MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2005
ROUND 1: ALGEBRA 2 SIMULTANEOUS EQUATIONS & DETERMINANTS

ANSWERS

A)_______________

B)_______________

C)_______________

A) If (a, b, c) is the solution to the following system, evaluate abc

$$\begin{cases} a + 2b + 3c = 3 \\ 3b + 2c = 7 \\ 3b + 4c = 17 \end{cases}$$

B) For real numbers s and t, $s + t = 17$ while $\sqrt{s}\sqrt{t} = 7$. In simplified radical form $s = a \pm b\sqrt{c}$ with $b>0$. Find the value of $c + b + a$.

C) For what value(s) of the constant c will the following system have no real solutions for (x, y, z)?

$$\begin{cases} x + 2y + 3z = 5 \\ 2x + cz = y + 1 \\ cx + 6z = 8 + 2y \end{cases}$$

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2005
ROUND 2: ALGEBRA ONE RATIONAL EXPONENTS/RADICALS

ANSWERS

A)_______________

B)_______________

C)_______________

A) Find the value of x if $\sqrt{1+\frac{4}{9}+\frac{9}{16}}=1+\frac{2}{3}+\frac{x}{4}$

B) Simplify $\left(1+2\sqrt{3}\right)^2+\sqrt{\frac{4}{27}}-\left(\sqrt{3}\right)^3+\frac{7}{3\sqrt{3}}$

C) Simplify $\frac{3^{n+4}+3\cdot 3^{n+1}}{3^{n+6}}$ Express your answer as a simplified fraction.

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2005
ROUND 3: ALGEBRA 2 POLYNOMIAL FUNCTIONS

ANSWERS

A)_______________

B)_______________

C)_______________

A) The third degree polynomial function p has a double zero at 2 and a single zero at -3. If its graph has a y-intercept of 6, find the sum of the coefficients of $p(x)$.

B) Give a polynomial equation of minimum degree whose roots are the reciprocals of the roots of $3x^4 + 5x^2 - 6x + 2 = 0$ Express your answer as a simplified fourth degree polynomial with integer coefficients.

C) Give a polynomial equation with integer coefficients and of minimum degree, three of whose root are 0, 1 + i, and $\frac{1}{2} + \sqrt{2}$. Express your answer in the form $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 4: ALGEBRA ONE ANYTHING

ANSWERS

A)_______________

B)__________ **Joules**

C)_______________

A) Find $\{ \text{ all real } x \mid \dfrac{2x-3}{x-3} = \dfrac{3x-6}{x-3} + 1 \ \}$.

B) The kinetic energy of a falling object is directly proportional to its mass and the square of its velocity. A quarter has three times the mass of a dime but 5 seconds after they are dropped simultaneously from the top of a building the lesser air resistance means the dime's velocity is 1.25 times that of the quarter. If the dime's kinetic energy at that moment is 250 Joules, what is the kinetic energy of the quarter?

C) If $\dfrac{a+c}{c} = \dfrac{d}{e}$ express $\dfrac{d+e}{ae-ec}$ as a quotient of polynomials containing only a and c

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 5: PLANE GEOMETRY ANYTHING

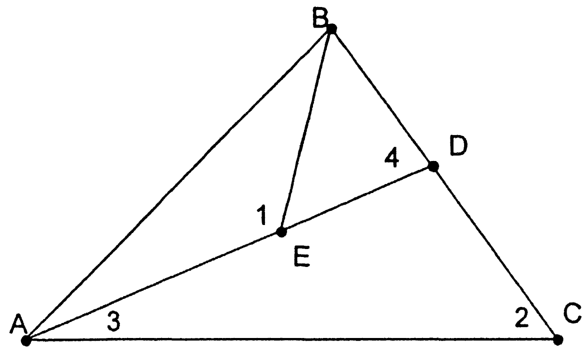
ANSWERS

A)_______________

B)_______________

C)_______________

A) Given AD > BC arrange the four numbered angles in descending order of size.



B) One diagonal of a rhombus is twice as long as the other. Express the area of the rhombus in terms of its perimeter p.

C) Right triangle ABC has right angle at B. D is on side $\overline{AC}$ so that AD : DC = 1:4 and E is on side $\overline{BC}$ so that BE : EC = 3 : 2 . If AB=7 and BC = 24 find the area of triangle DEC.

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2005
ROUND 6: ALGEBRA 2 PROBABILITY & BINOMIAL THEOREM

ANSWERS

A)______________

B)______________

C)______________

A) In the Junior class of 210 students 20 students were in the fall play and 28 were in the spring musical. If 12 students were in both events what is the probability that a Junior selected at random was in neither the fall play nor the spring musical? Express your answer as a simplified fraction.

B) A triangle is formed using the vertices of the octagon CREATION. What is the probability that the triangle's name contains at least one consonant? Express your answer as a simplified fraction

C) Consider the expansion of $\left(x+\dfrac{1}{2x}\right)^{2005}$. Divide the coefficient of the x^3 term by the coefficient of the x term and express the answer as a simplified fraction.

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2005
ROUND 7: TEAM QUESTIONS

******* NO CALCULATORS ON THIS ROUND ******

ANSWERS

A)______________ **D)**____**hrs**___**mins**

B)______________ **E)**______________

C)______________ **F)**______________

A) Find all pairs (x, y) for which $\begin{vmatrix} 1 & 2 \\ 3 & x \end{vmatrix} = \begin{vmatrix} x & 1 & y \\ 0 & 2 & 3 \\ 1 & 0 & x \end{vmatrix} = \begin{vmatrix} 3 & y \\ 1 & 2 \end{vmatrix}$

B) If $2^{x+y} = 4^{w+y} = 8^{x-w-2} = 16^{x+y-w-1}$ evaluate wxy.

C) If r, s, and t are the zeroes of $f(x) = 3x^3 - 39x^2 + 120x - 97$ find $r^2 + s^2 + t^2$

D) Marty's Masons charge \$150 plus \$60 an hour to build stonewalls while Bill' Builders charge \$350 plus \$45 an hour. For my small job Marty's cost was only 80% what Bill would have charged, but for my large job Bill's cost was only 80% of what Marty would have charged. Find the total time needed for the two jobs combined- answer in hours and minutes.

E) In ΔABC given AB = 8, AC = 14, and median AD = 7 find BC .

F) Find the last two digits of 3^{1000}

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2005 ANSWERS

Round 1:	A) 50	B) 102	C) $-6, 3.5$
Round 2	A) -1	B) $13 + 2\sqrt{3}$	C) $\frac{10}{81}$
Round 3	A) 2	B) $2x^4 - 6x^3 + 5x^2 + 3 = 0$	
	C) $4x^5 - 12x^4 + 9x^3 + 6x^2 - 14x = 0$ or an integral multiple thereof		
Round 4	A) { } or ϕ	B) 480	C) $\frac{a + 2c}{(a - c)c}$
Round 5	A) 1,4,2,3	B) $\frac{p^2}{20}$	C) 26.88
Round 6	A) 29 / 35	B) 13 / 14	C) 501 / 251
Team Round	A) (2.5, 9.5) (–3, 15)	B) -224	C) 89
	D) 82 hrs 5 mins	E) 18	F) 01

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2005 BRIEF SOLUTIONS

Round One:

A. c = 5 so b = − 1 so a = −10 so abc = 50.

B. st = 49 = s(17-s) Use quadratic formula to get $s = \dfrac{17 \pm \sqrt{93}}{2} = 8.5 \pm 0.5\sqrt{93}$

C. No solution if determinant of coef. matrix = 0 Solving $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & c \\ c & -2 & 6 \end{vmatrix} = 0$ we have $-6 + 2c^2 - 12 + 2c + 3c - 24 = 0$ so c = -6 or 3.5

Round Two:

A. $\sqrt{\dfrac{144 + 64 + 81}{144}} = \dfrac{12 + 8 + 3x}{12}$ so $\sqrt{289} = 17 = 20 = 3x$ so $x = -1$.

B. $1 + 4\sqrt{3} + 12 + \dfrac{2\sqrt{3}}{9} - 3\sqrt{3} + \dfrac{7\sqrt{3}}{9} =$

$13 + \dfrac{36 + 2 - 27 + 7}{9}\sqrt{3} = 13 + 2\sqrt{3}$

C. $\dfrac{3^{n+4}}{3^{n+6}} + \dfrac{3^{n+2}}{3^{n+6}} = \dfrac{1}{3^2} + \dfrac{1}{3^4} = \dfrac{3^2 + 1}{3^4} = \dfrac{10}{81}$

Round Three:

A. $p(x) = c\,(x+3)(x-2)^2 = c(x^3 - x^2 - 8x + 12)$ so c=0.5 to have a y-intercept of 6 and sum of coefficients is 0.5(1 − 1 − 8 + 12)=2

B. If the new equation were in y, then $\dfrac{3}{y^4} + \dfrac{5}{y^2} - \dfrac{6}{y} + 2 = 0$ which gives

$3 + 5y^2 - 6y^3 + 2y^4 = 0$

C. Using conjugate pairs of roots to obtain real coefficients:

$x(x - 1 - i)(x - 1 + i)(x - \frac{1}{2} - \sqrt{2})(x - \frac{1}{2} + \sqrt{2}) = x^5 - 3x^4 + \frac{9}{4}x^3 + \frac{3}{2}x^2 - \frac{7}{2}$

then multiply by 4 to obtain integer coefficients OR get two quadratics using sum and product of paired roots and multiply: $x(x^2 - 2x + 2)(x^2 - x + \frac{7}{4})$ etc.

Round Four:

A. If $x \neq 3$, we have $2x - 3 = 3x - 6 + x - 3$ so $6 = 2x$ thus $x = 3$. No solution.

B. $KE_q = CM_qV_q^2 = C(3M_d)(0.8V_d)^2 = 3(0.64)CM_dV_d^2 = 3.92(250) = 480$ J

C.

$$\frac{d+e}{ae-ec} = \frac{1}{a-c} \cdot \frac{d+e}{e} = \frac{1}{a-c} \cdot \left(1 + \frac{d}{e}\right) = \frac{1}{a-c} \cdot \left(\frac{c}{c} + \frac{a+c}{c}\right) = \frac{a+2c}{(a-c)c}$$

Round Five:

A. As exterior angles, 1>4 and 4>2. If AD>BC then surely AD>DC so 2>3

B. Let x = half the smaller diagonal. Area is then $0.5(2x)(4x) = 4x^2$ while perimeter is $4\sqrt{5}\,x$ so $x^2 = p^2/80$ and area is $p^2/20$.

C. Let x = area ΔABC. ΔBDC has base DC = (4/5)AC and same ht as ΔABC so area is (4/5)x. ΔCED has base CE = (2/5)CB and same ht as ΔBDC so area of ΔCED is (2/5)(4/5)x Since x = 84 answer is (0.32)(84) = 26.88

Round Six:

A. Prob (notF and notS)= (210 – 20 – 28 +12)/210 = 174/210 = **29/35**

B. There are ${}_8C_3 = 56$ triangles ; there are ${}_4C_3 = 4$ triangles with only vowels for vertices so the answer is 52/56 = **13/14**

C. The x^3 term is ${}_{2005}C_{1004}\, x^{1004}(1/2x)^{1001}$; the x term is ${}_{2005}C_{1003}\, x^{1003}(1/2x)^{1002}$ so

$$\frac{2005!}{1004!(1001!)2^{1001}} \div \frac{2005!}{1003!(1002!)2^{1002}} = \frac{2(1002)}{1004} = \frac{501}{251}$$

Team Round:

A. $x - 6 = 2x^2 + 3 - 2y = 6 - y$ so $y = 12 - x$ and substituting in the first equality simplifies to $2x^2 + x - 15 = 0$ so $x = -3$ or 2.5

B. Expressing all as powers of 2 yields the following system: $x+y = 2(w+y) = 3(x-w-2) = 4(x+y-w-1)$ which solves to x = 14, y = -2, and w = 8.

C. $r^2 + s^2 + t^2 = (r+s+t)^2 - 2(rs+rt+st)$; Since $x^3 - ax^2 + bx - c$ has roots whose sum is a, product is c, and sum of pair-wise products is b, we have r+s+t= 39/3=13 and (rs+rt +st) = 120/3 yielding 169 – 2(40)=89

D. 150+60x = 0.8(350+45x) gives x = 65 / 12 while 0.8(150+60x) = 350+45x gives x = 230/3. Sum is 82 and 1/12 or 82 hrs 5 mins.

E. Stewart's Thm: $(AB)^2(DC) + (AC)^2(BD) = (AD)^2(BC) + (BD)(DC)(BC)$ so if BD=x we have $64x + 196x = 49(2x) + 2x^3$ so $2x^3 - 162x = 0 = 2x(x+9)(x-9)$ so x=9 and BC=18 OR drop perpendicular from A to BD at E let AE=x BE=y ED=z so subtract $x^2+z^2=49$ from $x^2+(2z+y)^2=196$ to get (1) $y^2+4yz+3z^2=147$; From $x^2+y^2=64$ subtract $x^2+z^2=49$ to get $y^2-z^2=15$ added to (1) gives $2y^2+4yz+2z^2=162$ so $y^2+2yz+z^2=81$ whence y+z=9 and BC=18.

F. $3^{1000} = 9^{500} = (10-1)^{500}$ which when expanded influences the last two digits only in the last two terms as all other terms contain at least two factors of 10 so by Bin. Thm. we have … $+ {}_{500}C_{499}\,(10)(-1)^{499} + {}_{500}C_{500}\,(-1)^{500} = -5000+1$ so last two digits are 01.