MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2006
ROUND 1 ANALYTIC GEOMETRY ANYTHING

ANSWERS

A)_______________

B)_______________

C)_______________

A) Find the radius of the circle $x^2 + y^2 + 2(y - x + 3) = 40$.

B) Given two perpendicular lines $\ell_1 : ax + 6y = 3a$ and $\ell_2 : 2ax - 3y + a = 0$ for some constant $a > 0$, find the coordinates of the point of intersection of the two lines.

C) A parabola has $y + 3 = 0$ as its axis of symmetry. The parabola intersects $2x + y = 1$ twice, once at the parabola's vertex and once at the line's y-intercept. Find the coordinates of the parabola's x-intercept.

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2006
ROUND 2 ALGEBRA ONE: FACTORING & EQUATIONS

ANSWERS

A)____________

B)____________

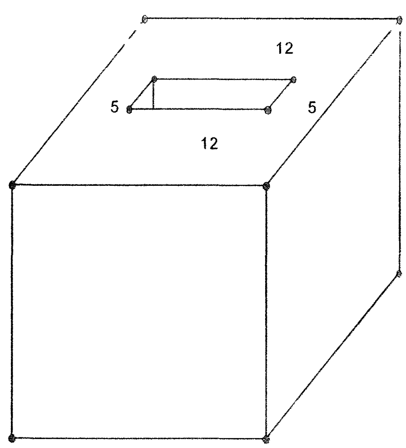
C)____________**cm**

All Factoring Is Over The Polynomials With Integer Coefficients

A) Find the largest integer g for which $2x^2 + gx - 15$ will be factorable.

B) Find the greatest common factor of $12x^2 - 42x + 18$ and $8x^2 + 20x - 12$.

C) A rectangular hole is cut all the way through a cube leaving side borders of 5 cm each and front and back borders of 12 cm as shown. If creating the hole removes exactly half of the volume of the cube, find all possible lengths for the side of the original cube.



MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2006
ROUND 3 TRIG: EQUATIONS WITH FEW SOLUTIONS

******* NO CALCULATORS ON THIS ROUND ******

ANSWERS

A)______________

B)______________°

C)______________

A) Find the sum of all x, $0 \le x < 2\pi$ for which $\sin(2x) = \cos(x)$

B) Find all x, $0° \le x < 360°$, for which $\dfrac{\tan(180 - x)}{\sin x} = 2$.

C) Find the <u>exact</u> sum of the five smallest positive solutions in radians to

$$\sec^2\left(\frac{x}{6}\right) + \sqrt{3}\tan\left(\frac{x}{6}\right) = \sqrt{3} + \tan\left(\frac{x}{6}\right) + 1$$

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2006
ROUND 4 ALGEBRA TWO: QUADRATIC EQUATIONS

ANSWERS

A)_______________

B)(_____,______,_____)

C)_______________

A) Find both real solutions for x: $x^2 + (x+1)^2 = (x+2)^2$

B) Find all real values of z if $2z^2 + yz + 3 = 0$ and $2y = y^2 - 35$

C) Shalomar owns several clothing stores. If she sells her sweaters at a price of \$100, her stores average 102 sales per month. Shalomar finds that for every \$5.00 she drops her price her stores sell on average 3 additional sweaters; each \$5 increase in price loses her three sales, however. What price will provide the greatest income for Shalomar?

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2006
ROUND 5 GEOMETRY: SIMILAR POLYGONS

ANSWERS

A)________ sq units

B)______________

C)______________

A) A 3-4-5 triangle is enlarged to make a similar triangle with hypotenuse 50 units long. What is the area of the enlarged triangle?

B) A right Δ has integer sides and one side has length 5. A second Δ with a perimeter of 1 is similar to the first Δ. Find the maximum possible difference between the areas of the two triangles. Express the answer as a simplified fraction $\frac{a}{b}$.

C) *ABCDEF* is a regular hexagon of side 10 cm. *M* is the midpoint of $\overline{AB}$ and *N* the midpoint of $\overline{CD}$. *X* is the intersection of $\overline{ME}$ and $\overline{NF}$. Find the exact length *MX* in simplified radical form.

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2006
ROUND 6 ALGEBRA ONE: ANYTHING

ANSWERS

A)______________

B)______________

C)______________

A) If $(a+b)^2$ is 12 more than $(a-b)^2$, find ab.

B) If $b = 5c - 3$ and $c = 5d - 3$ and $d = 5b - 3$, find c.

C) If $14p^2 + 15q^2 = 41pq$, find the sum of all possible values for $\dfrac{p}{q}$.

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2006
ROUND 7: TEAM QUESTIONS

ANSWERS

A)_____________ **D)**_____________

B)_____________ **E)**_______:_______

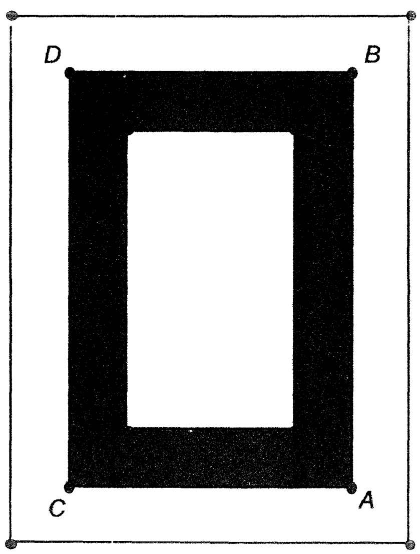
C)_____________ **F)**_____________

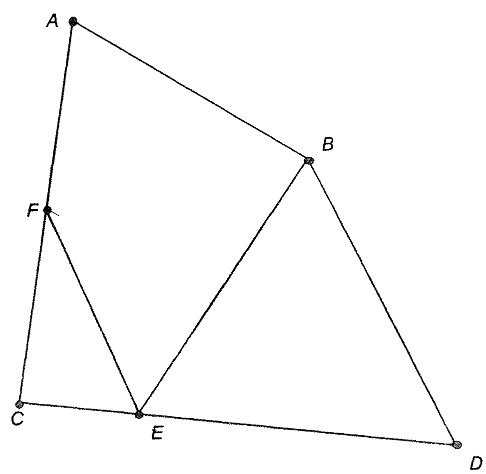
A) An ellipse with foci at (12, 16) and (12, 9) passes through the origin. If (w, 9) lies on the ellipse, find all possible values of w.

B) Factor $4(3b+1)^4+3(3b+1)^2+1$ over the polynomials with integer coefficients.

C) Find all pairs of complimentary angle measures x and y with $0<x<y$ such that x is a solution to $2\cos^2(6x) = \sin(12x)$ and y is a solution to $2\sin(12y)+2 = \cos^2(12y)$ Give your answers as ordered pairs (x, y).

D) The inner rectangle below is surrounded by two bands of the same uniform width. If the shaded region is one third the area of the largest rectangle and $ABDC$ is a 10 by 16 rectangle, what are both possible areas for the innermost rectangle?





E) In the above sketch on the right $ACDB \sim ABEF$ with $AC = DC = 12$ and $AB = 9$. Find the ratio of the area of ΔDFC to the area of ΔDAF as a simplified ratio of integers.

F) Sue and Bob each received money from Uncle Joe to buy mini music CDs; since they weren't the same age they got different amounts. SuperStore has a music club where you pay \$15 to join then pay \$2 per CD, while GrandSong charges just \$5 to join then \$6 per CD. Sue claimed her money would buy twice as many songs at one store than at the other and Bob claimed the same thing about his different amount of money! How much money did Uncle Joe send in total to Sue and Bob?

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2006 ANSWERS

Round 1:	A) 6	B) (1/5, 7/5) or (0.2, 1.4)	C) (3.5, 0)
Round 2	A) 29	B) 4x – 2 or 2(2x – 1)	C) 60 cm
Round 3	A) 3π	B) 120, 240	C) $36.5\ \pi$ or $73\pi/2$
Round 4	A) 3, –1	B) –3, -½, 1, 1.5	C) $135
Round 5	A) 600	B) 899/30	C $3\sqrt{13}$
Round 6	A) 3	B) 3/4	C) 41/14 or $2\frac{13}{14}$

Team Round A) –4.8, 16.8 B) $(18b^2+15b+4)(18b^2+9b+2)$ C) (7.5, 82.5) and (37.5, 52.5)

D) 16 or 55 E) 7 : 9 F) $52.00

MASSACHUSETTS MATHEMATICS LEAGUE

JANUARY 2006 BRIEF SOLUTIONS

Round One:

A. $x^2-2x+1+y^2+2y+1+4=40$ so $(x-1)^2+(y+1)^2=36$ and r = 6.

B. Slopes are $-a/6$ and $2a/3$ so $-2a^2/18=-1$.
Thus, $a=3$ Substitute and solve the system.

C. Intersection at $y=-3$ on line gives vertex as $(2,-3)$ so parabola is $x-2=a(y+3)^2$
Other intersection of (0,1) gives $a=-1/8$. Substitute this and $y=0$.

Round Two:

A. $(2x-a)(x+b)$ maximizes g when b is maximum, a minimum so b=15, a=1.

B. Factoring gives $6(2x-1)(x-3)$ and $4(2x-1)(x+3)$ common is $2(2x-1)$

C. $\frac{1}{2}x^3=x(x-24)(x-10)$ so $0=\frac{1}{2}x^3-34x^2+240=\frac{1}{2}x(x^2-68x+480)=$
$\frac{1}{2}x(x-8)(x-60)$. Only $x=60$ gives a large enough cube.

Round Three:

A. $2\sin(x)\cos(x)=\cos(x)$ so $\cos(x)=0$, $x=\pi/2, 3\pi/2$, or $\sin(x)=1/2$, $x=\pi/6, 5\pi/6$

B. $\tan(180-x)=-\tan(x)$ and $-\tan(x)/\sin(x)=-1/\cos(x)$ so $\cos(x)=-0.5$

C. $\tan^2(\frac{x}{6})+1+\sqrt{3}\tan(\frac{x}{6})=\sqrt{3}+\tan(\frac{x}{6})+1$ becomes

$\tan^2(\frac{x}{6})+\sqrt{3}\tan(\frac{x}{6})-\tan(\frac{x}{6})-\sqrt{3}=0=(\tan(\frac{x}{6})-1)(\tan(\frac{x}{6})+\sqrt{3})$ so
$x/6=\pi/4+n\pi$ thus $x=3\pi/2+6n\pi$ or $x/6=2\pi/3+n\pi$ thus $x=4\pi+6n\pi$ and the first five positive solutions are 1.5π, 4π, 7.5π, 10π, and 13.5π.

Round Four:

A. $x^2+x^2+2x+1=x^2+4x+4$ so $x^2-2x-3=0=(x-3)(x+1)$

B. Second equation gives $y=7$ or $y=-5$. $2z^2+7z+3=(2z+1)(z+3)$ so $z=-0.5$ or $z=-3$. $2z^2-5z+3=(2z-3)(z-1)$ so $z=1.5$ or $z=1$.

C. For n increases of \$5, price is $100+5n$ while sales is $102-3n$. Zeroes are at $n=-20$ and $n=34$, so vertex is at their average, $n=7$.

Round Five:

A. The larger triangle is scaled by 10 so its area is scaled by 100. The smaller triangle has area 0.5(3)(4) = 6.

B. Δ#1 is 3-4-5 or 5-12-13. Max difference comes from 5-12-13 whose area and perimeter are both 30. Δ#2 is 5/20, 12/30, 13/30 and area is 1/30.

C. $MN=15$ (midline) $\Delta MNX\sim\Delta EFX$ ratio 3:2 so MX is 3/5 of ME. AE twice altitude of equil Δ w/side $10=10\sqrt{3}$ and ME is hypotenuse of $\Delta AME=\sqrt{300+25}=5\sqrt{13}$,
so $MX=3\sqrt{13}$

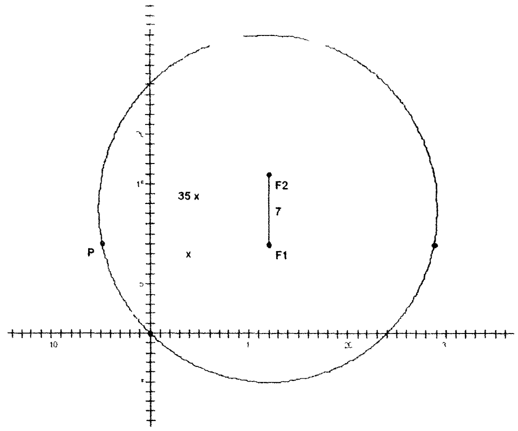
Round Six:

A. $a^2+2ab+b^2=12+a^2-2ab+b^2$ so $4ab=12$ and $ab=3$.

B. By symmetry, $b=c=d$ so $c=5d-3$ becomes $c=5c-3$ and $c=3/4$.

C. $14p^2-41pq+15q^2=(7p-3q)(2p-5q)$ so $7p=3q$, $p/q=3/7$ or $2p=5q$, $p/q=5/2$. Sum is 41/14.

Team Round:



A. Origin is 15 and 20 units from foci so sum of distances is 35. If $P=(w, 9)$ then ΔPF_1F_2 is a right triangle with legs of 7 and x and hypotenuse $(35-x)$. Pythagorean thm gives $x = 16.8$ So $a = 12 \pm 16.8$.

B. Let $x = 3b+1$. Factoring $4x^4 + 3x^2 + 1 = 4x^4 + 4x^2 + 1 - x^2 = (2x^2+1)^2 - x^2 = (2x^2 + 1 + x)(2x^2 + 1 - x)$ Replace x with $3b+1$ and simplify.

C. $2\cos^2(6x)=\sin(12x)=2\sin(6x)\cos(6x)$ so $\cos(6x)=0$, $6x=90+180n$, $x=15+30n$ or $\cos(6x)=\sin(6x)$, $6x=45+180n$, $x=7.5+30n$. $2\sin(12y)+2= \cos^2(12y)=1-\sin^2(12y)$, $\sin^2(12y)+2\sin(12y)+1=0$, $(\sin(12y)+1)^2=0$ so $\sin(12y)=-1$, $12y=270+360n$, $y=22.5+30n$. Complimentary pairs come from $x=7.5+30n$, $y=22.5+30n$.

D. Outer rectangle: $(10+2x)(16+2x)$; middle band: $160-(10-2x)(16-2x)=52x-4x^2$ Solve $3(52x-4x^2) = 160+52x+4x^2$ or $16x^2-104x+160=0$ or $8(x-4)(2x-5)=0$ If $x = 4$, inner is 2x8 =16; if $x = 2.5$, inner is 5x11=55.

E. $AF{:}AB = AB{:}AC$ so $AF = 27/4$. $FC = AB - AF = 21/4$. Area(ΔDFC)/ Area(ΔDAF) = FC/AF, since they have a common height.

F. If $15 + 2(k) = 5 + 6(2k)$ then $k = 1$ and there was $15 + 2(1)$ = \$17 to spend. If $15 + 2(2n) = 5 + 6(n)$ then $n = 5$ and there was $5 + 6(5)$ = \$35 to spend. Total was 35+17 = \$52.