MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2005
ROUND 1 ANALYTIC GEOMETRY ANYTHING

ANSWERS

A)_______________

B)_______________

C)_______________

A) The points (3,1) , (1, 3) , and (–2, 1) are three of the vertices of a parallelogram. Give all possible coordinates for the fourth vertex.

B) Given graphs of $x^2 + y^2 = 20$ and $(x - 5)^2 + (y - 5)^2 = 10$, find the coordinates of all points of intersection.

C) The second degree equation $6x^2 - 5xy - 6y^2 - 29x - 2y + 28 = 0$ represents a pair of perpendicular lines which intersect at P. Find the coordinates of P.

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 2 ALGEBRA ONE: FACTORING & EQUATIONS

ANSWERS

A)______________ft

B)______________

C)______________

A) A rectangular playground of area 560 square feet is built on a vacant lot 32 feet wide by 40 feet long. The playground is placed an equal distance from all four sides of the lot. Find the perimeter of the playground.

B) Find all real values of x for which: $2x = \dfrac{2 - x - x^2}{2 + x}$

C) Find all real values of x (no approximations!) for which $x^3 + 1 = x^2 + 4x + 3$

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 3 TRIG: EQUATIONS WITH FEW SOLUTIONS

******* NO CALCULATORS ON THIS ROUND ******

ANSWERS

A)_______________

B)_______________ °

C)_______________

A) Solve for $0 \le x < \pi$: $0 = (4\cos^2 x - 1)(4\cos^2 x - 3)(4\cos^2 x - 5)(4\cos^2 x - 7)(4\cos^2 x - 9)$

B) Given $\cot^3 z - \cot^2 z - \cot z = 2\cot z - 3$, $0° \le z < 360°$, find the <u>average</u> of all solutions for z .

C) Find the <u>exact</u> sum of all solutions θ , $0 \le \theta < 2\pi$ for

$$(\sin\theta - 0.35)(\tan\theta - 0.80)(\tan\theta - 1.25)(\sec\theta - 1.70) = 0$$

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2005
ROUND 4 ALGEBRA TWO: QUADRATIC EQUATIONS

ANSWERS

A)______________

B)(____,_____,____)

C)______________

A) Find all real x for which $x^2\sqrt{6} - 4x - 2\sqrt{6} = 0$

B) For integers a, b, and c the solutions for $ax^2 = 2x + b$ are the same as those for $(x + 0.5)(cx - 3) = 0$. Find the ordered triple (a, b, c).

C) If $N = 1 - \cfrac{1}{4 - \cfrac{1}{4 - \cfrac{1}{4 - \ldots}}}$ then N may be expressed in the form $A + \sqrt{B}$ Find A+B.

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 5 GEOMETRY: SIMILAR POLYGONS

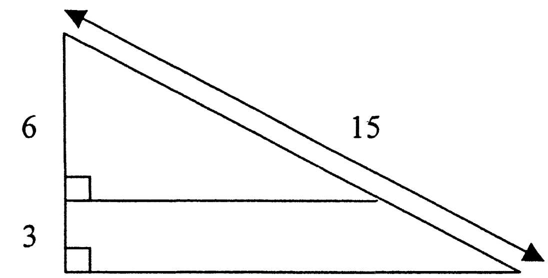
ANSWERS

A)________ sq units

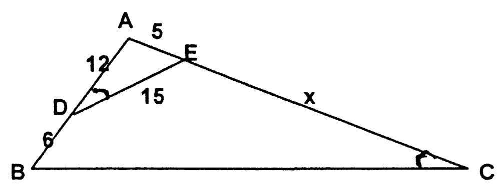
B)______________

C)______________

A) In the figure what is the area of the trapezoid?



B) If m$\angle$ACE = m$\angle$ADE and EC = x, express the exact value of x as a decimal.



C) The ratio of the perimeters of 2 regular hexagons is 4:3 . If the smaller diagonal of the smaller hexagon has length $4\sqrt{3}$ find the sum of the areas of the two hexagons in the simplified radical form $\dfrac{A\sqrt{B}}{C}$.

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2005
ROUND 6 ALGEBRA ONE: ANYTHING

ANSWERS

A)_______________

B)_______________

C)_______________

A) If $x = z^2 + z$ and $y = z^3 + z^2$ and $x = 2$, find all possible values for y.

B) If the line passing through (2,c) and (4, 5c) has slope –0.5 find the line's equation in y=mx+b form.

C) Sue and her Uncle Joe share the same birthday of January 1. Today (1/6/2005) Joe noticed that three years ago he was three times Sue's age but in 1996 he was <u>seven</u> times Sue's age! In what year will Sue be half Joe's age?

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2005
ROUND 7: TEAM QUESTIONS

ANSWERS

A)_____________ **D)**_____________

B)_____________ **E)**_____________

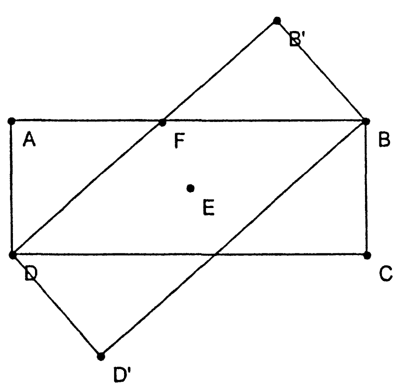
C)_____________ **F)**_____________

A) Given ΔABC with A(1, 2) B(4, 6) and C(–8, 11) If the angle bisector of $\angle$ ABC intersects $\overline{AC}$ at D find the coordinates of D.

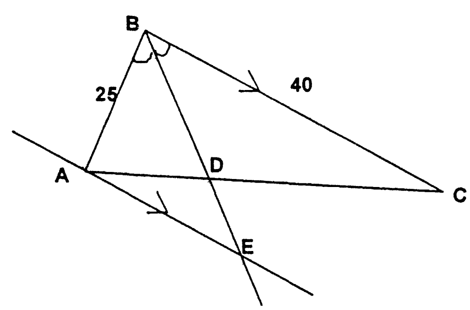
B) Factor : $x^{4n+3} + 4x^{2n+3} + 16x^{3}$

C) A student solved $cos(x) + sin(2x) - cos(3x) = 0$ by transforming it to the form $(n \cdot foo(x) + 1) \cdot foo(nx) = 0$ for some positive integer n and some basic trig function *foo*. Find the exact value of $n + foo(\frac{\pi}{3n})$.

D) Rectangle ABCD has AD=6 and AB>AD. Let E be the point of intersection of its diagonals. Rotate ABCD about E until A lands on D. If the area contained in the intersection of the rotated and original rectangular regions is 45 square units, what was the length of $\overline{AB}$?



E) In triangle ABC, AB = 25, BC = 40 and AC = 39. Ray BD bisects angle ABC and line AE is parallel to side BC. Find the exact perimeter of triangle ADE



F) Find all ordered pairs (a,b) which will make the solution to the following equation a set of three consecutive positive integers: $x^3 - 8 = (x - 2)(ax + b)$

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2005 ANSWERS

Round 1:	A) $(-4, 3)$ $(6, 3)$ $(0, -1)$ (in any order)	B) $(2,4)$ $(4,2)$	C) $(2, -1)$
Round 2	A) 96 ft.	B) 1/3	C) -1, $1 \pm \sqrt{3}$
Round 3	A) $\pi/6$, $\pi/3$, $2\pi/3$, $5\pi/6$	B) 165	C) 6π
Round 4	A) $\sqrt{6}$ or $\frac{-\sqrt{6}}{3}$	B) $(8, 3, 4)$	C) 2
Round 5	A) 30	B) 38.2	C $\frac{200\sqrt{3}}{3}$
Round 6	A) $2, -4$	B) $y = -0.5x + 0.75$	C) 2011
Team Round	A) $(-1.5, 4.5)$	B) $x^3(x^{2n} + 2x^n + 4)(x^{2n} - 2x^n + 4)$	C) 2.5
	D) 12	E) $40 + 5\sqrt{10}$	F) $(9, -8)$ and $(6, 1)$

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2005 BRIEF SOLUTIONS

Round One:

A. Given points A, B, C consider ABC?, ACB? and CAB? Thus covering all of the six permutations

B. Expanding #2 and subtracting #1 gives 10x+10y=60. Substituting into #1 for y gives a quadratic in x, x=2 or 4. Note #2 requires first quad solutions.

C. Factoring the first three terms gets us to (3x+2y+?)(2x–3y+?) whereas factoring the terms $6x^2 - 29x + 28$ gives (3x – 4)(2x – 7) so we see the complete factoring is (3x + 2y – 4)(2x – 3y – 7)so the intersection is **(2, –1)**

Round Two:

A. (32-2x)(40-2x)=560 becomes $x^2 - 36x + 180=0$. x=30 or 6. Nearest edge is 6. Playground is 20 X 28 so perimeter is 96 ft.

B. $2x = \dfrac{2-x-x^2}{2+x}$ becomes $2x = \dfrac{(2+x)(1-x)}{2+x}$ Since $x \neq -2$, $2x = 1 - x$ so x=1/3

C. Factoring each side: $(x+1)(x^2 - x + 1)=(x+1)(x+3)$ so x = – 1 or $(x^2 - x + 1)=(x+3)$ which by quad formula gives $x = 1 \pm \sqrt{3}$

Round Three:

A. Only first two factors have zeroes so x = π/6, π/3 , 2π/3, 5π/6

B. Factors to $(\cot z - 1)(\cot^2 z - 3)=0$ so z = 45, 225 or z = 30, 150, 210, 330 sum is 990, average 165.

C. (sin θ = 0.35) has solutions that are supplementary so sum to π; (tan θ = 0.80) means (cot θ = 1.25) and with (tan θ = 1.25) gives complementary first quadrant solutions and third quad solutions for sum of π/2 + (2π + π/2) and (sec θ = 1.70) gives solutions summing to 2π so final sum is 6π.

Round Four:

A. Use the quadratic formula or factor as $(\sqrt{6}x + 2)(x - \sqrt{6}) = 0$ so $x = \dfrac{-2}{\sqrt{6}} = \dfrac{-\sqrt{6}}{3}$ or $x = \sqrt{6}$

B. For integer coefficients use $(2x+1)(cx-3) = 2cx^2 + (c-6)x - 3 = ax^2 - 2x - b$ so b=3 and c–6 = –2 so c = 4 and a = 2c so a = 8.

C. View the expression as 1 – x and notice $x = \dfrac{1}{4-x}$ so $x^2 - 4x + 1 = 0$ and $x = 2 + \sqrt{3}$ so $1 - x = -1 + \sqrt{3}$ and the answer is **2**

Round Five:

A. The larger triangle is 9-12-15 so its area is 54. The smaller triangle is scaled by 2/3 so it's area is 4/9 the larger triangle; thus the trapezoid is 5/9 of 54 or **30**

B. By AA, ΔADE ~ ΔACB. Note carefully the order of the vertices. Thus, 5/(12 + 6) = 12/(x + 5) → x = 38.2

C. If the smaller hexagon has area A the larger has area 16/9 A. so sum is 25/9A. In the smaller hexagon the altitude of one of the 6 equilateral triangles is $2\sqrt{3}$ so the triangle's area is $4\sqrt{3}$ and the hexagon's area = $24\sqrt{3}$ so sum is $\dfrac{200\sqrt{3}}{3}$

Round Six:

A. $2 = z^2 + z$ has solutions of $z = 1$ and $z = -2$. Since $y = z(x) = 2z$, $y =$ 2 or -4

B. Slope $4c/2= -0.5$ so $c= -0.25$. Translation shows $(0, -3c)$ is also a point so y-intercept is 0.75.

C. Ages in 1996 were s and 7s; three years ago (2002) ages were s+6 and 7s+6. If $3(s+6) = 7s+6$ then $s = 3$ and Joe is 18 years older. Sue turns 18 in 2011.

Team Round:

A. The angle bisector divides AC into the ratio BC:BA or 13:5 so $x = \dfrac{(1)(13)+(-8)(5)}{5+13} = -1.5$ and $y = \dfrac{(2)(13)+(11)(5)}{5+13} = 4.5$

B. $x^3(x^{4n} + 4x^{2n} + 16) = x^3(x^{4n} + 8x^{2n} + 16 - 4x^{2n}) = x^3[(x^{2n}+4)^2 - (2x^n)^2] = x^3(x^{2n} + 2x^n + 4)(x^{2n} - 2x^n + 4)$

C. Since $\cos(a - b) - \cos(a + b) = 2\sin(a)\sin(b)$, $\cos x - \cos 3x + \sin 2x = \cos(2x-x) - \cos(2x+x) + \sin 2x = 2\sin 2x \sin x + \sin 2x = (2\sin x + 1)\cdot\sin 2x$ so $n = 2$ and *foo* is sin. Thus $n + foo(\pi/3n) = 2 + \sin(\pi/6) = 2.5$

D. If AD=6 and AB=x and FB=y then $6^2 + (x - y)^2 = y^2$ so $y = \dfrac{36+x^2}{2x}$ and the area of the triangles not in the shared region is $6(x-y)= 6x - \dfrac{108+3x^2}{x}$ which is also $6x - 45$ so $\dfrac{108+3x^2}{x} = 45$ so $3x^2 - 45x + 108 = 0$ and x=3 or 12. Since x>6, the answer is **12**

E. Let AD = x By angle bisector thm, $x/25 = 39\text{-}x/40 \rightarrow x = 15$. By Stewart's theorem*, $25^2\bullet24 + 40^2\bullet15 = BD^2\bullet39 + 39\bullet15\bullet24 \rightarrow BD^2 = 640$ so $BD = 8\sqrt{10}$ (or drop altitude to AC solve system to get h=24 and altitude divides the 15 into 7 and 8 making BD hypotenuse w. legs 24, 8). Triangle BAE is isos. so AE= 25 By AA, $\Delta ADE \sim \Delta CDB$ w ratio 5:8 so $DE = 5\sqrt{10}$ Thus, per=$15+25+5\sqrt{10}$

F. One solution is x =2; divide by $(x - 2)$ to get $x^2 + 2x + 4 = ax + b$. Solutions must be either 1&3 $(x^2 - 4x + 3) = 0$ so a=6,b=1 or 3&4$(x^2 - 7x + 12) = 0$ so a=9,b=−8

*Stewart's Theorem (Matthew Stewart, 1717-1785 Scottish Mathematician): In any triangle with sides a, b, c and any segment m from C dividing c into x and y, $a^2x+b^2y = m^2c+cxy$