MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2006
ROUND 1 ALGEBRA 2: SIMULTANEOUS EQUATIONS & DETERMINANTS

ANSWERS

A)_______________

B)_______________

C)(_____ , _____)

A) Find x if $\begin{vmatrix} 1 & 1 & 1 \\ x & 2 & x \\ 0 & x & 1 \end{vmatrix} = 5$

B) Find all ordered pairs (x, y) that satisfy this system:

$$\frac{-1}{1-x} = \frac{1}{2y+1}$$
$$(x-1)^2 + (2y+1)^2 = 50$$

C) If A is the sum of the x-coordinates of the ordered pairs (x, y) satisfying:

$$(1+x\sqrt{2})^2(1-x\sqrt{2})^2 = y^2$$
$$3x = y-1$$

and N is the <u>number</u> of ordered pairs satisfying the system, find (A, N).

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2006
ROUND 2 ALGEBRA 1: RATIONAL EXPONENTS/RADICALS

ANSWERS

A)________________

B)________________

C)________________

A) If $\sqrt{2^{3^2}} + \sqrt{2^{2^3}} = a + 8\sqrt{b}$, find the ordered pair (a, b).

B) Express the sum below as a simplified radical:

$$\frac{2}{2\sqrt{2}+\sqrt{7}} + \frac{2}{\sqrt{7}+\sqrt{6}} + \frac{2}{\sqrt{6}+\sqrt{5}} + \frac{2}{\sqrt{5}+2} + \frac{2}{2+\sqrt{3}} + \frac{2}{\sqrt{3}+\sqrt{2}}$$

C) Solve for x: $\dfrac{(1/4)^{3x}}{2(4)^7} = (8^{x+4})^x$

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2006
ROUND 3 ALGEBRA 2: POLYNOMIAL FUNCTIONS

ANSWERS

A) _______________

B) _______________

C) _______________

A) Determine k so that -1 is a root of $(k-3)x^3 + (2k-5)x^2 + (k-7)x + (k-10) = 0$.

B) The polynomial function $f(x)$ has exactly three distinct zeros at $x = 1$, $x = -4/3$ and $x = 3/2$. If $f(0) = -12$, find $f(-1)$.

C) The polynomial $P(x)$ has integer coefficients and leaves a remainder of -3 when divided by $(x-2)$. The remainder is 17 when $P(x)$ is divided by $(x+3)$.
What is the remainder when $P(x)$ is divided by $(x-2)(x+3)$?

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2006
ROUND 4 ALGEBRA ONE: ANYTHING

ANSWERS

A)______________

B)______________

C)_______________

A) Find the sum of the 4 numbers that form the coordinates for the intercepts of the line

$$20x + 30y = 24{,}072$$

B) A company makes school sweatshirts and sweatpants. Five sweatshirts and six sweatpants cost a total of \$147. For orders totaling more than 30 items, the company reduced by 40% the price of sweatshirts and cuts the price of sweatpants in half. Forty sweatshirts and forty sweatpants, therefore, cost a total of \$578. Find the original cost of a single pair of sweatpants.

C) If $|x - a| = a + 2$, $x > 0$ and $a \leq 668$, find the maximum possible value of $x + a$.

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2006
ROUND 5: PLANE GEOMETRY ANYTHING

ANSWERS

A)_______________

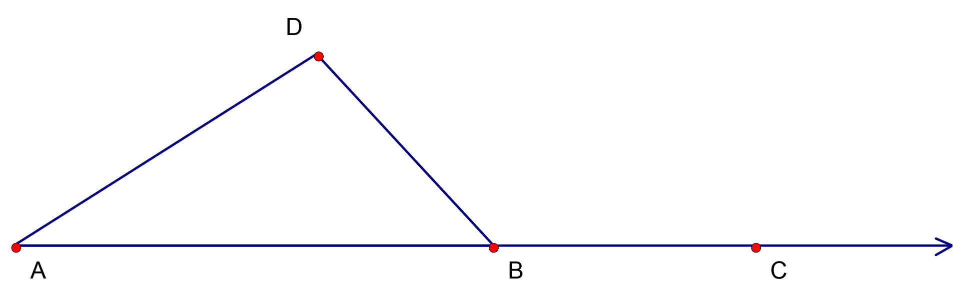
B)_______________

C)_______________

A) In isosceles triangle ABC, $m\angle B = 7\ (m\angle A)$. Find both possible measures for $\angle C$.

B) In ΔJKP, $m\angle P = 90$. M is on $\overline{JK}$ so that $\overline{PM} \perp \overline{JK}$ and N is on $\overline{KP}$ so that $\overline{MN} \perp \overline{KP}$. If $JP = 450$ and $KP = 600$, find MN.

C) In ΔABD, $AD = 12$, $DB = 8$ and $BA = 16$. The bisector of exterior $\angle DBC$ intersects line AD at E; F is on $\overline{AB}$ so that FDEB is a trapezoid. If $\overline{FE}$ intersects $\overline{BD}$ at G, find BG.



MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2006
ROUND 6: PROBABILITY & BINOMIAL THEOREM

ANSWERS

A)_______________

B)_______________

C)_______________

A) Suppose "numerical key" refers to: 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9 and "operation key" refers to: ^(raise to a power), ÷(divide), x(multiply), +(add) or –(subtract)

I pressed 4 keys on my TI-84: a numerical key, then an operation key, then a numerical key and then ENTER. The answer displayed on the screen was 16. If each key sequence that could generate this answer is equally likely, what is the probability that I pressed the 4 key twice?

B) If $(\sqrt{2}+\sqrt{3})^6 = a + b\sqrt{6}$, where a and b are integers, find the value of $a + b$.

C) Suppose we call a^n the <u>first</u> term in the expansion of $(a + b)^n$. Find both values of n, if the coefficients of the fifth, sixth and seventh terms in the expansion form an arithmetic sequence.

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2006
ROUND 7: TEAM QUESTIONS

******* NO CALCULATORS ON THIS ROUND ******

ANSWERS

A)_______________ **D)**_______________

B)_______________ **E)**_______________

C) (____,____,____) **F)**_______________

A) Let A be the product of all values for the constant k for which the system has no solutions for (x, y). Let B be the product of all values for the constant k for which the system has infinitely many solutions for (x, y). Find $A + B$.

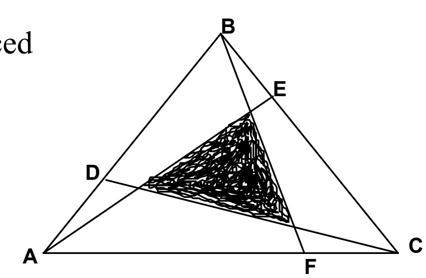
$$4x + k^2 y = -4 - 2k$$
$$(k^2 - 5)x - y = 2$$

B) Let A be a positive two-digit integer with the property that if the digits are reversed to form
The smaller integer B, then $A^2 - B^2$ is a perfect square. Find the sum of all values of A with this property.

C) The zeros of $y = f(x) = ax^3 + bx^2 + cx + 7$ are one more than the reciprocals of the zeros of
$y = g(x) = x^3 + x^2 - 5x + 2$. Determine (a, b, c).

D) $ABCD$ is a parallelogram. Three of the vertices are (1, 7), (-3, 1) and (9, 4). The fourth vertex has several possible locations. If P is the one furthest from the line $y = x$, exactly how far is P from the origin?

E) ΔABC is equilateral with $AB = 26$. Points D, E and F are placed so that $AD = ¼ (AB)$, $BE = ¼(BC)$ and $CF = ¼(CA)$ as shown. Find the exact area of the shaded region.



F) Assume n is a positive integer. Find the sum of all different values of n for which the expansion of $(4x^n + \frac{x^{-3}}{2})^{10}$ will contain an x-free term, i.e. a constant term.

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2006 ANSWERS

Round 1: Algebra 2 – Simultaneous Equations and Determinants

A) -3	B) (6, 2) (-4, -3)	C) (0, 4)

Round 2: Algebra 1 – Rational Exponents and Radicals

A) (16, 8)	B) $2\sqrt{2}$	C) -1, -5

Round 3: Polynomial Functions

A) 5	B) -10	C) $-4x + 5$

Round 4: Algebra 1 – Anything

A) 2006	B) $14.50	C) 2006

Round 5: Plane Geometry - Anything

A) 20, 84	B) 288	C) 16/3

Round 6: Algebra 2 – Probability & Binomial Theorem

A) 1/8	B) 683	C) 7, 14

Team Round

A) -4	B) 65	C) (-2, 11, -17)
D) $\sqrt{137}$	E) $52\sqrt{3}$	F) 51

MASSACHUSETTS MATHEMATICS LEAGUE
MARCH 2006 BRIEF SOLUTIONS

Round One:

A. $(2 + 0 + x^2) – (0 + x^2 + x) = 5$ means $2 – x = 5$, so $x = –3$.

B. First equation simplifies to $x – 1 = 2y + 1$; sub for $2y + 1$ in second to get $2(x – 1)^2 = 50$, so $x – 1 = \pm 5$. If $x = 6$, $y = 2$; if $x = – 4$, $y = – 3$.

C. First eqtn: $y^2 = [(1 + x\sqrt{2})(1 – x\sqrt{2})]^2$ so $y = \pm(1 – 2x^2)$ so $3x + 1 = 1 – 2x^2$ meaning $x = 0$ or $x = –3/2$; or $3x + 1 = 2x^2 –1$ meaning $x = 2$ or $x = –1/2$
The sum of the four numbers is 0.

Round Two:

A. $\sqrt{2^9} + \sqrt{2^8} = 2^{4.5} + 2^4 = 16 + 2^3 2^{1.5} = 16 + 8\sqrt{8}$ so $(a, b) = (16, 8)$.

B. Replace $2\sqrt{2}$ .with $\sqrt{8}$. Note $\dfrac{1}{\sqrt{x+1} + \sqrt{x}}\left(\dfrac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} - \sqrt{x}}\right) = \sqrt{x+1} - \sqrt{x}$ so

$$2\left(\sqrt{8} - \sqrt{7} + \sqrt{7} - \sqrt{6} + \sqrt{6} - \sqrt{5} + ... + \sqrt{3} - \sqrt{2}\right) = 2(\sqrt{8} - \sqrt{2}) = 2(2\sqrt{2} - \sqrt{2})$$

C. 2^(-6x) / 2^(15) = 2^(3(x + 4)x) so $–6x –15 = 3x^2 + 12x$ etc.

Round Three:

A. $(k –3)(-1)^3 + (2k –5)(-1)^2 +(k – 7)(-1) +(k – 10) = 0$ simplifies to $k – 5 = 0$

B. $k(x – 1)(3x + 4)(2x –3) = k(6x^3 – 7x^2 – 11x + 12) \rightarrow 12k = -12 \rightarrow k = -1$
$\rightarrow f(x) = -6x^3 + 7x^2 + 11x – 12 \rightarrow f(-1) = 6 + 7 – 11 – 12 = -10$.

C. $P(x) = Q_1(x)(x – 2)(x + 3) + ax + b$ [2nd degree divisor can leave a 1st degree remainder.]
$P(x) = Q_2(x)(x – 2) –3 \rightarrow P(2) = -3 = 2a + b$
$P(x) = Q_3(x) (x + 3) + 2 \rightarrow P(-3) = 17 = -3a + b \rightarrow a = -4, b = 5$

Round Four:

A. $30y = 24{,}072 \rightarrow y$-intercept = (0, 802.4) $20x = 24{,}072 \rightarrow x$-intercept = (1203.6, 0)

B. $5s + 6p = 147$ and $40(0.60s) + 40(0.5p) = 578$ or $24s + 20p = 578$; system solves to $s = 12$, $p = 14.5$.

C. If x – a is negative, $|x – a| = a – x = a + 2$ means $x = – 2$ violating $x > 0$. Thus, $x – a$ is nonnegative so $|x – a| = x – a = a + 2$, so $x + a = a + 2 + (2a) = 3a + 2$ maximized when $a = 668$, so $x = 2006$.

Round Five:

A. If A is a base angle $180 = A + A + 7A$. $A = C = 20$; if B is a base angle $180 = 7A + 7A + A$. $A = 12$, $C = B = 7(12) = 84$.

B. $JK = 750$ (Pythagoras, or 3-4-5 scaled by 150) $\Delta MKP \sim \Delta PKJ$ so $MK/600 = 600/750$ and $MK = 480$. $\Delta MNK \sim \Delta JPK$, so $MN/450 = 480/750$ and $MN = 288$.

C. Transversal DB gives m$\angle FDB$ = m $\angle DBE$; transversal FB gives m$\angle DFB$ = m$\angle CBE$, so $FB = DB$ (isos triangle) and DF is midline of ΔAEB, so $DE = AD$.
In ΔAEB, both BD and EF are medians so $BG = 2/3\ (BD)$.

Round Six:

A. The possible key sequences were: 4^2, 2^4, 4x4, 8x2, 8+8, and 2x8, 9+7 and 7+9, so prob =1/8.

B. Expand via binomial theorem or

$$\left(\sqrt{2}+\sqrt{3}\right)^{2(3)} = \left(5+2\sqrt{6}\right)^3 = 5^3 + 3(25)2\sqrt{6} + 3(5)4(6) + 8(6)\sqrt{6} = 485 + 198\sqrt{6}$$

C. If ${}_nC_6 - {}_nC_5 = {}_nC_5 - {}_nC_4$ then

$$\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{6!} = \frac{2n(n-1)(n-2)(n-3)(n-4)}{5!} - \frac{n(n-1)(n-2)(n-3)}{4!}$$

so $\frac{(n-4)(n-5)}{6(5)} = \frac{2(n-4)}{5} - \frac{1}{1}$ so $n^2 - 9n + 20 = 12(n-4) - 30$ … n = 7 or 14

Team Round:

A. Find k so coeff. matrix has determinant = 0: $k^2(k^2 - 5) + 4 = 0$ gives k = ±2 or ±1
Substitute to find inconsistent when k = 1 or – 2; dependent when k = 2 or –1.
$A = B = -2$, so sum is – 4.

B. $A = 10a + b \rightarrow B = 10b + a$. $A^2 - B^2 = 99(a-b)^2 = 9[(11)\,(a-b)^2] =$
$9[(11)(a+b)(a-b)]$. Since $a > b$ and a and b represent base 10 digits, the latter factor can be a perfect square, if $a + b$ is a multiple of 11 and $a - b = 1$, which only happens for $(a, b) = (6, 5)$.

C. Let the roots of $y = g(x)$ be r, s and t. Then: $r + s + t = -1$, $rs + rt + st = -5$ and $rst = -2$
If $f(x)$ has zeros: $1 + 1/r$, $1 + 1/s$ and $1 + 1/t$:

$$(1 + 1/r) + (1 + 1/s) + (1 + 1/t) = \frac{3rst + rs + rt + st}{rst} = \frac{-6 + (-5)}{-2} = \frac{11}{2}$$

$(1+1/r)(1+1/s) + (1+1/r)(1+1/t)+(1+1/s)(1+1/t) =$

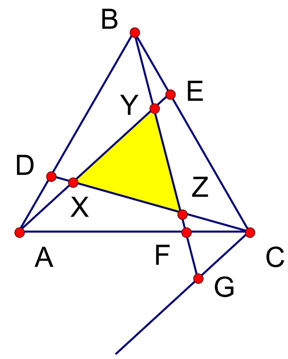
$$\frac{3rst + 2(rs + rt + st) + (r + s + t)}{rst} = \frac{-17}{-2} = \frac{17}{2}$$

$$(1+1/r)(1+1/s)(1+1/t) = 1 + \frac{1 + (r + s + t) + (rs + rt + st)}{rst} = 1 + \frac{1 + (-1) + (-5)}{-2} = \frac{7}{2}$$

$f(x) = k(x^3 - (11/2)x^2 + (17/2)x - 7/2 = -2x^3 + 11x^2 - 17x + 7$

D. The possible locations of the 4th vertex are: (13, 10), (-11, 4) and (5, -2). Note that A, B and C are midpoints of the triangle formed by connecting these three points. The one furthest from y = x is (-11, 4) which is $\sqrt{137}$ from the origin.

E. Area(ΔABE)= ¼ Area(ΔABC). To find Area(ΔABY), find ratio of bases AY to AE. Add parallel to AE from C, extend BF to G. $\Delta AYF \sim \Delta CGF$ gives $AY = 3CG$ $\Delta BYE \sim \Delta BGC$ gives $CG = 4YE$ so $AY = 12YE$ and Area(ΔABY) = (12/13) Area(ΔABE) = 3/13 Area(ΔABC). Removing 3 of these leaves Area(ΔXYZ) = (4/13) Area(ΔABC)= (4/13) $169\sqrt{3}$.



F. The k^{th} term in the expansion will be given by $\binom{10}{k}(4x^n)^{10-k}(\frac{x^{-3}}{2})^k$

$= C(x^{10n-nk-3k})$, where C is a numerical constant. x^0 insures that this is a constant term → $k = 10n/(n + 3) = 10 - 30/(n + 3)$ Thus, $n + 3$ must be a divisor of 30 = (2)(3)(5) The factors of 30 are: 1, 2, 3, 5, 6, 10, 15 and 30 so n may be 2, 3, 7, 12, and 27. The total is 51.