MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2005
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

ANSWERS

A)_______________

B)_______________

C)_______________

A) If $f(x) = 4x + 6$ and $f^{-1}(z) = z$ find z.

B) If $f(x) = 1 + f(x - 10)$ for all real x, and if $f(5) = 2$ find $f(2005)$.

C) Given $h(x) = x^3 + 3x + 4.5$, $g(x) = \frac{1}{x} + 3$, and $f(x) = \frac{x+3}{2}$ evaluate
$g \circ f \circ h \circ h^{-1} \circ g^{-1} \circ f^{-1}(2)$

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2005
ROUND 2 ELEMENTARY NUMBER THEORY

ANSWERS

A)_______________

B)_______________

C)_{___ ______}_

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positive

A) How many more factors are there for 5292 than for 520 ?

B) How many positive integers less than 200 have exactly three distinct (unrepeated) prime factors?

C) n = 1,111,200,311,112,004,111,1ab is a 22 digit number in base ten whose right-most digits are a and b. If n is divisible by 36 find the set of all possible values of the product of a and b.

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2005
ROUND 3 TRIG: IDENTITIES & INVERSE FUNCTIONS

ANSWERS

A)_______________

B)_______________

C)_______________

A) If $\sec^2(x) + \tan(x) = 1$ and $\tan(x) + \csc(x) = y$ find all exact real values for y.

B) Given $Sin(Sin^{-1}(2x+1)) = \dfrac{5}{27x+3}$ find exact values for all possible x.

C) Given $Cos^{-1}(Cos(2x+1)) = \dfrac{5}{x+2}$ find all possible <u>rational</u> values for x.

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2005
ROUND 4 ALGEBRA ONE: WORD PROBLEMS

ANSWERS

A)_________gallons

B)______________

C)__________hour

A) How many gallons of pure water must be added to a gallon of alcohol 75% pure to make a mixture only 15% pure?

B) I am half as old as my mother was when my brother was twelve years younger than I am now. My brother was born when my mother was 26. If the sum of my brother's and my own current ages is 36, how old was my mother when I was born?

C) I paddled my canoe upstream for 6 hours. I then rested for an hour as my canoe drifted downstream, then paddled back downstream to my starting point in just 2 hours. How much earlier would I have gotten back if I had headed back immediately rather than resting for that hour? Express your answer as a reduced fraction of an hour.

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2005
ROUND 5 GEOMETRY: CIRCLES

******* NO CALCULATORS ON THIS ROUND ******

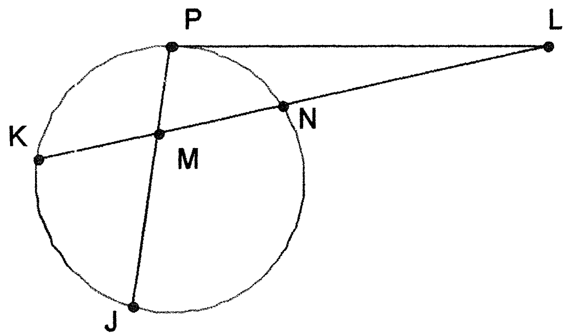
ANSWERS

A)______________

B)______________

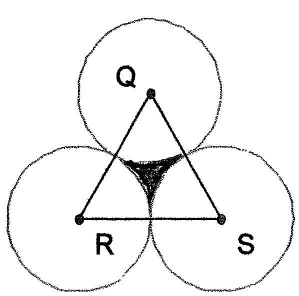
C)______________

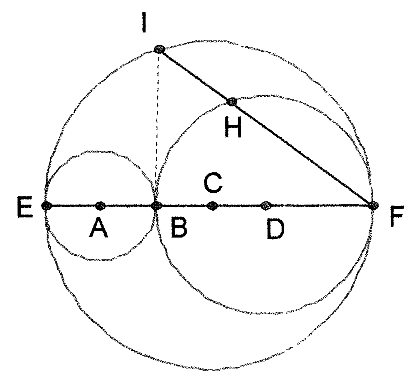
A) Given MJ=9, LN=13, M the midpoint of $\overline{KN}$, and PM=4 find the exact length of the tangent $\overline{PL}$



B) Circles Q, R, and S (as shown on the left below) are externally tangent and each has a radius of 3. Find the exact area of the total shaded region.

C) On the right below circles centered at A, C, and D are mutually tangent at E, B, and F. The largest circle has radius 12 and the smallest has radius 4. If $\overline{IB}$ is a tangent to the smaller circles find HF in simplified radical form.





MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2005
ROUND 6 ALGEBRA 2: SEQUENCES & SERIES

ANSWERS

A)_______________

B)_______________

C)_______________

A) Find the 2005[th] term of an arithmetic sequence whose third term is –2000 and whose fifth term is –1996.

B) For an arithmetic sequence a and a geometric sequence g, $a_9 = g_1$ while $a_{81} = g_3$. If $a_0 = 0$ and $g_1 = 6$ find all possible values for $a_2 + g_2$ as improper fractions.

C) At the beginning of each year Shauna adds \$100 to her bank account; at the end of each year the bank adds 8% interest to the account. At the beginning of every month Will adds \$20 to a shoe box in his closet. If each began with no money when they made their first deposits Jan 1 1990, who had the greater amount after interest was paid to Shauna at the end of 2004- and how much more did they have rounded to the nearest dollar?

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2005
ROUND 7: TEAM QUESTIONS

ANSWERS

A)____________ **D)**____________

B)____________ **E)**____________

C)____________ **F)**____________

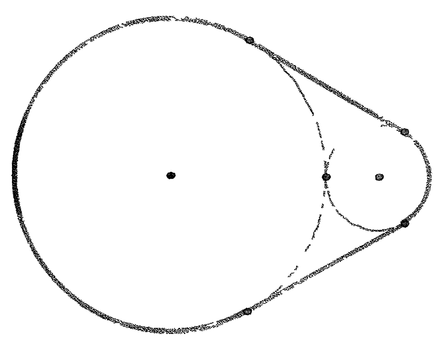
A) Suppose $f(x) = \dfrac{ax+7}{x-1}$ for some constant a, and $g(x) = \dfrac{x+1}{7}$. If one solution for $f^{-1}(x) = g^{-1}(x)$ is $x = 3$ find the exact value of the other solution.

B) Let T = 1 + 3 + 5 + . . + (2k – 1) for some k between 1 and 99 inclusive
Let S = 2 + 4 + 6 + ... + (2j) for some j between 1 and 99 inclusive
If k and j are distinct integers find the maximum value of k+j for which S–T is a multiple of 25.

C) If $Tan^{-1}(A) + Tan^{-1}(\frac{4}{A}) = Tan(\frac{-5}{3}) = n\pi$ for the function Tan^{-1}, real A, and integer n. find all possible ordered pairs (A, n).

D) On her road trip Sue drove part at 60 mph, part at 45 mph, and the rest at 36 mph. She spent twice as much time at 45 mph as at 60 mph but drove twice as far at 36 mph as she did at 60 mph. Her average speed for the trip was A/B mph where A and B are relatively prime. Find A+B.

E) A tight band is wrapped around two externally tangent circles as shown below. If the smaller circle's radius is one third that of the larger circle and the length of the band is exactly $36 + 14\pi\sqrt{3}$ find the area of the larger circle.



F) A sequence S has the unusual property that any four consecutive terms of the form $s_{2n}, s_{2n+1}, s_{2n+2}, s_{2n+3}$ form an arithmetic progression when n is even but form a geometric progression when n is odd. Although $s_2 = s_4 = -2$, S is <u>not</u> a constant sequence. Find in simplest form $10\,s_0 + s_{14}$

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2005 ANSWERS

Round 1:	A) -2	B) 202	C) 3.8
Round 2	A) 20	B) 19	C) $\{0, 42\}$
Round 3	A) $-1 \pm \sqrt{2}$	B) $-\frac{2}{3}$	C) $\frac{1}{2}$
Round 4	A) 4	B) 34	C) 9/13
Round 5	A) $5\sqrt{13}$	B) $\frac{45}{2}\pi + 9\sqrt{3}$	C) $\frac{16\sqrt{6}}{3}$
Round 6	A) 2004	B) 50/3, 58/3	C) Will \$668
Team Round	A) $-3/14$	B) 194	C) (1,1) (4,1)
	D) 829	E) 243π	F) $\frac{50}{49}$

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2005 BRIEF SOLUTIONS

Round One:

A. If $f^{-1}(z) = z$ then $f(z) = z$ too so $z = 4z + 6$ thus $z = -2$.

B. $f(2005) = 1 + f(1995) = \ldots = 200 + f(5) = 202$

C. $f^{-1}(x) = 2x - 3$; $g^{-1}(x) = 1/x + 3$; $f^{-1}(2) = 1$; $g^{-1}(1) = -0.5$; h cancels h^{-1}; $f(-0.5) = 1.25$; $g(1.25) = 3.8$

Round Two:

A. $520 = 2^3 x5x13$ so 4x2x2=16 factors. $5292 = 2^2 x3^3 x7^2$ so 3x4x3=36 factors, **20** more.

B. An organized list gives 2x3x5, 2x3x7, … 2x3x31 (9 values); then 2x5x7, …, 2x5x19 (5 values); then 2x7x11, 2x7x13 (2 values); 3x5x7,..3x5x13(3 values); for **19** values with exactly 3 distinct prime factors. Note that without the distinct requirement we would have additional values such as 2x2x3x5

C. Divisibility by 9 requires 23+a+b be a multiple of 9 or 5+a+b=9 or 18 so a+b=4 or 13. Divisibility by 4 requires ab be a multiple of 4. Only possibilities are 04, 40, and 76 so products are 0 or 42

Round Three:

A. $(\tan^{2(}x) + 1) + \tan(x) = 1 \Rightarrow \tan(x) = 0$ (but then csc(x) undefined) OR $\tan(x) = -1$ so $\csc(x) = \pm\sqrt{2}$

B. $2x + 1 = \dfrac{5}{27x + 3}$ so $54x^2 + 33x - 2 = 0$ so $x = -2/3$ or $1/18$. Since $2x + 1$ must be in the domain of the Sin^{-1} function, only $-2/3$ is valid.

C. $\dfrac{5}{x+2} = \pm(2x+1) + 2n\pi$ but x must be rational so n=0. $\dfrac{5}{x+2} = -(2x+1)$ yields no real solutions. $\dfrac{5}{x+2} = 2x + 1$ gives $5 = 2x^2 + 5x + 2$ so $x = ½$ or -3 making $\dfrac{5}{x+2} = 2$ or -5. Only the first is in the range of Cos^{-1} so the only solution is ½

Round Four:

A. The ratio of alcohol is $0.15 = \dfrac{0.75}{1+n}$ solving yields n = 4.

B. I am x now. "Then" my mother was 2x and my brother x-12 so $2x = x - 12 + 26$ so x=14 any my brother is 22. My mother was 8 yrs older when I was born or 34.

C. Let r= paddling speed in still water, s = speed of current. $6(r-s) = s + 2(r+s)$ so $s=(4/9)r$ and upstream rate is $(5/9)r$, downstream $(13/9)r$. Immediate return would have taken $\dfrac{Dist}{time} = \dfrac{6(r-s)}{r+s} = \dfrac{6(5/9)r}{(13/9)r} = \dfrac{30}{13}$ instead of the three hrs it took (1 resting)

Round Five:

A. Power of pt M = 9(4)=36=MK(MN) so MK=MN=6. Power of pt L= LK(LN) = 25(13) = LP^2 .

B. Equil. triangle of side 6 has area $9\sqrt{3}$ plus three 300° sectors = 3(5/6)9π

C. IB is alt to hyp of rt triangle EIF so IF is geom.. mean of FB & FE = $8\sqrt{6}$. HF:IF = DF:CF = 2:3 so HF = (2/3) $8\sqrt{6}$.

Round Six:

A. Constant difference is 4/2=2; $a_3 = a_0 + 3(2)$ so $a_0 = -2006$ and therefore $a_{2005} = -2006 + 2005(2)$.

B. $81d = g_0 r^3$ while $9d = g_0 r$; dividing gives $r = \pm 3$. If r = 3, $g_n = 2 \cdot 3^n$ and since $a_9 = 6$, $a_n = \frac{2}{3}n$ so $a_2 + g_2 = \frac{58}{3}$. If r = –3, $a_n = \frac{-2}{3}n$ and $a_2 + g_2 = \frac{50}{3}$.

C. Will accumulates \$20x12x15=\$3,600. Shauna has the geometric sum 100*1.08 + 100*(1.08)^2 + 100*(1.08)^{15} = 100*1.08*(1.08^{15} – 1)/ 0.08 ≈ \$2,932.43.

Team Round:

A. $g^{-1}(3) = 20$. If $f^{-1}(3) = 20$ then $f(20) = \frac{20a+7}{20-1} = 3$ so $a = 2.5$.

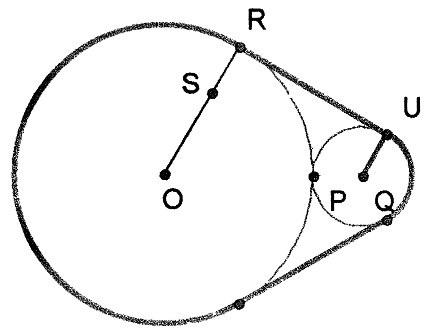
$f^{-1}(x) = \frac{x+7}{x-2.5} = g^{-1}(x) = 7x - 1$ so $(7x-1)(x-2.5) = (x+7)$ with solutions of x = 3 and x = –3/14.

B. T=k^2 while S=j(j+1) so S–T = $j^2 + j - k^2$. Maximize j at 99 gives S–T=9900–k^2 which is a multiple of 25 if k^2 is so k is divisible by 5 thus get max if k=95. The only pairs (j,k) left whose sum exceeds 99+95 are (98,97) and (97,98) both of which fail by inspection. Thus the answer is **194**

C. Take tangent of both sides and apply tangent of a sum theorem to get $\frac{A + 4/A}{1 - A(4/A)} = \frac{-5}{3}$ so $A + \frac{4}{A} = 5$ thus A= 4 or 1 making the left side of the original eqtn $\frac{\pi}{4} + Tan^{-1}(4)$ which is between $\frac{\pi}{2}$ and $\frac{3\pi}{4}$. Since $Tan^{-1}(\frac{-5}{3})$ is in the fourth quadrant, we need n=1.

D. Let x= time at 60mph . Total dist = 60x + 45(2x) +2(60x)=270x. Total time=x+2x+(120x/36)=(19/3)x . Avg speed is 270x / (19/3)x = 810/19.

E. If QU=x then OS = 2x, OQ = 4x so $\angle SOP=60°$, RU=$2x\sqrt{3}$ and band has length $(2/3)\pi 6x + (1/3)\pi 2x + 2RU = (14/3)\pi x + 4x\sqrt{3}$ so $x = \dfrac{36 + 14\pi\sqrt{3}}{4\sqrt{3} + {}^{14}\!/_{3}\,\pi} = 3\sqrt{3}$ so



larger circle's area of $9\pi x^2 = 243\pi$.

F. $s_4 = r^2 s_2$ and sequence is not constant so r = -1 and $s_3 = 2$. Working backwards we get $s_1 = -6$ and $s_0 = -10$. Working forward the sequence is: -10, -6, -2, 2, -2, 2, 6, 10, 50/3, 250/9, 350/9, 50, 450/7, 4050/49, and $s_{14} = 4950/49$.