MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2005
ROUND 1 COMPLEX NUMBERS

ANSWERS

A)__________________

B)__________________

C)________________

A) **If $x = a + bi$ for real a and b and if $x^2 = i$ find the product ab.**

B) **Simplify** $\left(i^9 - 5i^6 - 3i^8 + 7i^{11}\right)^2$ **as much as possible.**

C) Express in simplest form $(\sqrt{-6} - \sqrt{-2})^2 + \dfrac{16i}{1+\sqrt{-3}} - \left(\dfrac{4}{\sqrt{-2}}\right)^2$

MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2005
ROUND 2 ALGEBRA 1 ANYTHING

ANSWERS

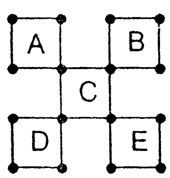
A)_$____________

B)______________

C)______________

A) The Arts Fund raised money by investing part of $30,000 in stocks paying 9% annual interest and the rest in safer bonds paying 8% annual interest. How much was invested in bonds if they made $2,500 in interest in one year?

B) The pattern below is superimposed over five dates on a monthly calendar. The total of dates covered is 70; A covers a Thursday date. Determine the ordered pair (p,q) where p denotes the center date (covered by C) and q is the day of the week (SUN MON TUE WED THU FRI SAT) on which the first of the month falls.



C) The sum of the digits of a 3-digit number is 14. The ten's digits is four less than the sum of the other 2 digits. If the ten's digit and unit's digit are interchanged the number's value is decreased by 18. Find the number.

MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2005
ROUND 3 GEOMETRY: AREA

ANSWERS

A)_______________

B)_______________

C)_______________

A) Square ABCD has area 400. Its diagonals intersect at E. Find the exact perimeter of triangle ABE.

B) Rectangle SROH has SR = 40 and SH = 30. Point E is on $\overline{RH}$ so that $\overline{SE} \perp \overline{RH}$ Find the area of concave pentagon SHORE.

C) The area of a kite is 168. The shorter diagonal is the axis of symmetry; the other diagonal has length 24. If the kite has integral sides, find its perimeter.

MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2005
ROUND 4 FACTORING

ANSWERS

A)______________

B)______________

C)______________

A) Given n is a positive integer and $x^2 + nx - 50$ is factorable. Find the sum of all possible values of n.

B) Factor completely over the integers: $6x^3 - 6 + 3x^2 - 12x$

C) Factor completely over the integers: $x^2(x^2 + x + 1) - (x^3 - 25)$

MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2005
ROUND 5 TRIG: FUNCTIONS OF 30, 45, 60 & 90

***** NO CALCULATORS ON THIS ROUND ****

ANSWERS

A)_______________

B)_______________°

C)_______________

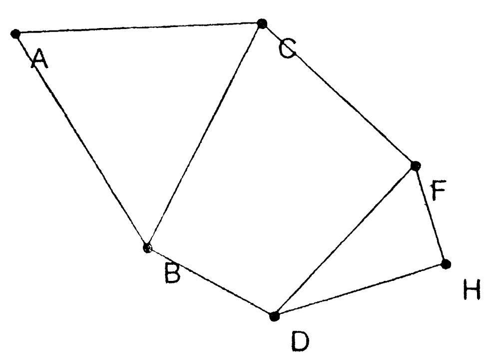
A) Find the exact value in simplified radical form of:

$$\sec\left(\frac{4\pi}{3}\right) - 2\sin^2\left(\frac{\pi}{12}\right) + \cot^2\left(\frac{11\pi}{6}\right) - 2\cos^2\left(\frac{\pi}{12}\right) - 2\csc\left(\frac{\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right)$$

B) Solve for all x, $0° \le x < 360°$: $\sin(2x) - \sin(-x) = 0$

C) In the figure below, find the value of DH in simplified radical form if:

$\sin(\angle FDH) = \cos(\angle A) = \cos(\angle ACB) = 0.5$, $CF = FD$, $AB = 10\sqrt{3}$,
$\cot(\angle CFD) = \cos(\angle CBD) = \cot(\angle H) = 0$, and $\cot(\angle BDH) = -1$



MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2005
ROUND 6 PLANE GEOMETRY: ANGLES

ANSWERS

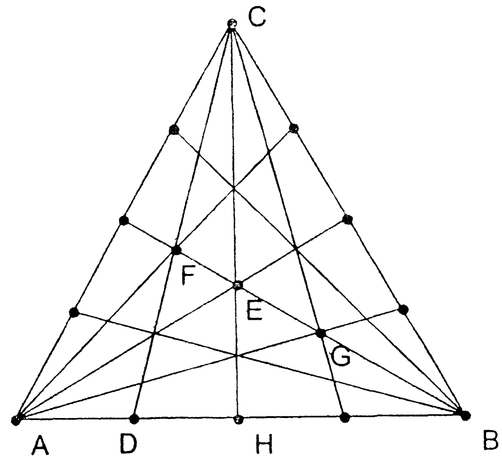
A)______________

B)______________

C)______________

A) **Given WXYZ a trapezoid with legs WX=8 and YZ=6. WZ : XY = 9: 5 . The bisectors of $\angle$ W and $\angle$ Z happen to intersect on $\overline{XY}$ Find WZ.**

B) **Each angle of an equilateral triangle is divided into 4 equal angles as shown. Find the sum of the measures of $\angle$ADF, $\angle$AEH, $\angle$FGC, and $\angle$AFD.**



C) **If $P_1P_2P_3...P_n$ are the vertices of a regular n-gon find in terms of n the measure of the acute angle formed by the intersection of P_1P_3 and P_2P_4**

MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2005
ROUND 7: TEAM QUESTIONS

ANSWERS

A)______________ **D)**______**by**______

B)______________ **E)**______________**m**

C)______________ **F)**______________

A) If $(1 + i)^{2006} = a + bi$ for real a and b, find the larger of a and b.

B) Given A, B, C, and D positive integer with A:B = 2:3 B:C = 5:8 and C:D = 20:27 express $\dfrac{AB}{BD + AD}$ as a simplified fraction.

C) A trapezoid has bases of 2 and 5 and legs of 1 and 3. Its area can be simplified to $\dfrac{a}{b}\sqrt{c}$ Find the sum $a + b + c$.

D) If a rectangle with side of length $3x - 4$ has an area of $12x^2 + 14x - 40$ and a perimeter of 194, find the dimensions of the rectangle.

E) A surveyor standing at a point on the ground so his eye is level with the bottom of a building measures the angle of elevation to the top of the building to be 60°. He backs up 30 meters and finds the angle of elevation has decreased by 15°. Find the exact height of the building in meters assuming the building is perpendicular to the ground.

F) ΔABC is isosceles with base $\overline{BC}$ The bisector of $\angle ABC$ intersects $\overline{AC}$ at D and intersects the bisector of the exterior angle from C at E. If ΔADB is also isosceles, find $m\angle BEC$.

MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2005 ANSWERS

Round 1: A) 1/2 B) $-32 - 24i$ C) 4i $2\sqrt{3} + 4i$

Round 2 A) \$20,000 B) (14, SAT) C) 653

Round 3 A) $20 + 20\sqrt{2}$ B) 816 C) 56

Round 4 A) 77 B) $3(2x+1)(x^2 - 2)$

C) $(x^2 - 3x + 5)(x^2 + 3x + 5)$

Round 5 A) -3 B) 120, 180, 240, 360 C) $5\sqrt{6}$

Round 6 A) 126/5 or 25.2 B) 240 C) 360/n

Team Round A) 0 B) $\frac{5}{27}$ C) 54

D) 35 by 62 E) $15\sqrt{3} + 45$ F) 18 °

MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2005 BRIEF SOLUTIONS

Round One:

A. $(a+bi)^2 = a^2 - b^2 + 2abi = 0 + 1i$ so 2ab=1.

B. $(i+5-3-7i)^2 = 4 - 24i - 36 = -32 - 24i$

C. $-6 - 2\sqrt{12} - 2 + \dfrac{16i\left(1 - i\sqrt{3}\right)}{1-(-3)} - (-8) = -4\sqrt{3} + \dfrac{16i + 16\sqrt{3}}{4} = 4i$

Round Two:

A. a+b=30000; 0.09a + 0.08b=350 so a= \$10,000 and a = **\$20,000**

B. C is the average of A and E and also of B and D so sum = 5C so p=14 so Saturday was day 15 and 8 and 1.

C. H+T+U=14; H+U=T+4; subtract. T=10-T so T=5. H+50+U-(H+10U+5)=18 so U=3.

Round Three:

A. Side is 20, diagonal is $20\sqrt{2}$.

B. ΔSER ~ ΔHSR ratio 4:5. Area ΔSER is 16/25 of ΔHSR = 384 . Subtract from 1200.

C. Area implies other diagonal is 14. Thus $12^2 + x^2 = a^2$ while $12^2 + (14-x)^2 = b^2$, Integer solutions suggest 9-**12**-15 and 5-**12**-13 triangles (14=9+5)

Round Four:

A. n is the difference of factors of 50 so a = 50 – 1 or 25 – 2 or 10 – 5.

B. $3(2x^3 + x^2 - 4x - 2) = 3[x^2(2x+1) - 2(2x+1)] = \mathbf{3(2x+1)(x^2-2)}$

C. Simplify to $x^4 + x^2 + 25 = x^4 + 10x^2 + 25 - 9x^2 = (x^2+5)^2 - (3x)^2$ etc.

Round Five:

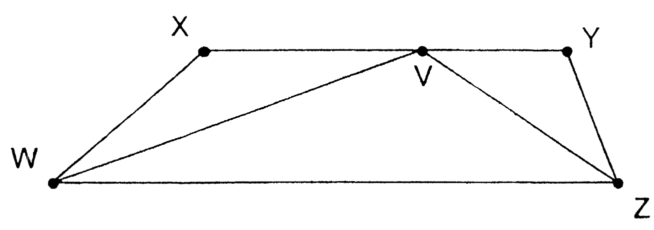
A. $(-2) - 2(\sin^2 \frac{\pi}{12} + \cos^2 \frac{\pi}{12}) + (-\sqrt{3})^2 - 2(1) = -3$

B. 2sinx cos x + sin x = sin x (2cos x +1) so sin x =0 or cos x = -0.5 thus x = 180, 360 or 120, 240

C. ABC equilateral so BC=$10\sqrt{3}$. BCD 30-60-90 so CD=20. CFD isos rt so FD=$10\sqrt{2}$. FDH 30-60-90 so DH = $5\sqrt{6}$

Round Six:

A. ΔWXV and ΔZYV are isosceles so XY=6+8=14 so WZ=(9/5)14.



B. $\angle$AEH = 60, $\angle$FGC=45, $\angle$ADF=105, $\angle$AFD=30.

C. Drawing all such diagonals P_jP_{j+2} creates vertices of a smaller regular n-gon whose exterior angles have measure 360/n

Team Round:

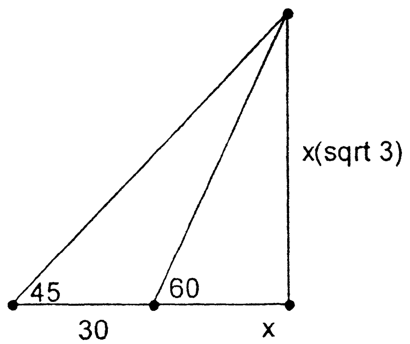
A. $\left((1+i)^2\right)^{1003} = (2i)^{1003} = 2^{1003}i^{1003} = -2^{1003}i = 0 + -2^{1003}i$ so a>b

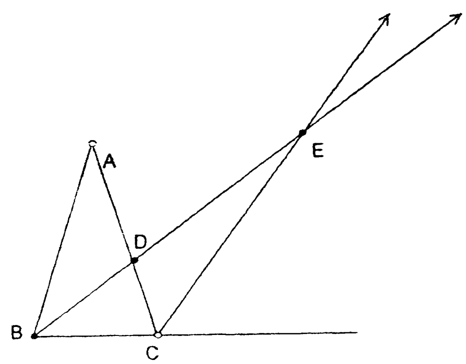
B. Multiply all 3 eqtns: A:D = 200:672 = 25:81. Multiply second and third eqtn gives B:D = 100:216 = 25:54(BD+AD)/AB = D/A + D/B = 54/25+81/25=135/25 so reciprocal is 35/135 = 5/27.

C. Draw hts from ends of shorter base, solve $x^2 + h^2 = 1$ $(3\text{-}x)^2 + h^2 = 9$ so $x = 1/6$ and $h = \frac{\sqrt{35}}{6}$ and area is $\frac{7}{12}\sqrt{35}$

D. Since $12x^2 + 14x - 40 = (3x - 4)(4x + 10)$ half the perimeter is $(3x - 4)+(4x + 10) = 97$ so x = 13 and dimensions are 35 by 62.

E. $30 + x = x\sqrt{3}$ so $x = \frac{30}{\sqrt{3}-1} = \frac{30(\sqrt{3}+1)}{3-1} = 15\sqrt{3} + 15$ see left sketch





F. If m∠A = x m∠ABD=x so angles ABC and BCA are each 2x so 5x=180 x = 36. m∠EBC=36, m∠BCE=72 + 108/4=126 so m∠BEC=18. See above sketch