MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006
ROUND 1 VOLUME & SURFACES

ANSWERS

A) ______________________________

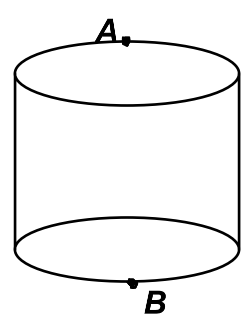
B) ___________________________cm^3

C) ______________________________

A) The volume of a right circular cylinder is 48π and the circumference of the base is 8π. Find the total surface area in terms of π.

B) Six slices, each with a uniform thickness of half a centimeter, are removed from a wooden cube, one slice per face, reducing the volume of the original cube by 169 cm^3. What is the volume of the resulting smaller cube in cm^3?

C) Points A and B are on diametrically opposite "sides" of the cylinder. Find the exact shortest possible distance from A to B along the surface of the cylinder. The diameter and height of the cylinder are 4 units and 1 unit respectively.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006
ROUND 2 PYTHAGOREAN RELATIONS IN RECTILINEAR FIGURES

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) The diagonal of a rectangle with integer dimensions is 25. List all possible perimeters of this rectangle.

B) A right triangle has legs of length 20 and 21. Find the exact length of the perpendicular from the vertex of the right angle to the hypotenuse. If necessary, express your answer in simplified radical form.

C) The area of a trapezoid with bases of lengths 10 and 25 is 294. Its diagonals are perpendicular and have integer length. Determine the length of each diagonal.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) $ ____________________________

A) Farmer Euclid MacDonald states: "I have 360 livestock, horses and chickens. The total number of legs, excluding my own, is 1100." How many chickens does farmer MacDonald have?

B) If (2006, b) and (a, 2006) are two points on the line $y = \frac{1}{4}x + 17$, what is the numeric value of $\frac{2006-a}{2006-b}$?

C) A shopper went into 10 stores and at each spent half the money she had upon entering the store PLUS an additional 25¢. Her purchases at the last store took all the money she had left. How much money did she have after leaving the third store?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

******* NO CALCULATORS ON THIS ROUND *******

ANSWERS

A) ______________________________

B) ______________________________

C) (_______, _______)

A) Determine the positive difference between A and B.

$$A = 2-\frac{1}{3+\frac{1}{4}} \qquad B = \frac{\frac{1}{3}+\frac{1}{4}}{\frac{2}{3}-\frac{1}{4}}$$

B) Solve for x: $$\left(\frac{1+x}{1-x}\right)=4-3\left(\frac{1-x}{1+x}\right)$$

C) The fraction $\frac{17}{25}$ can be expressed as the sum of three unit fractions, the first of which is $\frac{1}{2}$, that is $\frac{17}{25}=\frac{1}{2}+\frac{1}{A}+\frac{1}{B}$. Find (A, B), where $0 < A < B$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) It cost a publishing company \$250 to setup for the printing of a brochure. After that, the cost of printing, materials, etc. is 19¢ per brochure. If the publisher sells the brochure for 40¢ per copy, how many copies minimum must be sold before any profit is realized?

B) How many <u>integer</u> solutions are there that satisfy both $|2x - 1| \leq 23$ and $|x + 1| > 4$?

C) Find the domain of the real-valued function defined by $f(x) = \sqrt{\dfrac{x+4}{12-4x-x^2}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006
ROUND 6 ALG 1: EVALUATIONS

ANSWERS

A) _______________________________

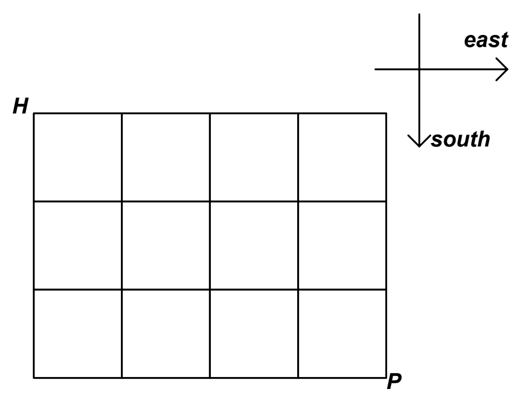
B) _______________________________ %

C) _______________________________

A) Consider the following binary operation: a ♣ b = $2a(3 - b)$.
Determine the positive difference between (0.5) ♣ 4 and 4 ♣ (0.5).

B) What is percent error (to the nearest 0.1%) when $0.\overline{42}$ is approximated as $\frac{2}{5}$?

C) H denotes Jack's house and P the local playground. If Jack walks only east or south, how many different routes can he take from his house to the playground?
Note: *SSSEEEE* represents one such route traveling along the left side and then the bottom side of the rectangle.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006
ROUND 7 TEAM QUESTIONS

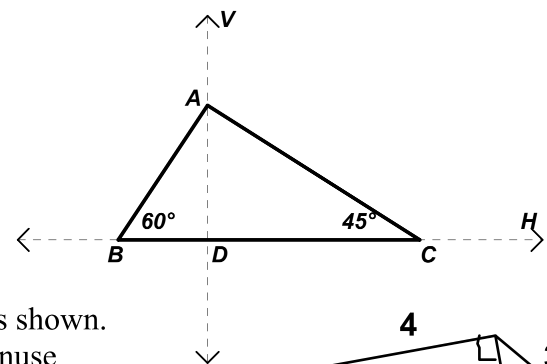
ANSWERS

A) ____________________________ D) ____________________________

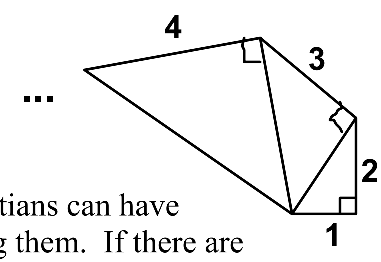
B) ____________________________ E) ____________________________

C) ____________________________ F) ____________________________

A) Two solids are generated, one by rotating the triangle pictured below about the horizontal axis, the other by rotating about the vertical axis. If $BC = 10$, determine the positive difference between the volumes of the two solids. Give an exact answer.



B) An infinite sequence of right triangles is formed as shown. Determine the length of the first (smallest) hypotenuse with integer length.



C) Scientists have discovered life on Venus and have learned that Venutians can have 2, 3 or 4 legs. There are 26 Venutians in a crater with 68 legs among them. If there are at least three 4-legged Venutians for every 3-legged Venutian and at least one of each type, how many 2-legged Venutians are there?

D) How many irreducible fractions $\frac{a}{b}$ lie between $\frac{4}{5}$ and $\frac{5}{6}$, where a and b are integers and $a + b = 2400$?

E) Find the value of A for which the distance d between the two points of intersections of the graphs of $y = \frac{1}{2}(x + |x|)$ and $|x| + |y| = A$ is $2\sqrt{5}$ units .

F) Consider the following recursive definition, defined for <u>any</u> integer values of n.

$$T_n = T_{n-1} + 2T_{n-2} \qquad \text{,where } T_2 = 1.5 \text{ and } T_4 = 1.5T_0$$

Determine the exact value of T_{-1}

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006 ANSWERS

Round 1 Geometry Volumes and Surfaces

A) 56π B) 343 cm^3 C) 5

Round 2 Pythagorean Relations

A) 70, 62 B) $\frac{420}{29}$ C) 21 and 28

["or", "and" acceptable as well]

Round 3 Linear Equations

A) 170 B) -4 C) $63.50

Round 4 Fraction & Mixed numbers

A) $\frac{19}{65}$ B) 0, $\frac{1}{2}$ C) (6, 75)

Round 5 Absolute value & Inequalities

A) 1191 B) 15 C) $x < -6$ or $-4 \leq x < 2$

{-11,-10,…,-6,4,5,…,12}

Round 6 Evaluations

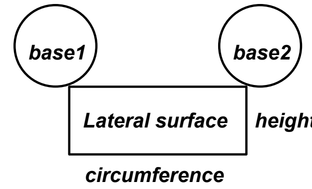
A) 21 B) 5.7% C) 35

Team Round

A) $250\pi(3\sqrt{3}-5)$ B) 70 C) 17

D) 6 E) $2\sqrt{2}$ F) $-\frac{57}{44}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006 SOLUTION KEY

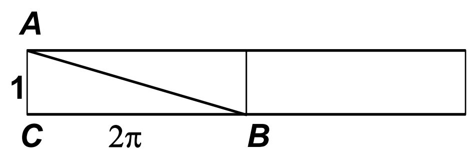


Round 1

A) $C = 2\pi r = 8\pi \rightarrow r = 4$, $V = \pi r^2 h = 48\pi \rightarrow h = 3$,
$TSA = 2(\pi r^2) + (2\pi r)h = 32\pi + 24\pi =$ **$\underline{56\pi}$**

B) Let x denote the edge of the original cube. Then $(x - 1)$ denotes the edge of the smaller cube. The decrease in volume is $x^3 - (x-1)^3 = 169 \rightarrow x^3 - (x^3 - 3x^2 + 3x - 1) = 169$
$\rightarrow 3x^2 - 3x - 168 = 3(x^2 - x - 56) = 3(x-8)(x+7) = 0 \rightarrow x = 8$
Thus, the volume of the smaller cube is $7^3 =$ **$\underline{343}$** cm^3.

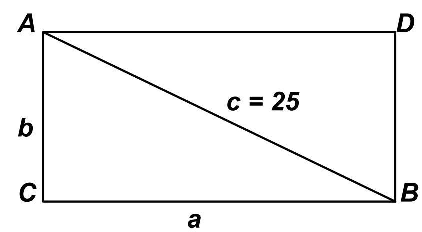
C) A path from point B along the lateral surface from the bottom base to the top base (perpendicularly) and then through the center of the top base to point A has length $1 + 4 = 5$
Cutting the surface of the cylinder along a line through point A perpendicular to a base and rolling out the lateral surface of the cylinder into a plane, we get a rectangle whose width equals the height of the cylinder and whose length equals the circumference of the base of the cylinder. The shortest path along this surface is the hypotenuse of ΔABC.
Using the Pythagorean Theorem, we have $AB = \underline{\sqrt{1+4\pi^2}}$



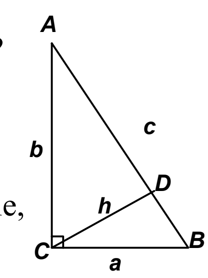
Since $5 < \sqrt{1+4\pi^2}$, the shortest distance is **$\underline{5}$**
An interesting question is under what circumstances is one path shorter than the other?

Round 2

A) Method I – Trial and Error
Since $a^2 + b^2 = 25^2 = 625$ and both a and b are integers, it remains to test integers from 1 to 24 inclusive. The only winners are $15^2 + 20^2 = 225 + 400 = 625$ and $7^2 + 24^2 = 49 + 576 = 625$. The corresponding perimeters $= 2(a + b)$ are $2(35) =$ **$\underline{70}$** and $2(31) =$ **$\underline{62}$**.

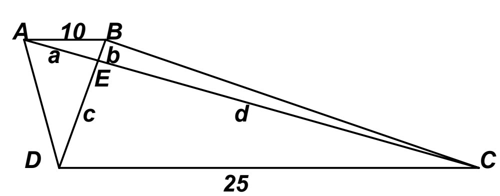


Method II
We all know the ordered triple (3, 4, 5) satisfies the Pythagorean theorem, i.e. are sides of a right triangle. But so are multiples!
$\rightarrow$ (15, 20, 25) $\rightarrow 2(15 + 20) =$ **$\underline{70}$**
The Pythagorean Triple (PT) (7, 24, 25) $\rightarrow 2(7 + 24) =$ **$\underline{62}$** $(7^2 + 24^2 = 49 + 576 = 25^2 = 625)$
Some other common Pythagorean triples worth remembering:
(5, 12, 13) *(7,24,25)* (9, 40, 41) (11, 60, 61) (13, 84, 85) … Do you see a pattern?
Another sequence: (8, 15, 17) (12, 35, 37) (16, 63, 65) (20, 99, 101) …



B) Since $\text{Area}(\Delta ABC) = \frac{1}{2}ab = \frac{1}{2}hc$, it follows that $h = \frac{ab}{c}$. Using the Pythagorean Theorem, or noting that (20, 21, 29) is a primitive Pythagorean Triple, the hypotenuse $c = 29$. Thus, $h = \boxed{\mathbf{\frac{420}{29}}}$.

C) The fact that $\Delta AEB \sim \Delta CED$ implies $\frac{AE}{CE} = \frac{BE}{DE} = \frac{AB}{CD} = \frac{10}{25} = \frac{2}{5}$ or $d = (5/2)a$ and $c = (5/2)b$. Thus, $AC = a + d = (7/2)a$ and $BD = b + c = (7/2)b$. Requiring the diagonals to have integer lengths implies both a and b must be integers. Since $a^2 + b^2 = 10^2$, it follows that $(a, b) = (6, 8)$ and $(c, d) = (20, 15)$ or vice versa. Thus, the diagonals have lengths $6 + 15 =$ **$\underline{21}$** and $8 + 20 =$ **$\underline{28}$**.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006 SOLUTION KEY

Round 3

A) Let H and C denote the number of horses and chicks respectively.
Then
$H + C = 360 \rightarrow H = 360 - C$
$4(360 - C) + 2C = 1100 \rightarrow 2C = 1440 - 1100 = 340 \rightarrow C = \underline{\mathbf{170}}$

B) Since the slope of the line is $\frac{1}{4}$, $\frac{b-2006}{2006-a} = \frac{1}{4}$. Inverting both sides and multiplying through by -1, it follows that $\frac{2006-a}{2006-b} = \underline{\mathbf{-4}}$.

C) If she has $\$x$ entering a store, she has $\$\frac{x}{2} - \frac{1}{4}$ when she leaves.
Thus, we half her money and then subtract 0.25. To make life easier, we'll work <u>backwards</u>, by starting with \$0 and first adding 0.25, then doubling!

Upon leaving store #10	9	8	7	6	5	4	3
$0 \rightarrow$	0.5	$\rightarrow$ 1.5	$\rightarrow$ 3.5	$\rightarrow$ 7.5	$\rightarrow$ 15.5 $\rightarrow$	31.5 $\rightarrow$	**<u>63.5</u>**

Round 4

A) $A = 2 - 4/13 = \frac{22}{13}$ and $B = \frac{4+3}{8-3} = \frac{7}{5}$ Computing the difference, $\frac{22(5)-7(13)}{13(5)} = \boxed{\frac{\mathbf{19}}{\mathbf{65}}}$

B) Let $A = \left(\frac{1+x}{1-x}\right)$ Then the equation becomes $A = 4 - 3(1/A) \rightarrow A^2 - 4A + 3 = (A-3)(A-1) = 0$
$\rightarrow A = 1$ or 3. Substituting for A, $1 + x = 1 - x \rightarrow x = \underline{\mathbf{0}}$ or $1 + x = 3(1 - x) \rightarrow 4x = 2 \rightarrow x = \underline{\mathbf{\tfrac{1}{2}}}$

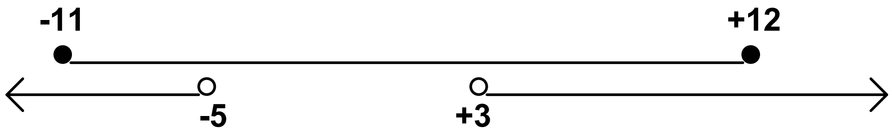
C) Method 1:
$\frac{1}{A} + \frac{1}{B} = \frac{17}{25} - \frac{1}{2} = \frac{9}{50} \rightarrow 50A + 50B = 9AB \rightarrow B = \frac{50A}{9A-50}$
B is an integer if and only if $9B$ is an integer. $9B = \frac{450A}{9A-50} = 50 + \frac{2500}{9A-50}$
Thus, $9A$- 50 must be a positive factor of 2500. The smallest possible value of A for which this is true is $A = 6$ and $B = 75 \rightarrow$ **<u>(6, 75)</u>**. In fact, the only ordered pairs are (6, 75) and (75, 6).
Method 2:
$\frac{17}{25} - \frac{1}{2} = \frac{9}{50} = 0.18$
The next largest unit fraction which is less than or equal to 0.18 is $\frac{1}{6} \approx 0.83 \rightarrow A = 6$
$\frac{9}{50} - \frac{1}{6} = \frac{4}{300} = \frac{1}{75} \rightarrow B = 75$ Therefore, $(A, B) =$ **<u>(6, 75)</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006 SOLUTION KEY

Round 5

A) $40x \geq 19x + 25000 \rightarrow 21x \geq 25000 \rightarrow x \geq 1190.47+ \rightarrow x_{\text{min}} = \underline{\mathbf{1191}}$

B) $|2x - 1| \leq 23 \rightarrow -23 < 2x - 1 < +23 \rightarrow -11 \leq x \leq +12$
$|x + 1| > 4 \rightarrow x + 1 < -4$ <u>or</u> $x + 1 > +4 \rightarrow x < -5$ or $x > +3$



Thus, the overlap contains integers from -11 to -6 inclusive as well as integers from 4 to 12 inclusive, a total of 6 + 9 = **<u>15</u>** integers.

C) The expression under the square root, i.e. the radicand, must be non-negative.

$\dfrac{x+4}{12-4x-x^2} = \dfrac{x+4}{(6+x)(2-x)}$ The critical values are -4, -6 and +2.
Two factors $(x + 4)$ and $(6 + x)$ are negative for values of x less than the critical value and positive for values of x greater than the critical value.
For $(2 - x)$ the situation is reversed.
The following diagram summarizes this situation:

```
(x + 4) ----------------------------------0++++++++++++++++++++++++++++++
(6 + x) ------------------0+++++++++++++++++++++++++++++++++++++++++++++
(2 - x) +++++++++++++++++++++++++++++++++++++++++++++++0------------
Section    1                2                    3                   4
________________-6________-4________________________2________
        2 neg          1 neg                  0 neg              1 n
```

Thus, in section 1 (**<u>$x < -6$</u>**) and section 3 (**<u>$-4 \leq x < 2$</u>**), the quotient is non-negative.
Note: Only -4 is included, since the other critical values would cause division by zero.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006 SOLUTION KEY

Round 6

A) $(0.5) \clubsuit 4 = 2(.5)(3-4) = \ = -1$

$4 \clubsuit (0.5) = 2(4)(3 - .5) = 24 - 4 = 20 \rightarrow 20 - (-1) =$ **<u>21</u>**

B) Let $n = 0.\overline{42}$ Then $100n = 42.\overline{42}$ Subtracting $99n = 42 \rightarrow n = \dfrac{14}{33}$

The error (i.e. the difference) is $\dfrac{14}{33} - \dfrac{2}{5} = \dfrac{14(5) - 2(33)}{33(5)} = \dfrac{4}{5(33)}$

The percent error is $\dfrac{\text{difference}}{\text{original}} \cdot 100\%$. $\dfrac{4/(5 \cdot 33)}{14/33} = \dfrac{2}{35} \approx 0.05714+ \rightarrow$ **<u>5.7%</u>**

C) Any traversal (path) from the house *H* to the playground *P* is simply an arrangement of the letters *SSSEEEE*. If the letters were distinct, they could be arranged in 7! ways. But since they are not, the number of arrangements must be divided by 3!, since there would be 3! = 6 ways of arranging S_1, S_2 and S_3, if they were distinguishable. ($S_1S_2S_3$, $S_1S_3S_2$, $S_2S_1S_3$, $S_2S_3S_1$, $S_3S_1S_2$, $S_3S_2S_1$ are indistinguishable, if the subscripts are dropped.)
Similarly, we must divide by 4! = 24, because the 4 *E*'s are indistinguishable.

Thus, the number of distinct paths is $\dfrac{7!}{3! \cdot 4!} = \dfrac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} =$ **<u>35</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006 SOLUTION KEY

Team Round

A) Rotating around the vertical axis, a single cone(call it #1) with height AD is produced by ΔADC. Rotating around the horizontal axis, two cones sharing a common base and heights BD and DC (call them #2 and #3 respectively) are produced by Δs ABD and ADC.

Let $DC = AD = x$. $DB = 10 - x = \dfrac{x}{\sqrt{3}}$ → $x = \dfrac{10\sqrt{3}}{\sqrt{3}+1} = 15 - 5\sqrt{3} = 5(3 - \sqrt{3})$

Using V(cone) = $\dfrac{1}{3}\pi r^2 h$, V(cone #1) = $\dfrac{1}{3}\pi(DC)^2(AD) = \dfrac{1}{3}\pi x^2 \cdot x = \dfrac{\pi x^3}{3}$,

V(cone #2) = $\dfrac{1}{3}\pi(AD)^2(BD) = \dfrac{1}{3}\pi x^2(\dfrac{x}{\sqrt{3}}) = \dfrac{\sqrt{3}}{9}\pi x^3$

V(cone #3) = $\dfrac{1}{3}\pi(AD)^2(DC) = \dfrac{1}{3}\pi x^2 \cdot x = \dfrac{\pi x^3}{3}$

Thus, the difference is $\dfrac{\sqrt{3}}{9}\pi x^3 = \dfrac{125\pi\sqrt{3}}{9}(3-\sqrt{3})^3 = \dfrac{125\pi\sqrt{3}}{9}(54-30\sqrt{3}) = \underline{\mathbf{250\pi(3\sqrt{3}-5)}}$

B) $1^2 + 2^2 = 5$, $5 + 3^2 = 14$, $14 + 4^2 = 30$, $30 + 5^2 = 55$
The difference between successive terms is the square of the next integer.
→ 91, 140, 204, 285, 385, 506, 650, 819, 1015, 1240, 1496, 1785, 2109, 2470, 2870, 3311, 3795, 4324, and finally, $4324 + 24^2 = 4900 = 70^2$ → **<u>70</u>**

C) Let (a, b, c) denote the number of 2-, 3- and 4-legged Venutians respectively.
Then $2a + 3b + 4c = 68$, $a + b + c = 26$, $c \geq 3b$ and $a, b, c \geq 1$. The first two equations give us $b + 2c = 16$ or $c = \dfrac{16-b}{2}$. Substituting in the inequality, we have $\dfrac{16-b}{2} \geq 3b$, hence $16 \geq 7b$.
Thus, $b = 1$ or 2 and because $2c = 16 - b$, it follows that b must be even, so $b = 2$.
This implies $c = 7$ and, therefore, $a = 26 - (2 + 7) =$ **<u>17</u>**.

Team Round - continued

D) $\frac{4}{5} < \frac{a}{2400-a} < \frac{5}{6}$ → $24(2400 - a) < 30a < 25(2400 - a)$

→ $54a > 24(2400)$ → $a > \frac{4}{9}(2400) \approx 1066.6+$ → min 1067

→ $55a < 25(2400)$ → $a < \frac{5}{11}(2400) \approx 1090.9+$ → max 1090

An even integer in the numerator would produce an even number in the denominator and the fraction would be reducible.
Likewise for numerators that are multiples of 5.
Thus, the following fractions must be examined for reducibility:

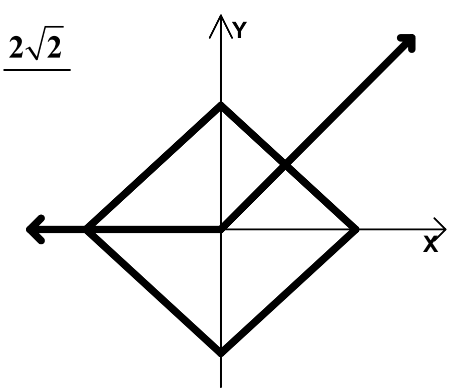
$\frac{1067}{1333}, \frac{1069}{1331}, \cancel{\frac{1071}{1329}}, \frac{1073}{1327}, \cancel{\frac{1077}{1323}}, \frac{1079}{1321}, \frac{1081}{1319}, \cancel{\frac{1083}{1317}}, \frac{1087}{1313}, \cancel{\frac{1089}{1311}}$

Each of the crossed out fractions is reducible, since both numerator and denominator are divisible by 3. The remaining **<u>6</u>** fractions are irreducible.
How do you verify this with a calculator? without a calculator?

E) For the first equation, if $x \le 0$, $y = 0$ and for $x \ge 0$, $y = x$.
The second equation is a square with vertices on the axes at $(\pm A, 0)$ and $(0, \pm A)$
The points of intersection are at $(-A, 0)$ and $(A/2, A/2)$. Using the distance formula,

$d = \sqrt{\left(\frac{3A}{2}\right)^2 + \left(\frac{A}{2}\right)^2} = \frac{1}{2}\sqrt{10A^2} = 2\sqrt{5}$ → $10A^2 = 80$ → $A = \underline{\mathbf{2\sqrt{2}}}$

($-\mathbf{2\sqrt{2}}$ is rejected, since A is the sum of two absolute values.)



F) By definition, $T_1 = T_0 + 2T_{-1}$ → $T_{-1} = \frac{T_1 - T_0}{2}$ Thus, we set out to find these two values.

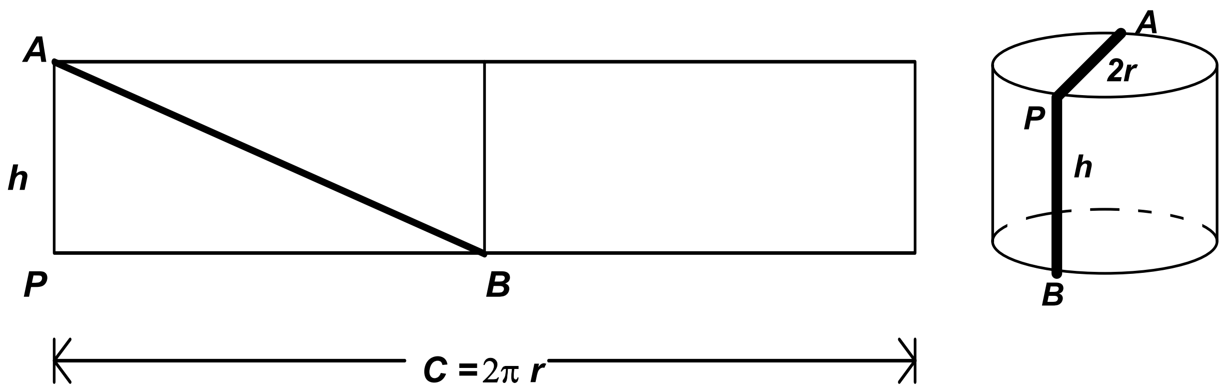
Since we were given $T_n = T_{n-1} + 2T_{n-2}$, $T_2 = 1.5$ and $T_4 = 1.5T_0$
(1) $T_2 = T_1 + 2T_0 = 1.5$ → $T_1 = 1.5 - 2T_0$
(2) $T_4 = T_3 + 2T_2 = T_3 + 2(T_1 + 2T_0) = T_3 + 2(1.5 - 2T_0 + 2T_0) = T_3 + 3 = 1.5T_0$
(3) $T_3 = T_2 + 2T_1 = 1.5 + 2(1.5 - 2T_0) = 4.5 - 4T_0$
Thus, $1.5T_0 - 3 = 4.5 - 4T_0$ → $5.5T_0 = 7.5$ → $T_0 = 15/11$
Substituting in (1) above, $T_1 = 1.5 - 30/11 = -27/22$

$T_{-1} = \frac{-\frac{27}{22} - \frac{15}{11}}{2} = \frac{-27-30}{44} = \underline{\mathbf{-\frac{57}{44}}}$

Addendum

Discussion question (Round 1 Question C)

Under what circumstances is the spiraling distance around the lateral surface of a cylinder shorter than the 'up and over the top' distance?



$\sqrt{h^2+\pi^2r^2} < h + 2r$? Squaring both sides,

$$h^2 + \pi^2r^2 < h^2 + 4hr + 4r^2 \rightarrow \pi^2r^2 - 4r^2 = (\pi^2 - 4)r^2 < 4hr \rightarrow h > \frac{(\pi^2-4)r}{4}$$

Thus, with a radius of 2, the spiral distance is shorter than the 'up and over the top' distance as long as the height is greater than approximately 2.93 .
Since this was not the case in round 1 question C, the answer was simply 1 + 4 = 5.