MASSACHUSETTS MATHEMATICS LEAGUE
OCTOBER 2005
ROUND 1 VOLUME & SURFACES

ANSWERS

A)________cubic in.

B)_____________cm

C)_______________

A) The base of a right pyramid is a square with perimeter 10”; the pyramid’s altitude is 9”. Find the exact volume of the pyramid.

B) ABCD is a square of side 8 cm in a horizontal plane. $\overline{DK}$ is a 7 cm. line segment perpendicular to the plane. Q is the midpoint of $\overline{AB}$. Find the exact length of the longest edge of pyramid KDAQ.

C) A right triangle with legs of 30 and 40 is rotated around each of its three sides. Rotation about the shorter leg produces cone #1; rotation about the longer leg produces cone #2; rotation about the hypotenuse produces two cones sharing a common base. Find the sum of the volumes of the four cones. Leave answer in terms of π.

MASSACHUSETTS MATHEMATICS LEAGUE
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ROUND 2 PYTHAGOREAN RELATIONS

ANSWERS

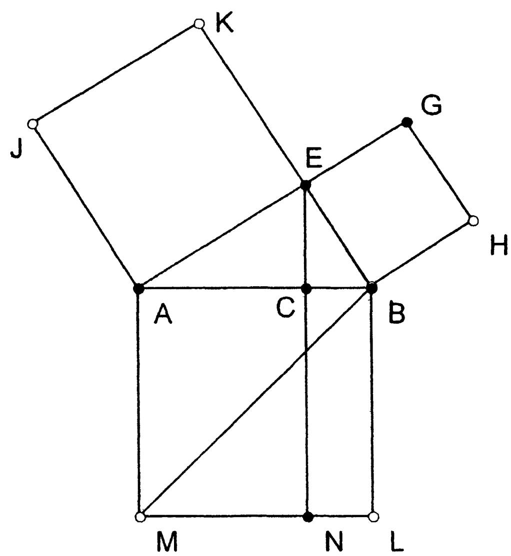
A)___________ft.

B)_____________

C)_____________

A) The shortstop fields a batted ball at a point 1/3 the distance from second base to third base. To the nearest 0.1 ft. how far must she throw the ball to get it to first base? Assume the infield is a square 90 ft. on a side.

B) Squares are constructed on each side of right ΔABE as shown. $\overline{EN} \perp \overline{AB}$
Find the area of JAEK if NL=4 and AC=12



C) Two kites each have sides of 15 and 41 and a smaller diagonal of 18. However, one kite is concave and the other convex. Find the difference in their areas.

MASSACHUSETTS MATHEMATICS LEAGUE
OCTOBER 2005
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A)______________

B)___________ **mph**

C)______________

A) If $[r]$ represents the largest integer which is less than or equal to r and

$$\frac{x+2}{9} - \frac{x-1}{16} = 1 \quad \text{find} \quad [x]$$

B) I took 6 hours to reach my destination. After averaging 60 mph for the first 2.5 hours, bad weather forced me to reduce my speed for the remainder of the trip. If my overall average speed was 39 mph, what was my average speed for the second part of my trip?

C) Suppose for all x, $3Ax - ABx + 15 - 5B = 13(4x + 5)$ with A and B real numbers. Find the value of A+B.

MASSACHUSETTS MATHEMATICS LEAGUE
OCTOBER 2005
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

******* NO CALCULATORS ON THIS ROUND ******

ANSWERS

A)_______________

B)_______________

C)_______________

A) **A ream (500 sheets) of letter size paper (eight and a half by eleven inches) is $2\frac{1}{8}$ inches thick. If the volume of a single sheet as a simplified fraction is $\frac{a}{b}$ cubic inches, find $a+b$.**

B) **Egyptians wrote fractions as sums of unit fractions ($\frac{1}{n}$). If we write $\frac{5}{18}$ as $\frac{1}{a}+\frac{1}{b}$ and $\frac{4}{9}$ as $\frac{1}{b}+\frac{1}{a}+\frac{1}{b}$, find the value of $a+b$.**

C) **My hose will fill my pool in 14 hours; the pool's drain empties the pool in 8 hours. At 6 a.m. my pool was empty so I closed the drain and turned on the hose. When the pool was half full, the drain accidentally opened! At what time did my pool become empty again? (Specify a.m. or p.m.)**

MASSACHUSETTS MATHEMATICS LEAGUE
OCTOBER 2005
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

ANSWERS

A)_______________

B)_______________

C)_______________

A) **Find all real x for which $4x^2 + 20x + 25 \le 0$**

B) **Find the sum of all solutions to**

$$|x(x-13)| = 30$$

C) **Find all real x for which**

$$13x^3 > 50x^2 - 44x - 8$$

MASSACHUSETTS MATHEMATICS LEAGUE
OCTOBER 2005
ROUND 6 ALG 1: EVALUATIONS

ANSWERS

A)_______________

B)_______________

C)_______________

A) **If we define** $n \oplus m$ **as** $\dfrac{nm}{n+m}$ **, find the exact value of** $\dfrac{1}{2 \oplus 3}$

B) **Find the exact value of** $\dfrac{4x-3y}{2x+y}$ **if** $\dfrac{x}{y} = 0.249\overline{99}$ **.**
Express your answer as a simplified fraction.

C) **In the expression below the letters** ***a, b, c, d, e, f,*** **and** ***g*** **are to be assigned positive integer values of 1 through 7 in some order. If** M **is the highest possible value of the expression we can obtain and** N **is the minimum, express** $M - N$ **as a simplified improper fraction.**

$$a + \cfrac{b}{c + \cfrac{d}{e + \cfrac{f}{g}}}$$

MASSACHUSETTS MATHEMATICS LEAGUE
OCTOBER 2005
ROUND 7: TEAM QUESTIONS

ANSWERS

A)__________cm D)__________

B)__________ E)__________

C)__________ F)__________

A) A pile of four cue balls is formed such that each ball is tangent to the other three. If the radius of each ball is 2 cm. find the exact height of the pile.

B) Acute ΔABC has integral sides with BC = 27 and AB = 2AC. The altitude from A divides $\overline{BC}$ into two segments of integral length. Find the exact length of that altitude in simplified radical form.

C) Given $\frac{1}{x}+\frac{x}{y}+\frac{6}{z}=14-y$ and $y=2x$ find z in terms of y assuming $xyz \neq 0$

D) A basic carpenter with an apprentice can frame a wall in 10 hours. A master carpenter with an apprentice can do it in 8 hours. A master carpenter working with a basic carpenter and no apprentices can do it in 6 hours. If a basic carpenter with apprentice began framing at 7 a.m. and the master carpenter joined them at 9 a.m. when did the three of them finish the job? Give answer to the nearest minute.

E) The minimum value of the following expression is $\frac{2}{3}$ and it occurs at $x=\frac{1}{2}$. Find all possible ordered pairs (a, b)

$$|ax+2|+|bx-1|$$

F) Keeping the normal order of operations we use $\ni$ and $\therefore$ to replace two of the usual arithmetic operations (addition, subtraction, multiplication, or division). If for any $a>0$

$$a \ni (a \ni a) \ni (a \ni a) = a \ni 2 \therefore a \ni 2 = a$$

find $8 \therefore 6 \ni 2$

MASSACHUSETTS MATHEMATICS LEAGUE
OCTOBER 2005 ANSWERS

Round 1 Geometry Volumes and Surfaces: A) 18.75 B) $\sqrt{129}$ C) 37,600 π

Round 2 Pythagorean Relations A) 94.9 B) 192 C) 216

Round 3 Linear Equations A) 14 B) 24 mph C) – 6

Round 4 Fraction & Mixed numbers A) 11,179 B) 15 C) 10:20 p.m.

Round 5 Absolute value & Inequalities A) –2.5 B) 26

C) $x > -2/13$, $x \neq 2$

Round 6 Evaluations A) 5/6 B) –4/3 C) 42,082 / 4,075

Team Round A) $4 + \frac{4\sqrt{6}}{3}$ B) $4\sqrt{14}$ C) $\frac{12y}{-2y^2 + 27y - 4}$ or equivalent

D) 1:05 p.m. E) $(-4, 10/3)$ and $(-4, 2/3)$ F) 11

MASSACHUSETTS MATHEMATICS LEAGUE
OCTOBER 2005 BRIEF SOLUTIONS

Round One:

A. Base has side 2.5, area 6.25 Vol = 9(6.25)/3 = 18.75

B. Hypotenuse KA = $\sqrt{98}$ while DQ and AQ are each $\sqrt{8^2 + 4^2} = \sqrt{80}$ Since $\overline{KD} \perp \overline{DQ}$ hypotenuse KQ = $\sqrt{80 + 49} = \sqrt{129}$.

C. #1 ht = 30, radius = 40, vol = 16000π. #2 ht = 40, radius = 30, vol = 12000π. Other 2 have radius = 24 (40x30/50) and hts of 18 and 32, vol=9600π.

Round Two:

A. Throws along hypotenuse of rt triangle (from shortstop to second to first). $\sqrt{90^2 + 30^2} = \sqrt{9000} = 94.8683... \approx 94.9$

B. $AE^2 = 12^2 + EC^2$; $EC^2 = EB^2 - 4^2$; $EB^2 = 16^2 - AE^2$ combined gives $AE^2 = 144 + 256 - AE^2 - 16$ so $AE^2 = 192$ (or AE is geom. mean of AC and AB)

C. Two diagonals create rt triangles with leg of 9 and hypotenuse of 41 or 15 so other leg is 40 or 12 so area is ½(52)(18) or ½(28)(18) difference is 216.

Round Three:

A. $16(x + 2) - 9(x - 1) = 144$; $7x + 33 = 144$; $7x = 103$; $x = 14\frac{5}{7}$

B. x = second leg avg. speed. Total distance was 60(2.5) + x(3.5) = 39(6) . Solve x= 24

C. If x=0, 15 – 5B = 65 so B = –10. if x=1, 3A + 10A + 65 = 117, A = 4. Sum is –6

Round Four:

A. $(8\frac{1}{2} \times 11 \times 2\frac{1}{8}) / 500 = (17 \times 11 \times 17) / (2 \times 6 \times 500) = \frac{3179}{8000}$ which is irreducible.

B. 1/b = 4/9 – 5/18 =1/6 so b=6 , a =9.

C. ½ = n/8 – n/14 yields n = 28/3 = 9 and 1/3 hours after 1 p.m.

Round Five:

A. $(2x + 5)^2 \le 0$ only if $(2x + 5)^2 = 0$ so $2x + 5 = 0$, $x = -2.5$

B. $x^2 - 13x = 30$ so $(x - 10)(x - 3) = 0$ or $x^2 - 13x = -30$ so $(x - 15)(x + 2) = 0$ sum is $10 + 3 + 15 - 2$

C. By synthetic division testing or calculator table $x-2$ is a factor of $13x^3 - 50x^2 + 44x + 8 = (x - 2)(x - 2)(13x + 2)$ If $x \neq 2$ first two are positive product so $13x + 2 > 0$ if $x > -2/13$

Round Six:

A. $1/(6/5) = 5/6$

B. If $x/y = ¼$ then $y = 4x$. Substitute to get $(4x - 12x)/(2x + 4x) = -8x/6x$

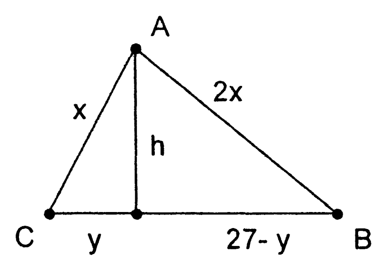
C. $$M = 7 + \cfrac{6}{1 + \cfrac{2}{5 + \cfrac{4}{3}}} = 7 + \cfrac{6}{1 + \cfrac{2}{19/3}} = 7 + \cfrac{6}{1 + \cfrac{6}{19}} = 7 + \frac{6(19)}{25} = \frac{289}{25}$$

Similarly $$N = 1 + \cfrac{2}{7 + \cfrac{6}{3 + \cfrac{4}{5}}} = \frac{201}{163} \text{ so } M - N = \frac{289(163) - 201(25)}{25(163)} = \frac{42082}{4075}$$

Team Round:

A. Centers of balls form reg. tetrahedron of edge 4. Altitude hits base incenter 2/3 way from vertex so ht = $\sqrt{4^2 - \left(\frac{2}{3} \cdot 2\sqrt{3}\right)^2} = \frac{4\sqrt{6}}{3}$ Add on radius of top ball and bottom layer.

B. $x^2 = h^2 + y^2$ and $4x^2 = h^2 + y^2 + 729 - 54y$ subtracting gives $3x^2 = 729 - 54y$ or $x^2 = 243 - 18y$ with x^2 a perfect square by trial and error or calculator table y is 1, $x^2 = 225$ or $y = 9$, $x^2 = 81$. Only the first fits the problem so $h = \sqrt{225 - 1} = 4\sqrt{14}$



C. $x = y/2$ so subst. to get $2/y + ½ + 6/z = 14 - y$ so $6/z = 27/2 - y - 2/y = \frac{27y - 2y^2 - 4}{2y}$ so $\frac{z}{6} = \frac{2y}{-2y^2 + 27y - 4}$ and $z = \frac{12y}{-2y^2 + 27y - 4}$

D. $1/b + 1/a = 1/10$; $1/m + 1/a = 1/8$; $1/m = 1/B = 1/6$; sum to get $2/b + 2/a + 2/m = 47/120$ so together $1/b + 1/a + 1/m = 47/240$. After 2 hrs. 4/5 job remains. 4/5 divided by 47/240 = $4\frac{4}{47} \approx 4\frac{4}{48} = 4hrs5$ min

E. The minimum value will occur at a vertex of the graph where one of the two AV expressions is zero and the minimum value is just the other expression. If the first has value zero at x= 1/2 then a= – 4 and to get the given value $\left|b(\frac{1}{2})-1\right| = \frac{2}{3}$ so b= $\frac{2}{3}$ or $\frac{10}{3}$. This gives possible locations for the other vertex of the graph as x=3/2 and expression there is 4 or x=3/10 and expression there is 4/5 so both agree with stated minimum. Similarly if second expression is zero at x=1/2 then b=2 and a = –16/3, x= 3/8, expression is ¼, less than minimum; or a = –8/3, x=3/4, expression is ½ ,again less than minimum.

F. If the first expression equals a then $\ni$ is either subtraction or division. If it is subtraction no substitute for $\therefore$ works in the second identity so $\ni$ is division and $\therefore$ must be addition. 8 + 6 / 2 = 11