MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2004
ROUND 1 TRIG: RT ANGLE, LAWS SINES & COSINES

ANSWERS

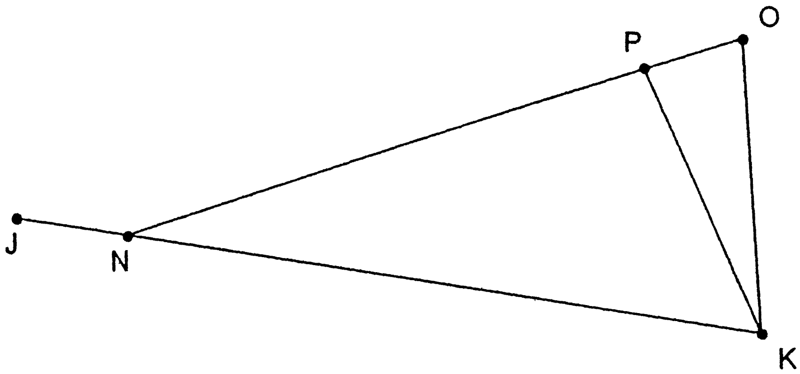
A)______________

B)______________

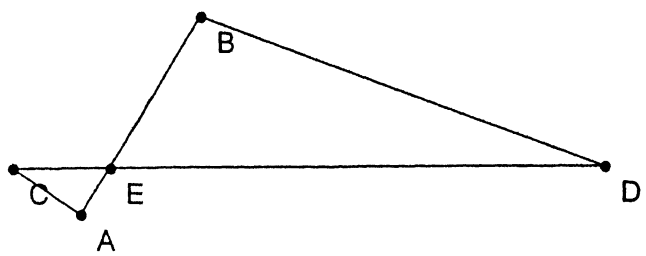
C)______________

A) If sec(x)=2.2 and tan(x)< cos(x) – 2, find the exact value of csc(x) in simplified radical form

B) If ΔNOK is isosceles with NO = NK = 18, OP = 2, and cos($\angle$ JNO)= -0.75 find PK in simplified radical form.



C) Given DB = 91, AC = 7, EC =8, $\angle$D = 30°, and $\overline{AC} \perp \overline{AB}$ find the exact length of $\overline{AB}$.



MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2004
ROUND 2 ELEMENTARY NUMBER THEORY

ANSWERS

A)_______________

B)_______________

C)_______________

A) Three men who are no longer teenagers find the product of their current ages is 26,390. Find the sum of their current ages.

B) How many positive integers less than 500 each have exactly three different positive integer divisors?

C) Find all primes of the form $8n^3 - 2197$

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2004
ROUND 3 ANALYTIC GEOM OF LINE

ANSWERS

A)______________

B)______________

C)______________

A) $\overline{AB}$ has endpoints A(1,0) and B(61, 45). Find the coordinates of the points on the segment that trisect it.

B) Let $y = 2x + b$ represent a line with x- and y-intercepts at P and Q respectively. Let O represent the origin. Find all possible values for b so that the area of ΔPQO is 100 square units.

C) If $k>0$, find the sum of the coordinates of the point on $x - 2y + k = 0$ closest to the origin. Express your answer in terms of k.

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2004
ROUND 4 ALG 2: LOGS & EXPONENTIAL FUNCTIONS

******* NO CALCULATORS ON THIS ROUND ******

ANSWERS

A)______________

B)______________

C) ______________

A) Find all real solutions for: $\log_2 x + \log_2 (x - 6) = 4$

B) Solve for x: $8^{\frac{2\log_4 3}{3}} - e^{\ln 5} = x^2 - 7^{(\log_7 3 + \log_7 x)}$

C) The graph of the exponential function $f(x) = a\,b^x$ passes through the points $(1, \frac{1}{2})$ and $(3, \frac{3}{8})$. Find the exact value of a+b

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2004
ROUND 5 ALG 1: RATIO PROPORTION VARIATION

ANSWERS

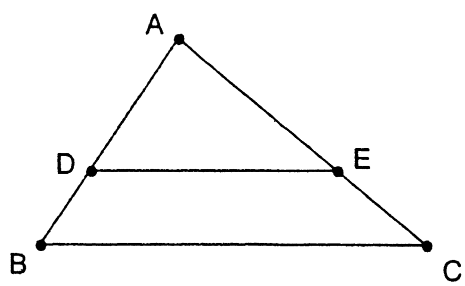
A)______________**lbs**

B)______________

C)______________**pm**

A) The safe load limit for a beam varies jointly as the thickness and the square of the depth of the beam and inversely as the length between the supports. The safe load limit is 2500 lbs for a beam 18 feet long, 6 inches thick, and 15 inches deep. What is the safe load limit for a beam of the same material but half as long, half as thick, and half as deep?

B) If the area of ΔABC is exactly 2.5 times the area of trapezoid DECB and AB=82.175 find AD rounded to the nearest thousandth.



C) It would take Sue 6 hours to rake her entire yard by herself. She began at 11:00 a.m. At noon Rana joined her and they raked together for an hour. Sue then finished the job herself, ending at 4:24 p.m. When would they have finished if Rana had stayed and they worked until the job was done?

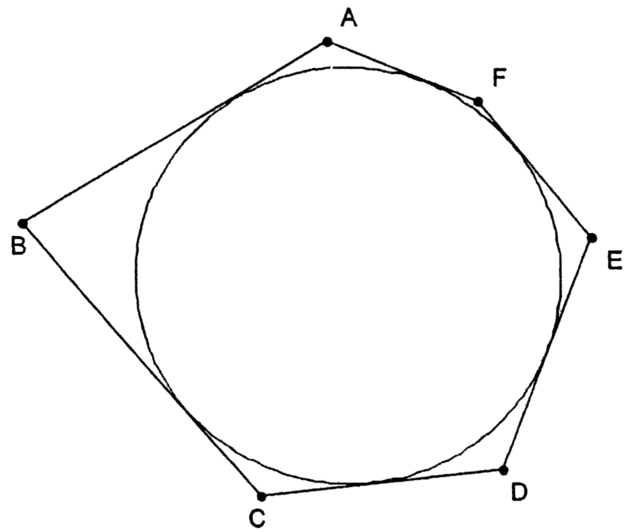
MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2004
ROUND 6 GEOMETRY: POLYGONS
ANSWERS

A)______________

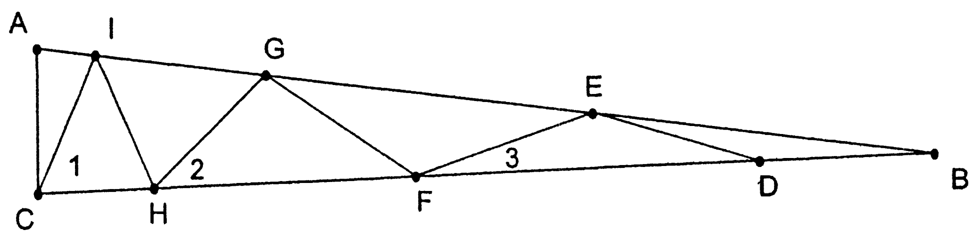
B)______________

C)______________

A) Given the circumscribed hexagon ABCDEF with AF=2, EF=5, ED=8, and BC=10 find AB + CD



B) ΔABC is isosceles with AC=CI=IH=HG=GF=FE=ED=BD. Find $\angle 1 + \angle 2 + \angle 3$



C) The number of diagonals in a regular polygon is exactly 20.9 times the measure of an interior angle divided by the measure of an exterior angle of the polygon. How many sides does the polygon have?

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2004
ROUND 7: TEAM QUESTIONS

ANSWERS

A)____________ **D)**____________

B)____________ **E)**__________**p.m.**

C)____________ **F)**____________

A) Find the perimeter of a regular dodecagon (a 12-gon) whose area is $96 + 48\sqrt{3}$.

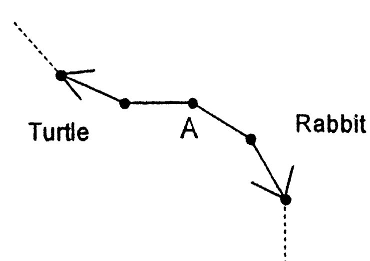
B) If $907_{2k-1} = 709_{2k+1}$ find the value of 3724_k expressed in base 10.

C) A rectangle has vertices $(0,0)$, $(5,0)$, $(5,3)$, and $(0,3)$ A line through $(1,b)$ in the interior of the rectangle divides the rectangle into two regions of equal area. Find the slope of the line in terms of b.

D) Find all exact values of x for which $(\log_s 5)(\log_x 3) + 3\log_5 x = \log_{\sqrt{5}} 5 + \log_5 25$

E) Two candles are the same length but burn at different rates. If the first were lighted at 7 a.m. and the second at 10 a.m. both would be burn out at 7 p.m. Instead both were lighted at noon. At what time will one candle be 2/3 the length of the other?

F) A turtle starts at point A <u>facing west</u> and runs at 1 ft/min repeating this plan: run 10 ft, turn right 24°. A rabbit starts at point A <u>facing east</u> and runs at 11 ft/min repeating this plan: turn right 30°, run 10 ft. After starting they <u>first</u> meet at a point B after k minutes; they then meet often but first meet back <u>at point A</u> again after m minutes. Find $k + m$.



MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2004 ANSWERS

Round 1:	A) $\frac{-11\sqrt{6}}{24}$	B) $2\sqrt{37}$	C) $52+\sqrt{15}$
Round 2	A) 90	B) 8	C) 547
Round 3	A) (21,15) and (41, 30)	B) ± 20	C) $\frac{k}{5}$
Round 4	A) 8	B) 1, 2	C) $\frac{5\sqrt{3}}{6}$
Round 5	A) 625 lbs	B) 63.652	C) 3:07:30 pm
Round 6	A) 15	B) 144	C) 22
Team Round	A) 48	B) 2004	C) (3–2b)/3 or 1 – (2/3) b
	D) 5, $\sqrt[3]{5}$	E) 6:00 p.m.	F) 6610

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2004 BRIEF SOLUTIONS

Round One:

A. Since tan(x)<0 we are in fourth quadrant with right triangle having sides of 11,5, and $4\sqrt{6}$ so csc(x) = $\dfrac{-11\sqrt{6}}{24}$.

B. Law of Cosines: $PK^2 = 16^2 + 18^2 - 2(16)(18)(0.75) = 148$ so PK=$2\sqrt{37}$

C. Pythagorus gives AE=$\sqrt{15}$ Law of Sines gives BE= sin $\angle$ D (BD) / sin $\angle$ BED. Since sin $\angle$ CEA = 7/8, BE = 0.5 (91)(8/7) = 52

Round Two:

A. 26,390=2x5x7x13x29 so 2x13 + 5x7 + 29 = 90.

B. The desired numbers must be the squares of primes so we have 2^2=4 up to 19^2=361 or 8 such numbers

C. $8n^3 - 2197 = (2n)^3 - 13^3 = (2n - 13)(4n^2 + 26n + 169)$ which is prime only if one of the factors is one. Thus n=7 and the number is 547.

Round Three:

A. (61-1)/3=20; (45-0)/3=15 Find (1+n(20) ,0+ n(15)) for n=1 and 2.

B. OQ = |b|; OP = |–b/2| ; so |b| times |–b/2| = 200, thus $b^2 = 400$

C. Perpendicular through origin is y = –2x; system solves to $\left(\dfrac{-k}{5},\dfrac{2k}{5}\right)$

Round Four:

A. x (x – 6) = 16 so x = 8 or x = –2 Since –2 has no log the solution is x = **8**.

B. Use log properties simplify to $3 - 5 = x^2 - 3x$ to $0 = (x - 1)(x - 2)$

C. Divide $3/8 = a b^3$ by ½ = a b to get ¾=b^2 so b=$\sqrt{3}/2$, a=$\sqrt{3}/3$

Round Five:

A. Scale by 0.5 for thickness, $(0.5)^2$ for depth, 2.0 for length. Net scaling 0.25

B. If area DECB=x, area ABC=2.5x and area ADE=1.5x so similar triangles have ratio $\sqrt{0.6}$ and AD=$\left(\sqrt{0.6}\right)$82.175 = 63.6524... ≈ 63.652

C. 1/6 + (1/6 + 1/x) + 204/60 (1/6) =1 so x =10. 5/6=T(1/6 + 1/10) so T=25/8

Round Six:

A. Number sides consecutively. Sum of even sides = sum of odd sides so 2 + 8 + 10 = 5 + AB + CD thus AB + CD = 15

B. If $\angle$B=x, $\angle$EDF=2x, $\angle$GEF=3x etc $\angle$A=7x so sumΔABC gives x=12. Since $\angle$1=6x, $\angle$2=4x, and $\angle$3=2x, their sum is 144.

C. Ratio of angles is (180–360/n) / (360/n) simplifies to (n–2)/n. Solving (n/2)(n–3) = 20.9 (n–2)/n so $10n^2 - 30n = 209n - 418$ so n = 22 (or 1.9)

Team Round:

A. Decompose into 12 isos triangles with vertex 30° base w leg r. Each has area of $0.5\, r^2 \sin 30 = r^2/4 = (96 + 48\sqrt{3})/12$ so $r^2 = 32 + 16\sqrt{3}$ Law Cosines gives $w^2 = r^2 + r^2 - 2\, r\, r \cos 30° = 2(32 + 16\sqrt{3}) - 2\,(32 + 16\sqrt{3})(\sqrt{3}/2) = 16$ so $w=4$ and $12w = 48$.

B. If $n = 2k-1$ then $9n^2+7=7(n+2)^2+9$ so $2n^2 - 28n - 30 = 0$ so $n=15$ or -1 and $k=8$ (or $k=0$, impossible) so $3724_k = 2004$.

C. By symmetry the line must also pass through $(4, 3-b)$ so its slope is $(3-2b)/3$

D. If $\log_x 5 + 3\log_5 x = 2 + 2$ then if $\log_5 x = A$ we have $1/A + 3A = 4$ or $1 + 3A^2 = 4A$ yielding $A=1$ or $1/3$ so $x = 5$ or $\sqrt[3]{5}$.

E. $(1 - t/9) = 2/3\,(1 - t/12)$ gives $t = 6$ so at 6:00 p.m.

F. Turtle travels a 15-gon of side 10; rabbit a 12-gon of side 10. Rabbit covers 11 sides in 10 minutes while turtle covers one side in 10 minutes. They meet at B in k =10 minutes. Turtle hits A every 150 minutes; rabbit every 120/11 minutes. First common multiple is 11(5)(4)(30) so m = 6600 minutes.