MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2006
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

ANSWERS

A)________________

B)________________

C)________________

A) If $f(x) = 2x^2 + 1$, express $f(2y^2 + 1)$ as a simplified expression in terms of y.

B) Suppose $G(x) = 2x^2$ and $F(x) = 1 - 3x$. Find all x for which

$$F(G(x)) - G(F(x)) = x.$$

C) Given $h(x) = 2x + 1$, and $g(x) = 3 - 2x$, find all x for which $h^{-1}(x) + g^{-1}(x) = \dfrac{h(x) \cdot g(x)}{h(x) + g(x)}$.

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2006
ROUND 2 ELEMENTARY NUMBER THEORY

ANSWERS

A)_______________

B)_______________

C) (______, ______)

A) If abc is a three digit prime, find the sum of the second largest prime factor and the second smallest prime factor of the six digit number $abcabc$.

B) If A © B is defined as the sum of all composite numbers <u>strictly</u> <u>between</u> A and B, that is, including neither A nor B, evaluate:

(15 © 21) © (28 © 33)

B) If x and y are integers satisfying $2xy - 4x - y - 1 = 0$, which ordered pair (x, y) is furthest from the origin?

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2006
ROUND 3 TRIG: IDENTITIES & INVERSE FUNCTIONS

ANSWERS

A) $Y =$ ____________

B) ______________

C) ______________

A) Suppose $\text{Arctan}(\sqrt{x}) = d$, where $0° < d < 90°$. If $d = \text{Arcsec}(Y)$, express Y in terms of x.

B) Simplify $\dfrac{\sin\theta}{2(1+\cos\theta)} + \dfrac{1+\cos\theta}{2\sin\theta}$ to obtain a single trigonometric function of θ.

C) If $\sin(4\theta)$ is written in the form $A\sin\theta\cos\theta(B + C\sin^2\theta)$ for integers A, B and C, find $A^2 + BC$.

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2006
ROUND 4 ALGEBRA ONE: WORD PROBLEMS

ANSWERS

A)______________

B)___________mins

C)_______________

A) I have a mixture of quarters and Sacagawea dollar coins the same size. I gave \$16.50 worth of these coins to a teller, but he mistakenly counted 3 of the quarters as dollars. If he credited me X **cents** for these coins and the minimum number of coins I could have given the teller is K, find the numeric value of $X + K$.

B) I jog at 12 feet per second and my little sister jogs at a constant slower rate. If we run in opposite directions on a quarter mile track, we pass each other every minute. If we run in the same direction, how many minutes will it take me to lap her? (Recall: 1 mile = 5280 feet!)

C) A chemist adds 20 liters of an alcohol and water solution that is 30% alcohol to 10 liters of an original solution of alcohol and water. He finds the percentage of alcohol in the resulting mixture is 6 percentage points higher than in the original solution. What was the percentage of alcohol of the original solution?

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2006
ROUND 5 GEOMETRY: CIRCLES

******* NO CALCULATORS ON THIS ROUND ******

ANSWERS

A)________________

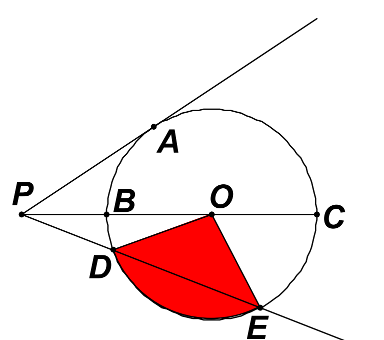
B)________________

C)________________

A) A chord of length 96 cm is 20 cm from the center of the circle.
How far is the midpoint of the chord from the furthest point on the circle?

B) Two chords $\overline{AB}$ and $\overline{CD}$ intersect at E. If $AE = 5x - 3$, $CE = 3x - 1$, $BA = 6x - 2$, and $DC = 5x - 1$, find all possible lengths for AE.

C) In the dagram (not to scale) $\overline{PA}$ is tangent to the circle with center O.
$PO = 7\sqrt{7}$, $PD = DE$ and AP = $7\sqrt{6}$. Find the exact area of sector ODE.



MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2006
ROUND 6 ALGEBRA 2: SEQUENCES & SERIES

ANSWERS

A)_________minutes

B)_______________

C)_______________

A) Sal gets 4 hours of homework every school night. On day 1 he is exceptionally motivated and does all his homework. However, on each successive school night he does only half as much homework as he did on the previous school night. At the end of the school year (180 days) to the nearest minute, how much total homework will Sal have done?

B) For an arithmetic sequence *a*, we find a_{2006} is twice a_{2004} and a_{2006} is 500 more than three times a_{2000}. Find a_{2005} .

C) The sum of the first three terms of a geometric series is 296, while the infinite sum is 80 less than twice that amount. Find the fifth term of the series.

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2006
ROUND 7: TEAM QUESTIONS

ANSWERS

A)_______________ **D)**_______________

B)_______________ **E)**_______________

C)_____________°_ **F)**_______________

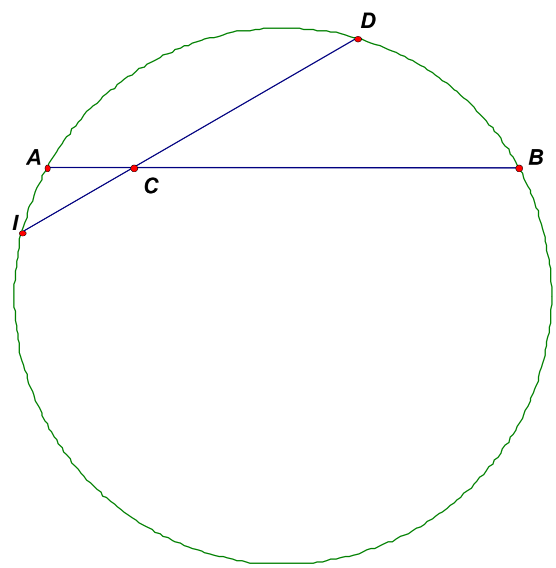
A) Suppose $f(x) = \dfrac{2x-1}{x-3}$ For what values of x does $2 \cdot f(x) \cdot f^{-1}(x) = 2$?

B) Suppose m and n are twin primes (i.e. primes differing by 2) and suppose the digits used to form m and n are distinct. If the smallest such pair of numbers and the largest such pair of number are all added together, what is the sum?

C) In ΔABC, $\angle B = \csc^{-1}(\frac{\sqrt{34}}{3})$ and $\angle A = \tan^{-1}(0.25)$. Find m$\angle C$ in degrees.

D) I am half as old as my mother was when my brother was twelve years younger than I am now. My brother was born when my mother was 26. If the sum of my brother's and my own current ages is 36, how old was my mother when I was born?

E) In the diagram at the right, m$\angle DCB = 30°$, $AC = 4$, $IC = 6$ and $BC = 18$. The exact positive difference between the distances of the two chords from the center of the circle is $a - b\sqrt{c}$ for integers a, b, and c. Evaluate $b^2c - a$.



F) $T_n = 3n + 2$. For some integers j and k, $j > k > 6$, T_k will be the geometric mean between T_6 and T_j. Find the smallest possible value of the sum $j + k$.

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2006 ANSWERS

Round 1:Algebraic Functions

A) $8y^4 + 8y^2 + 3$ B) 1/8, 1/3 C) ½

Round 2: Elementary Number Theory

A) 24 B) 286 C) (1, 5)

Round 3: Trig – Identities and Inverse Functions

A) $\sqrt{x+1}$ B) $\csc\theta$ C) 14

Round 4: Algebra 1 – Word Problems

A) 1896 B) 11 C) 21%

Round 5: Geometry - Circles

A) 72 B) 2 or 12 C) $\frac{49\pi}{3}$

Round 6: Algebra 2 – Sequences and Series

A) 480 B) 150 C) 40.5

Team Round

A) 1, –1 B) 128 C) 135°

D) 34 E) 280 F) 57

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2006 BRIEF SOLUTIONS

Round One:

A. $f(2y^2 + 1) = 2(2y^2 + 1)^2 + 1 = 2(4y^4 + 4y^2 + 1) + 1 = 8y^4 + 8y^2 + 3$.

B. $G(F(x)) = 2(1 - 3x)^2 = 18x^2 - 12x + 2$. $F(G(x)) = 1 - 6x^2$. Substitute to get $24x^2 - 11x + 1 = x$, so $(8x - 1)(3x - 1) = 0$.

C. $h^{-1}(x) = \frac{1}{2}x - \frac{1}{2}$; $g^{-1}(x) = -\frac{1}{2}x + 1\frac{1}{2}$ so $h^{-1}(x) + g^{-1}(x) = 1$. $h(x) + g(x) = 4$. $h(x)\cdot g(x) = -4x^2 + 4x + 3$. If $1 = (-4x^2 + 4x + 3)/4$ then $4x^2 - 4x + 1 = (2x - 1)^2 = 0$.

Round Two:

A. $abcabc = abc(1001) = abc(11)(7)(13)$. $13 + 11 = 24$.

B. 15©21 = 16 + 18 + 20 = 54; 28©33 = 30 + 32 = 62 ;
54©62 = 55 + 56 + 57 + 58 + 60 = 286

C. $2xy - y = 4x + 1$ so $y = \dfrac{4x+1}{2x-1} = \dfrac{2(2x-1)+3}{(2x-1)}$ so $2x - 1$ is a factor of 3 (±1 or ±3).
Thus, x must be 0, ±1 or 2. This yields the ordered pairs (1, 5), (0, -1), (2, 3) and (-1, 1). The first of these is furthest from the origin.

Round Three:

A. Right triangle has opposite side $\sqrt{x}$, adjacent 1, hypotenuse $\sqrt{1+x}$

B. Common denominator gets
$$\frac{\sin^2\theta + (1+\cos\theta)^2}{2\sin\theta(1+\cos\theta)} = \frac{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta}{2\sin\theta(1+\cos\theta)} = \frac{2(1+\cos\theta)}{2\sin\theta(1+\cos\theta)} = \frac{1}{\sin\theta}.$$

C. $\sin(2\cdot 2\theta) = 2\sin(2\theta)\cos(2\theta) = 2(2\sin\theta\cos\theta)(1 - 2\sin^2\theta) = 4\sin\theta\cos\theta(1 + -2\sin^2\theta)$, so $A = 4$, $B = 1$, $C = -2$.

Round Four:

A. The absolute minimum number of coins would be 16 dollars and 2 quarters, but since there must be at least 3 quarters, we have 15 dollars and 6 quarters → $K = 21$.
The teller's mistake credited my account 3(75) = 225 extra cents → $X = 1875$.

B. In one minute 12(60)+x(60)=1320 so sister jogs at 10 ft/sec. In same direction I gain 2 ft/sec or 120 ft/minute. 1320/120 = 11.

C. Original mix was $n/100$. $\dfrac{n+6}{100} = \dfrac{(n/100)10 + .30(20)}{30}$ Solving, $n = 21$.

Round Five:

A. Rt. triangle with radius as hypotenuse has legs of 20 and ½ (96), so hypotenuse is 52 [4x(5 - 12 - 13) triangle] . 20 + 52 = 72

B. $(AE)(BE)=(DE)(CE)$, so $(5x - 3)(x + 1) = (3x-1)(2x)$. Solve quadratic to get $x = 1$ or 3. Both give all positive lengths so $AE = 2$ or 12.

C. Rt ΔPOA gives $OA = 7$. If $DE = x$, $x(2x) = (7\sqrt{6})^2 = 294$, so $x = DE = 7\sqrt{3}$
Thus, $m\angle DOE = 120°$ and sector is 1/3 of the circle.

Round Six:

A. HW done = 4 + 2 + 1 + …. + a geometric progression of 180 terms with r = ½. The difference between the sum of 180 terms and the sum of an infinite sequence is considerably less than 1 minute, so use $a/(1-r)$ → $4/(1-½) = 8$ hrs = 480 min

B. $a_{2006} = a_{2000} + 6d = 2(a_{2000} + 4d)$, so $a_{2000} = -2d$, while $a_{2000} + 6d = 500 + 3a_{2000}$ so $a_{2000} = 3d - 250$. Thus, $-5d = -250$ → $d = 50$, $a_{2000} = -100$ and $a_{2005} = -100 + 5(50) = 150$.

C. $a + ar + ar^2 = 296$, while $a/(1-r) = 512$. $a = 296/(1 + r + r^2) = 512(1-r)$, so $296/512 = (1 + r + r^2)(1-r) = 1 - r^3$ → $r^3 = 1 - 296/512 = 216/512$ → $r = 3/4$ and $a = 128$.

Team Round:

A. $f^{-1}(x) = \dfrac{1-3x}{2-x}$ so if $f(x)\cdot f^{-1}(x) = \dfrac{2x-1}{x-3}\cdot\dfrac{1-3x}{2-x} = \dfrac{-6x^2+5x-1}{-x^2+5x-6} = 1$ then $5x^2 = 5$, so $x = \pm 1$.

B. Smallest such pair is 3 and 5. Largest such pair is 59 and 61. (Note that twin primes with three or more digits either share the most significant digit or the smaller has 9 as both its one and tens digit) Sum is 128.

C. Draw a right triangle to find $\tan(B) = 3/5$. $\tan A \cdot \tan B \cdot \tan C = \tan A + \tan B + \tan C$ gives $\tan C = -1$ OR use tangent sum identity to find $\tan(A + B) = 1$.

D. I am x years old now. "Then" my mother was $2x$ and my brother $x - 12$. Thus, $2x - (x - 12) = 26$ → $x = 14$. I am 14 and my brother is 22 now. My mother is 26 + 22 = 48 now and thus, 34 when I was born.

E. (See diagram below) $CD = AC(CB)/IC = 12$; $CG = .5(18) - 6 = 3$. Since ΔCGJ is a 30-60-90, $CJ = 2\sqrt{3}$, so $JE = .5(22) - 4 - CJ$.

$GF = GJ + 2(JE) = \sqrt{3} + 2(7 - 2\sqrt{3}) = 14 - 3\sqrt{3}$, while $EF = \sqrt{3}JE = 7\sqrt{3} - 6$

Difference: $20 - 10\sqrt{3}$

F. $\sqrt{20(3j+2)} = 3k + 2$. Square and simplify to $20j = 3k^2 + 4k - 12$ which must be a multiple of 4, so k is even. (WHY?)

If we substitute $k = 2n$ we have $3n^2 + 2n - 3 = 5j$.

Trial and error yields $n \cong 3 \bmod 5$ or $n = 3, 8, 13, \ldots$ → $k = 6, 16, \ldots$ Since $k > 6$, $k = 16$ and $j = 41$.