

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2004
ROUND 1: SIM. EQUATIONS & DET.

ANSWERS

A) _____

B) _____

C) _____

A) A rectangular cafeteria has a perimeter of 800 feet, and an area of 30,000 square feet. Find the number of feet in the length and in the width of the cafeteria.

B) Lia is offered two different salary options to sell health club memberships. Plan A is: \$500 weekly in addition to 6% of her sales; while Plan B is: \$700 weekly in addition to 1% of her sales. Lia realized that she could get the same salary for a week under both plans. What is that salary?

C) Solve for x:
$$\begin{vmatrix} 1 & -3 & x \\ x & 4 & -1 \\ -2 & 4 & 1 \end{vmatrix} = 40$$

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2004
ROUND 2: EXPONENTS & RADICALS

ANSWERS

A) _____

B) _____

C) _____

A) Find the exact value of x : $\sqrt{4 + \frac{1}{4} + \frac{4}{9}} = 2 + \frac{1}{2} + \frac{x}{3}$

B) Convert to simplified radical form: $\frac{\sqrt{6} - \sqrt{2}}{\sqrt{3} + 1} + \frac{2\sqrt{3}}{\sqrt{2}}$

C) Solve for x . $8^{\frac{x+2}{x}} = 16^{\frac{x+2}{4}}$

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2004
ROUND 3: POLYNOMIAL FUNCTIONS

ANSWERS

A) _____

B) _____

C) _____

A) Given $P(x) = 2x^3 - 5x^2 + ax + b$. Calculate the value of $(a + b)$ if $P(-1) = 8$, and the remainder when $P(x)$ is divided by $x - 2$ is -22 .

B) Find all roots of. $2x^4 + 3x^3 - 3x^2 - 5x - 6 = 0$

C) If r , s , and t are the roots of $3x^3 + 7x^2 - 4 = 0$, calculate the value of

$\frac{1}{r-1} + \frac{1}{s-1} + \frac{1}{t-1}$ as a simplified fraction

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2004
ROUND 4: ALGEBRA I ANYTHING

ANSWERS

A) _____

B) _____

C) _____

A) The difference of the squares of two consecutive positive even integers is 2004. What is the sum of these two integers?

B) What is the exact time after 11 o'clock when the hands of a clock first form a ninety degree angle?

C) Two persons A and B together have fifteen dollars less than a third person C. If A gives \$4 to B, then A's amount of money will be one-third of C's. If, instead, B gives \$4 to A, then A will have twice as much money as B. How many dollars does each person have?

**MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2004
ROUND 5: GEOMETRY ANYTHING**

ANSWERS

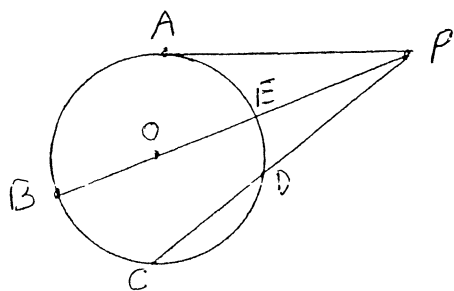
A) _____

B) _____

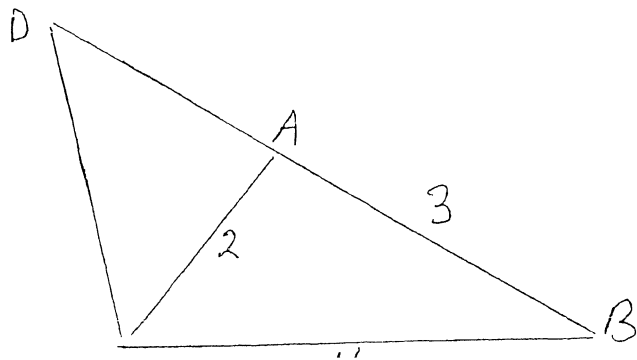
C) _____

A) The length of a diagonal of a cube is the same as the length of a shorter diagonal of a regular hexagon. The ratio of the total surface area of the cube to the area of the hexagon is A/B . Compute A/B in simplified radical form.

B) \overline{PA} is tangent to circle O at A . $PEOB$ and PDC are secants to circle O . $AP = PE + 4$, $PD = CD + 2$. The circumference of circle O is 10π . In simplified form, find PD/PA .



C) Given $\triangle ABC$, $AC = 2$, $AB = 3$, and $BC = 4$. \overline{BA} is extended to D so that $\triangle CAD \sim \triangle BCD$. Find the perimeter of $\triangle BCD$.



MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2004
ROUND 6: PROB & BINOMIAL THEOREM

ANSWERS

A) _____

B) _____

C) _____

A) A fair coin is tossed ten times. What is the probability of getting exactly six heads?
Express the answer as a simplified fraction.

B) In the expansion of $(x^5 - 3)^{12}$ there is a term of the form kx^{40} . *WHERE K IS A CONSTANT.* In simplified form, find the value of k.

C) A test has eight multiple choice questions, each with four answer choices. What is the probability of answering exactly five questions correctly by random guessing?

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2004
ROUND 7: TEAM QUESTIONS: NON- CALCULATOR
ANSWERS

A) _____ D) _____

B) _____ E) _____

C) _____ F) _____

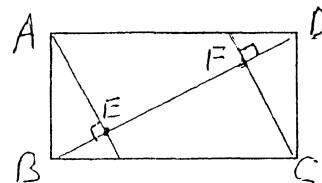
A) Solve for x:
$$\begin{vmatrix} x & 2 & 1 \\ 1 & x & 1 \\ -x & 3 & x \end{vmatrix} = \begin{vmatrix} 2x + 1 & 1 \\ x + 7 & 2 \end{vmatrix}$$

B) A box contains one dozen sugar cookies, eight chocolate chip cookies, and six oatmeal raisin cookies. If five cookies are chosen at random, what is the probability that two are chocolate chip, and three are sugar cookies?

C) Given $P(x) = x^3 + cx + d$, c and d are integers, and $P(3 + i) = 0$. Solve the equation $x^3 + cx + d = 3x^2 - 16x + 36$ for all possible roots.

D) A man drives due north at a constant rate for six hours, and then drives due east for five hours at rate that is four mph slower. He could have reached the same point travelling directly in seven hours and thirty minutes going at the original rate. What was his original rate?

E) Given rectangle ABCD, \overline{AE} and $\overline{CF} \perp \overline{BD}$, $BE = FD = 1$, and $EF = 2$. Find the area of ABCD in simplified radical form.



F) Express in simplified radical form $\sqrt{9 + \sqrt{72}} + \sqrt{14 - 8\sqrt{3}}$

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2004
ROUND 1: SIM. EQUATIONS & DET.

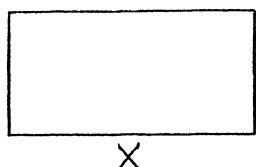
ANSWERS

A) $l = 300, w = 100$

B) 740

C) 2, -19/4

A) A rectangular cafeteria has a perimeter of 800 feet, and an area of 30,000 square feet. Find the number of feet in the length and in the width of the cafeteria.



$$x + y = 400, \quad xy = 30,000$$

$$x(400 - x) = 30,000, \quad x^2 - 400x + 30,000 = 0$$

$$(x - 100)(x - 300) = 0$$

$$x = 300, \quad y = 100$$

B) Lia is offered two different salary options to sell health club memberships. Plan A is: \$500 weekly in addition to 6% of her sales; while Plan B is: \$700 weekly in addition to 1% of her sales. Lia realized that she could get the same salary for a week under both plans. What is that salary?

$$y = .06x + 500, \quad y = .01x + 700$$

$$.06x + 500 = .01x + 700, \quad .05x = 200, \quad x = 4000$$

$$y = 740$$

C) Solve for x:
$$\begin{vmatrix} 1 & -3 & x \\ x & 4 & -1 \\ -2 & 4 & 1 \end{vmatrix} = 40$$

$$\begin{array}{ccc|ccc} 1 & -3 & x & 1 & -3 & \\ x & 4 & -1 & x & 4 & \\ -2 & 4 & 1 & -2 & 4 & \end{array}$$

$$4 - 6 + 4x^2 + 3x + 4 + 8x = 40$$

$$4x^2 + 11x + 2 = 40, \quad 4x^2 + 11x - 38 = 0$$

$$(4x + 19)(x - 2) = 0$$

$$x = 2 \text{ or } -19/4$$

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2004
ROUND 2: EXPONENTS & RADICALS

ANSWERS

A) $x = -1$

B) $2\sqrt{2}$

C) $x = -2 \text{ or } 3$

A) Find the exact value of x : $\sqrt{4 + \frac{1}{4} + \frac{4}{9}} = 2 + \frac{1}{2} + \frac{x}{3}$

$$\sqrt{\frac{144 + 9 + 16}{36}} = \sqrt{\frac{169}{36}} = \frac{13}{6} = \frac{5}{2} + \frac{x}{3}, \quad \frac{13 - 15}{6} = \frac{x}{3}$$

$$\frac{x}{3} = -\frac{2}{6} = -\frac{1}{3}, \quad x = -1$$

B) Convert to simplified radical form $\frac{\sqrt{6} - \sqrt{2}}{\sqrt{3} + 1} + \frac{2\sqrt{3}}{\sqrt{2}}$

$$\frac{(\sqrt{6} - \sqrt{2})(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)} + \sqrt{2}\sqrt{3} = \frac{3\sqrt{2} - 2\sqrt{6} + \sqrt{2} + 2\sqrt{6}}{2} =$$

$$\frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

C) Solve for x $8^{\frac{x+2}{x}} = 16^{\frac{x+2}{4}}$

I. $2 \frac{3(x+2)}{x} = 2 \frac{4(x+2)}{4}$ $3x+6 = x(x+2), \quad 3x+6 = x^2+2x,$

$$x^2 - x - 6 = 0, \quad (x+2)(x-3) = 0, \quad x = -2, 3.$$

II $x = -2$ since $8^0 = 16^0 = 1$, Then $8^{\frac{1}{x}} = 16^{\frac{1}{4}} = 2, \quad x = 3$

III. $8^{\frac{x+2}{x}} = 8^{\frac{4}{3}}, \quad \frac{x+2}{4} = \frac{x+2}{3} \quad x = -2 \text{ or } 3$

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2004
ROUND 3: POLYNOMIAL FUNCTIONS

ANSWERS

- A) -7
 B) $-2, 3/2, \frac{-1 \pm i\sqrt{3}}{2}$
 C) $-23/6$

A) Given $P(x) = 2x^3 - 5x^2 + ax + b$. Calculate the value of $a + b$ if $P(-1) = 8$, and the remainder when $P(x)$ is divided by $x - 2$ is -22 .

$$P(-1) = -2 - 5 - a + b = 8, \quad -a + b = 15$$

$$P(2) = 16 - 20 + 2a + b = -22, \quad \frac{2a + b = -18}{3a = -33, \quad a = -11}$$

$$a + b = (2a + b) - a = -18 - (-11) = -7$$

B) Solve for x : $2x^4 + 3x^3 - 3x^2 - 5x - 6 = 0$

$$\begin{array}{r|rrrrr} -2 & 2 & -1 & -1 & -3 & 0 \\ 3/2 & 2 & 2 & 2 & 0 & \end{array}$$

$$\frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

C) If r , s , and t are the roots of $3x^3 + 7x^2 - 4 = 0$, calculate the value of

$\frac{1}{r-1} + \frac{1}{s-1} + \frac{1}{t-1}$ as a reduced fraction

$$\begin{array}{r|rrrr} \textcircled{1} & 3 & 7 & 0 & -4 \\ 1 & 3 & 10 & 10 & \textcircled{6} \\ 1 & 3 & 13 & \textcircled{23} & \\ 1 & 3 & 16 & & \\ 1 & 3 & & & \end{array} \quad \text{ANS} = \frac{-23}{6}$$

$\textcircled{2}$ ORIG roots $-1, -2, \frac{2}{3}$
 roots $-1 = -2, -3, -\frac{1}{3}$
 recip. are $-\frac{1}{2}, -\frac{1}{3}, -3$
 Sum = $-\frac{3}{6} - \frac{2}{6} - \frac{18}{6} = -\frac{23}{6}$

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 4: ALGEBRA I ANYTHING

ANSWERS

A) 1002

B) 11:10 $\frac{10}{11}$

C) A = 22, B = 17, C = 54

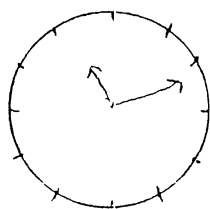
A) The difference of the squares of two consecutive positive even integers is 2004. What is the sum of these two integers?

$$x, x+2 \quad (x+2)^2 - x^2 = 2004, \quad (x+2+x)(x+2-x) = 2004$$

$$2(2x+2) = 4(x+1) = 2004, \quad x+1 = 501, \quad x = 500,$$

$$x+2 = 502, \quad \underline{\text{ANS}} \quad 1002$$

B) What is the exact time after 11 o'clock when the hands of a clock first form a ninety degree angle?



$$(x+5) - \frac{x}{12} = 15$$

$$x - \frac{x}{12} = 10$$

$$12x - x = 120$$

$$11x = 120$$

$$x = 10 \frac{10}{11}$$

ANS 11:10 $\frac{10}{11}$

C) Two persons A and B together have fifteen dollars less than a third person C. If A gives \$4 to B, then A's amount of money will be one-third of C's. If, instead, B gives \$4 to A, then A will have twice as much money as B. How many dollars does each person have?

$$A + B = C - 15$$

$$A + B - C = -15$$

$$2A - B = 27$$

$$A - 4 = \frac{1}{3}C$$

$$3A - C = 12$$

$$\underline{A - 2B = -12}$$

$$A + 4 = 2(B - 4)$$

$$A - 2B = -12$$

$$A + B = 39$$

$$3A = 66, \quad A = 22$$

$$B = 17$$

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2004
ROUND 5: GEOMETRY ANYTHING

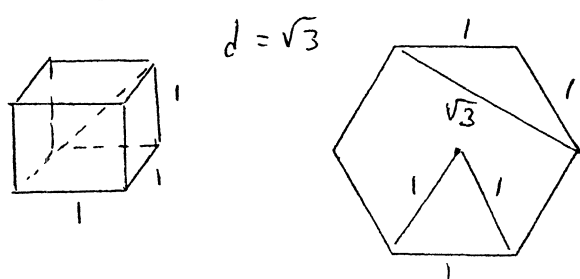
ANSWERS

A) $\frac{4\sqrt{3}}{3}$

B) $\frac{3}{4}$

C) 10

A) The length of a diagonal of a cube is the same as the length of a shorter diagonal of a regular hexagon. The ratio of the total surface area of the cube to the area of the hexagon is A/B. Compute A/B in simplified radical form.

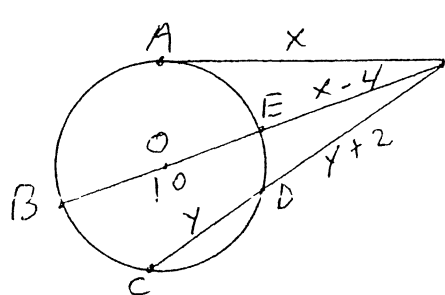


$$S.A. = 6$$

$$A = 6 \cdot \frac{1}{4} \sqrt{3}$$

$$\frac{6}{\frac{6}{4} \sqrt{3}} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

B) \overline{PA} is tangent to circle O at A. \overline{PEOB} and \overline{PDC} are secants to circle O. $AP = PE + 4$, $PD = CD + 2$. The circumference of circle O is 10π . In simplified form, find PD/PA .



$$d = 10$$

$$x^2 = (x-4)(x+6)$$

$$x^2 = x^2 + 2x - 24$$

$$2x = 24$$

$$x = 12 = PA$$

$$(y+2)(2y+2) = 144$$

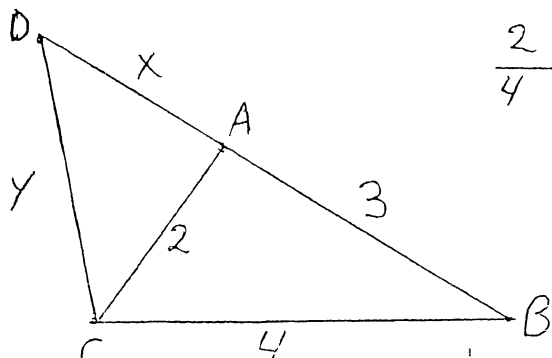
$$(y+2)(y+1) = 72$$

$$y^2 + 3y - 70 = 0$$

$$(y+10)(y-7)$$

$$PD = 9$$

C) Given $\triangle ABC$, $AC = 2$, $AB = 3$, and $BC = 4$. \overline{BA} is extended to D so that $\triangle CAD \sim \triangle BCD$. Find the perimeter of $\triangle BCD$.



$$\frac{2}{4} = \frac{x}{y} = \frac{y}{x+3}, \quad \frac{P_{\triangle CAD}}{P_{\triangle BCD}} = \frac{1}{2} =$$

$$\frac{x+y+2}{x+y+7}, \quad \text{so } 2(x+y)+4 = x+y+7$$

$$\text{and } x+y = 3, \quad P_{\triangle BCD} = 3+7 = 10$$

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2004
ROUND 6: PROB & BINOMIAL THEOREM

ANSWERS

A) $\frac{105}{512}$

B) 40095

C) $\frac{189}{8192}$

A) A fair coin is tossed ten times. What is the probability of getting exactly six heads?
Express the answer as a simplified fraction.

$$\frac{\binom{10}{6}}{2^{10}} = \frac{210}{1024} = \frac{105}{512}$$

B) In the expansion of $(x^5 - 3)^{12}$ there is a term of the form kx^{40} . In simplified form, find the value of k

$$\binom{12}{r} (x^5)^{12-r} (-3)^r \quad 50 \quad x^{60-5r} = x^{40} \text{ and } r=4$$

$$k = \binom{12}{4} (-3)^4 = 495 \cdot 81 = 40095$$

C) A test has eight multiple choice questions, each with four answer choices. What is the probability of answering exactly five questions correctly by random guessing?

$$\binom{8}{5} \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^3 = \frac{56 \cdot 27}{4^8} = \frac{14 \cdot 27}{4^7} = \frac{7 \cdot 27}{2^{13}} = \frac{189}{8192}$$

MASSACHUSETTS MATHEMATICS LEAGUE

MARCH 2004

ROUND 7: TEAM QUESTIONS - SOLUTIONS

ANSWERS

A) $x=1, 2, -4$ D) 40 mph

B) $28/299$ E) $4\sqrt{3}$

C) $x=2, 4, -3$ F) $2\sqrt{2} + \sqrt{3}$

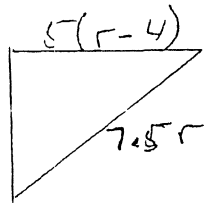
A) $\begin{vmatrix} x & 2 & 1 \\ 1 & x & 1 \\ -x & 3 & x \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \\ 2x+1 & 1 \\ x+7 & 2 \end{vmatrix} = 4x+2-x-7 = 3x-5$
 $x^3 - 2x + 3 - 2x - 3x + x^2 = x^3 + x^2 - 7x + 3$
 $x^3 + x^2 - 7x + 3 = 3x - 5, \quad x^3 + x^2 - 10x + 8 = 0$

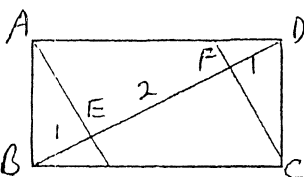
$$\begin{array}{r|rrrr} & 1 & 1 & -10 & 8 \\ 2 & 1 & 2 & -8 & 0 \\ & -4 & 1 & 0 & \end{array}$$

B) $\frac{{}^1C_3 \cdot {}^2C_2 \cdot {}^3C_0}{{}^6C_5} = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} \cdot \frac{1 \cdot 7}{2 \cdot 1} \cdot \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22} = \frac{4 \cdot 7}{13 \cdot 23} = \frac{28}{299}$

C) $3-i$ is a root, so 3rd root is -6 , Factors of $p(x)$ are
 $(x+6)(x^2-6x+10) = x^3 + 26x + 60 = 3x^2 - 16x + 36$

$$\begin{array}{r|rrrr} & 1 & -3 & -10 & 24 \\ 2 & 1 & -1 & -12 & 0 \\ & 4 & 3 & 0 & \\ -3 & 1 & 0 & & \end{array}$$

D)  Times 2, $144r^2 + 100(r^2 - 8r + 16) = 225r^2$
 $19r^2 - 800r + 1600 = 0$
 $(19r - 40)(r - 40) = 0 \quad r = 40 \text{ only}$

E)  $\frac{BE}{AB} = \frac{AB}{BD}, \quad \frac{1}{AB} = \frac{AB}{4}, \quad AB = 2$
 $\frac{AE}{AD} = \frac{AD}{BD}, \quad \frac{3}{AD} = \frac{AD}{4}, \quad AD^2 = 12, \quad AD = 2\sqrt{3}$
 Area = $4\sqrt{3}$

F) $\sqrt{3} \sqrt{3+2\sqrt{2}} + \sqrt{2} \sqrt{7-4\sqrt{3}} = \sqrt{3}(1+\sqrt{2}) + \sqrt{2}(2-\sqrt{3}) =$
 $\sqrt{3} + \sqrt{6} + 2\sqrt{2} - \sqrt{6} = 2\sqrt{2} + \sqrt{3}$