine the 142nd positive integer divisible by three or five.

300 is the 100th div by 3, and the 60th div by 5, but the 20th div by 15. So 300 is the $100+60-20 = 140$th div by 3 or 5.

ANS 305

C) Express $\cos^2 \frac{7\pi}{24} - \sin^2 \frac{7\pi}{24}$ in simple radical form.

$= \cos \frac{7\pi}{12} = \cos 105^\circ = \cos(60^\circ + 45^\circ) = \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}-\sqrt{6}}{4}$

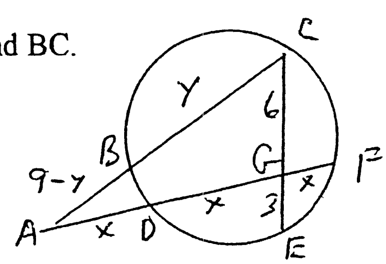
D) The hundred's digit of a three-digit number is one more than twice the units digit, and the tens digit is three less than the units digit. If the digits were reversed, the number obtained would be 396 less than the original number. Find the original number.

$h = 2u+1,\ T = u-3,\ 100(2u+1) + 10(u-3) + u = 100u + 10(u-3) + (2u+1) + 396$

$200u + 100 + u = 102u + 397$

$99u = 297,\ u = 3;\ T = 0,\ h = 7$

E) In the figure, AC = 9, GC = 6, GE = 3, and AD = DG = GF. Find BC.



$x^2 = 3 \cdot 6 = 18,\ x = 3\sqrt{2}$

$9(9-y) = 3\sqrt{2}(9\sqrt{2}) = 54$

$9 - y = 6,\ y = 3$

F) An infinite geometric series has a sum of 54 while the sum of the first three terms is 52. What is the first term?

$\frac{a}{1-r} = 54,\ a + ar + ar^2 = 52$, so $54(1-r)(1+r+r^2) = 52$.

$54(1-r^3) = 52,\quad 54r^3 = 2,\ r^3 = \frac{1}{27},\ r = \frac{1}{3},\ a = \frac{54}{1}\cdot\frac{2}{3} = 36$