MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2006
ROUND 1 ALG 2: COMPLEX NUMBERS (No Trig)

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Expand and give your answer in $a + bi$ form: $i(2 + 3i)(1 - 4i)$

B) Find $\sqrt{2i}$ in $a + bi$ form, where $b > 0$.

C) Evaluate in $a + bi$ form: $\sum_{n=1}^{n=3} (1 - i\sqrt{3})^{(2^n)}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006
ROUND 2 ALGEBRA 1: ANYTHING

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Evaluate: $4^2 + 3 \cdot 7 - 8^{-2} \cdot \frac{2^8}{5 \cdot 9} \div \frac{1}{4 \cdot 3} \cdot 6^2$

B) Determine all values of x for which $\left(\frac{1+x}{2}\right)^2 - 3\left(\frac{1+x}{2}\right) = 18$

C) If he were still alive in the year I was born, Ramanujan on his birthday would have been one year older than I was on my birthday this year. By the end of this year (2006), the sum of our ages would be 178. In what year was I born?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

ANSWERS

A) ________________________________

B) _______________ : _______________

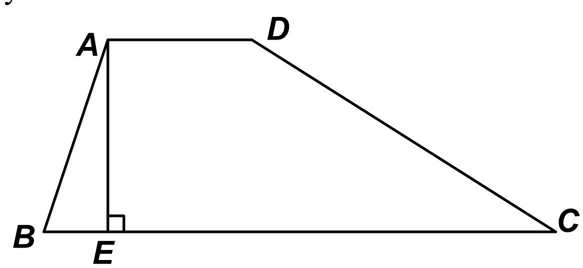
C) ________________________________

A) Find the exact area of trapezoid *ABCD*, with bases $\overline{AD}$ and $\overline{BC}$, given:

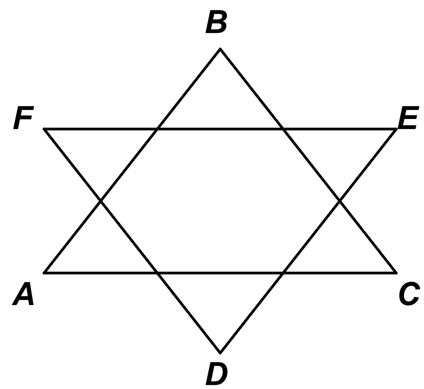
$AB = 25$, $BC = DC = 40$, $AE = 24$

$AD < BC$ and E is between B and C

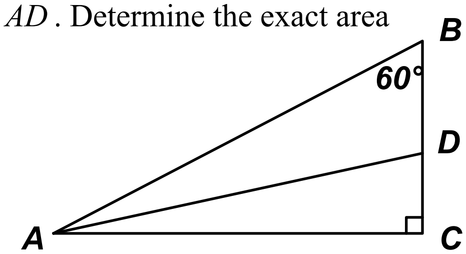
The diagram is not necessarily drawn to scale.



B) A six-pointed star is formed by taking equilateral ΔABC, flipping it over a horizontal line to form ΔDEF, and placing it on top of the ΔABC so that all of its sides are trisected by the intersection points. Express (in simplest form) the ratio of the area of the entire star to the area of the original ΔABC.



C) The area of ΔABC is 6 units^2. The 30° angle is bisected by $\overline{AD}$. Determine the exact area of ΔADC.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006
ROUND 4 ALG 1: FACTORING AND ITS APPLICATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) $x =$ ____________________________

A) Factor completely $(a^2 - 3a + 1)^2 - 1$

B) Factor completely: $(x + 1)^3 - (x + 1)^2 - 9(x + 1) + 9$

C) Given: $x(x + 2y) = 1$ and $x < 0$. Solve for x in terms of y.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006
ROUND 5 TRIG: FUNCTIONS OF SPECIAL ANGLES

******* NO CALCULATORS ON THIS ROUND ******

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

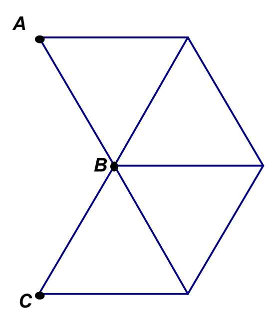
A) If $\cos A = \frac{1}{4}$ and $\tan A < 0$, find the exact value of $\sin(90 - A) \cdot \cos(90 - A)$.

B) If $\sin^2(\frac{\pi}{9}) + \sin^2(\frac{2\pi}{9}) + \sin^2(\frac{3\pi}{9}) + \sin^2(\frac{4\pi}{9}) = \frac{a}{b}$, then what is the value of

$$\cos^2(\frac{\pi}{9}) + \cos^2(\frac{2\pi}{9}) + \cos^2(\frac{3\pi}{9}) + \cos^2(\frac{4\pi}{9}) \ ?$$

Express your answer as a single simplified fraction.

C) The figure in the diagram consists of 4 equilateral triangles each with side of length 6. A square pyramid is formed by joining sides $\overline{AB}$ and $\overline{BC}$. Let θ be the angle each face makes with the base. Find $\sin(\theta)$. If necessary, express your answer as a simplified radical.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006
ROUND 6 PLANE GEOMETRY: ANGLES, TRIANGLES AND PARALLELS

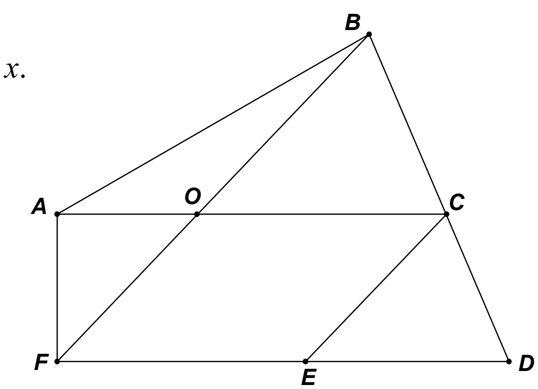
ANSWERS

A) ________________________________ °

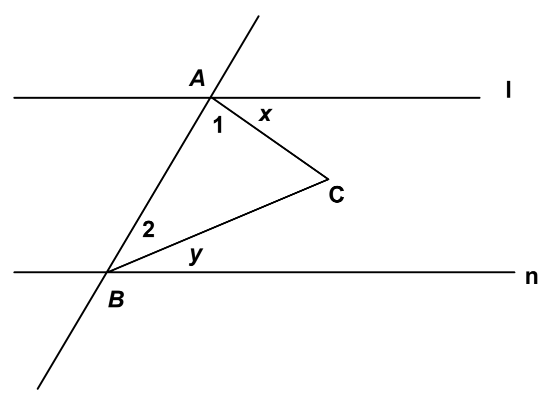
B) y = ______________________________

C) _________________________________

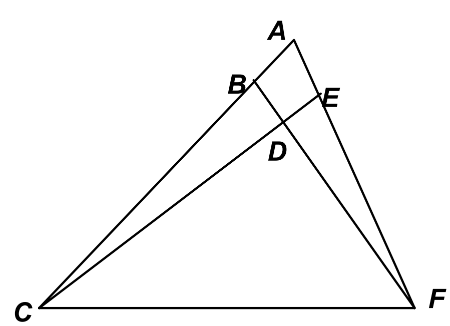
A) Let $x = m\angle FAB$.
If $OC = FE$, $EC = ED$, $m\angle OCE = 42°$, $m\angle D = 69°$, $m\angle AFE = 93°$ and $m\angle ABO = 20°$, determine the value of x.
The diagram is not necessarily drawn to scale.



B) If $l \parallel n$ and the measure of angle C is three times the sum of the measures of angles 1 and 2, find y in terms of x.



C) In ΔACF, $AB = AE$, $AE = EF/3$, $m\angle ACE = m\angle AFB$, $BC = 6$ and $AC = CF/2 + 6$. Find CF.
The diagram is not necessarily drawn to scale.
Your answer must be exact and expressed in simplified form.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ____

B) _______________________mph E) ____________________________

C) ____________________________ F) ____________________________

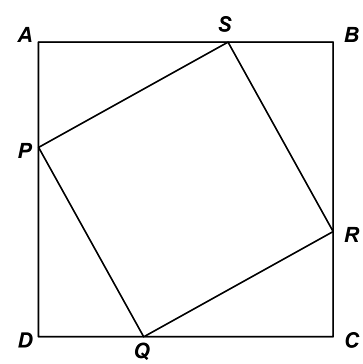
A) Given: $z = 3 + 4i$

For some complex constant c, $\bar{z} + \frac{1}{z} = \sqrt{z} + c$.

If $\sqrt{z}$ denotes a complex number in quadrant 1, determine the value of c.

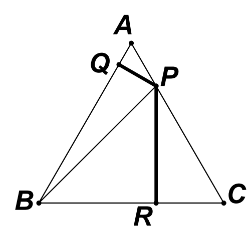
Note: $\bar{z}$ denotes the conjugate of z.

B) The upstream rate of an amateur kayaker is 80% of his downstream rate. If he kayaks 6 miles upstream and drops off some maps and immediately returns to his original starting point downstream in a total of 3 hours, determine the kayak's rate in still water. Assume no loss of time in dropping off the maps and turning around.



C) Squares $ABCD$ and $PQRS$ have areas 144 and 128 respectively. Determine $SA^2 + SB^2 + SC^2 + SD^2$.

D) Factor the polynomial $-x^{10} + x^4 + x - x^7$ over the integers.

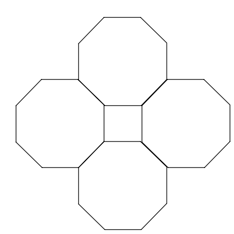


E) Given: ΔABC is equilateral with side of length 6

$m\angle PBC = 45°$, $\overline{PQ} \perp \overline{AB}$ and $\overline{PR} \perp \overline{BC}$. Determine $PQ + PR$.

F) A regular polygon of m sides is exactly enclosed by m regular polygons of n sides each, as illustrated for $m = 4$ and $n = 8$.

Specify all <u>other</u> ordered pairs (m, n) for which this statement is true?



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006 ANSWERS

Round 1 Algebra 2: Complex Numbers (No Trig)

A) $5 + 14i$ B) $1 + i$ C) $-138 - (122\sqrt{3})i$ or equivalent

Round 2 Algebra 1: Anything

A) -1.4 B) 11, -7 C) 1947

Round 3 Plane Geometry: Area of Rectilinear Figures

A) 492 B) 4 : 3 C) $12\sqrt{3} - 18$ or $6(2\sqrt{3} - 3)$

Round 4 Algebra 1: Factoring and its Applications

A) $a(a-3)(a-2)(a-1)$ B) $x(x+4)(x-2)$ C) $x = -y - \sqrt{1+y^2}$

Round 5 Trig: Functions of Special Angles

A) $-\frac{\sqrt{15}}{16}$ B) $\frac{4b-a}{b}$ C) $\frac{\sqrt{6}}{3}$

Round 6 Plane Geometry: Angles, Triangles and Parallels

A) 109 B) $y = 135 - x$ C) 4

Team Round

A) $1.12 - 5.16i$ (or $\frac{28-129i}{25}$) B) 4.05 mph C) 544

D) $x(1+x)^2(1-x)(1-x+x^2)^2(1+x+x^2)$

E) $3\sqrt{3}$

F) (3, 12), (6, 6), (10, 5)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006 SOLUTION KEY

Round 1

A) $i(2 + 3i)(1 - 4i) = i(2 - 8i + 3i - 12i^2) = i(2 - 5i + 12) =$ **<u>5 + 14i</u>**

B) Let $\sqrt{2i} = \sqrt{0+2i} = \sqrt{(a+bi)^2} = \sqrt{(a^2-b^2)+(2ab)i} \rightarrow a^2 - b^2 = 0$ and $ab = 1$
Thus, $b > 0 \rightarrow b = 1$ and $a = 1 \rightarrow$ **<u>1 + *i*</u>**

C) $(1-i\sqrt{3})^2 = -2 -2i\sqrt{3} = -2(1 + i\sqrt{3})$
$(1-i\sqrt{3})^4 = [-2(1 + i\sqrt{3})]^2 = 4(-2 + 2i\sqrt{3}) = -8(1 - i\sqrt{3})$
$(1-i\sqrt{3})^8 = [-8(1 - i\sqrt{3})]^2 = 64(-2 - 2i\sqrt{3}) = -128(1 + i\sqrt{3})$
Thus, the sum is $(-2 - 8 - 128) + (-2 + 8 - 128)i\sqrt{3} =$ **<u>$-138-(122\sqrt{3})i$</u>**

Round 2

A) $4^2+3\cdot 7-8^{-2}\cdot\frac{2^8}{5\cdot 9}\div\frac{1}{4\cdot 3}\cdot 6^2 = 16 + 21 - \frac{1}{2^6}\cdot\frac{2^8}{5\cdot 3^2}\cdot 2^2\cdot 3\cdot 2^2\cdot 3^2 = 37 - \frac{2^6\cdot 3}{5} = 37 - \frac{192}{5}$
$= 37 - 38.4 =$ **<u>-1.4</u>**

B) Let $a = \left(\frac{1+x}{2}\right)$ Think $a^2 - 3a - 18 = 0 \rightarrow (a - 6)(a + 3) = 0 \rightarrow a = 6$ or -3
Substituting for a, $1 + x = 12$ or $-6 \rightarrow x =$ **<u>11 or -7</u>**

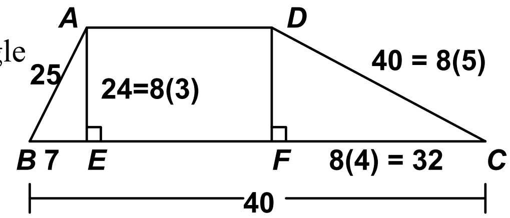
C)

	Now	In year of my birth
Me	x	0
Ramanujan	$2x + 1$	$x + 1$

According to the chart, $x + (2x + 1) = 3x + 1 = 178 \rightarrow x = 59 \rightarrow 2006 - 59 =$ **<u>1947</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006 SOLUTION KEY

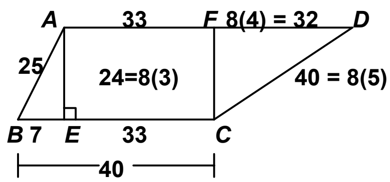
Round 3

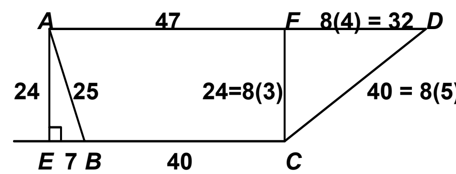


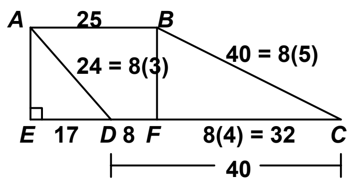
A) Drop perpendiculars from A and D to base $\overline{BC}$, creating a rectangle and two special right triangles as indicated in the diagram.
$EF = 40 - (7 + 32) = 1 \rightarrow AD = 1$. Thus, Area(trapezoid) =
$\frac{1}{2}h(b_1 + b_2) = \frac{1}{2}(24)[40+1] =$ **<u>492</u>**.

Failing to specify that $AD < BC$ and that $\overline{AD}$ and $\overline{BC}$ are bases, allows additional solutions.

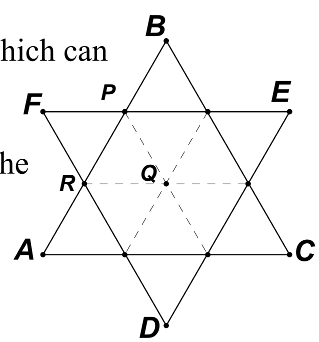
$\frac{1}{2}(24)[40+33+32] =$ **<u>1260</u>** $\qquad \frac{1}{2}(24)[40+47+32] =$ **<u>1428</u>** $\qquad \frac{1}{2}(24)[25+40+17] =$ **<u>984</u>**



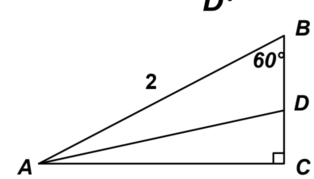




Are there others?



B) Note that the intersection of the two equilateral triangles is a regular hexagon, which can be subdivided into 6 congruent equilateral triangles by drawing the 3 indicated diagonals. It's easy to argue that $FPQR$ is a parallelogram and, therefore, $\Delta FPR \cong \Delta QRP$ and all 12 equilateral triangles are congruent. Thus, the ratio of the area of the entire star to the area of the original ΔABC is 12 : 9 = **<u>4 : 3</u>**



C) Triangles ADC and ADB have the same altitude from point A and, therefore, their areas are in the ratio of bases DC and DB.

By the angle bisector theorem, $\frac{DC}{\sqrt{3}} = \frac{DB}{2} \rightarrow \frac{DC}{DB} = \frac{\sqrt{3}}{2}$

$\text{Area}(\Delta ADC) + \text{Area}(\Delta ADB) = \sqrt{3}x + 2x = 6 \rightarrow x = \frac{6}{2+\sqrt{3}} = 6(2 - \sqrt{3})$

and Area(ΔADC) = $\sqrt{3}x$ = **<u>$12\sqrt{3}$ - 18</u>** or **<u>$6(2\sqrt{3}$ - 3)</u>**

Round 4

A) As the difference of perfect squares, $(a^2 - 3a + 1)^2 - 1 = (a^2 - 3a)(a^2 - 3a + 2)$
= **<u>$a(a-3)(a-2)(a-1)$</u>** - in any order

B) Let $A = (x + 1)$. Then, grouping in pairs, $A^3 - A^2 - 9A + 9 = A^2(A - 1) - 9(A - 1)$
$= (A - 1)(A^2 - 9) = (A - 1)(A + 3)(A - 3)$ Substituting for A, we have **<u>$x(x + 4)(x - 2)$</u>**.

C) Treat the equation as a quadratic equation in x and complete the square.

$x(x + 2y) = 1 \rightarrow x^2 + (2y)x + \mathbf{y^2} = 1 + \mathbf{y^2} \rightarrow (x + y)^2 = 1 + y^2 \rightarrow x + y = \pm\sqrt{1+y^2}$

$x = -y \pm\sqrt{1+y^2}$

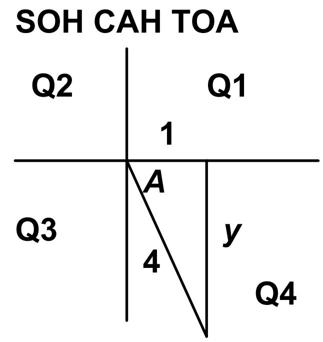
Since $\sqrt{1+y^2} > \sqrt{y^2} = |y|$, it follows that $\sqrt{1+y^2} > y$ or $\sqrt{1+y^2} - y > 0$ and the only solution giving $x < 0$ is $x =$ **<u>$-y - \sqrt{1+y^2}$</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006 SOLUTION KEY

Round 5

A) $\angle A$ must be located in quadrant 4, since the cosine is positive and the tangent is negative. $y^2 = 16 - 1 = 15$ and $y < 0 \rightarrow y = -\sqrt{15}$
Using cofunction identities (or complementary angle relationships),

$\sin(90 - A) \cdot \cos(90 - A) = \cos(A) \cdot \sin(A) = \dfrac{\sin A}{4} = \underline{\dfrac{\mathbf{-\sqrt{15}}}{\mathbf{16}}}$

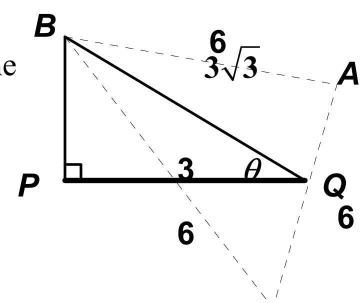


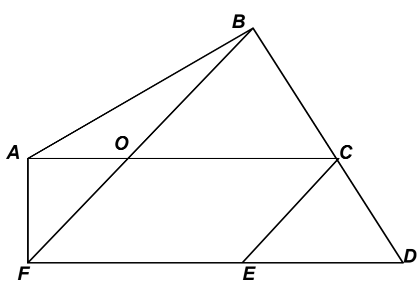
B) Since $\sin^2\theta + \cos^2\theta = 1$,

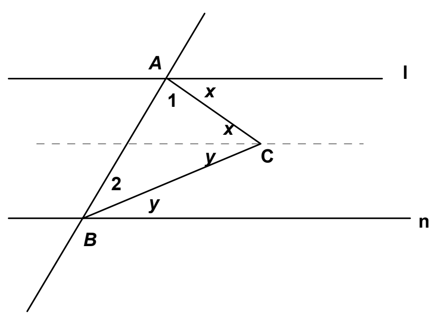
$$\cos^2(\tfrac{\pi}{9}) + \cos^2(\tfrac{2\pi}{9}) + \cos^2(\tfrac{3\pi}{9}) + \cos^2(\tfrac{4\pi}{9}) \ +$$
$$\sin^2(\tfrac{\pi}{9}) + \sin^2(\tfrac{2\pi}{9}) + \sin^2(\tfrac{3\pi}{9}) + \sin^2(\tfrac{4\pi}{9}) = 4$$

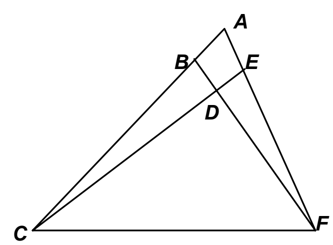
and $\cos^2(\tfrac{\pi}{9}) + \cos^2(\tfrac{2\pi}{9}) + \cos^2(\tfrac{3\pi}{9}) + \cos^2(\tfrac{4\pi}{9}) = 4 - \dfrac{a}{b} = \underline{\dfrac{\mathbf{4b - a}}{\mathbf{b}}}$

C) The slant height of the pyramid is the altitude from B in each equilateral triangle. Let P denote the center of the square base and Q the foot of one of these altitudes. Then $AB = 6 \rightarrow BQ = 3\sqrt{3}$ and $PQ = 3$
Using the Pythagorean Theorem, $BP^2 = 27 - 9 = 18 \rightarrow BP = 3\sqrt{2}$
Thus, the angle formed by a face with the base of the pyramid is $\angle\theta$ as indicated in the diagram at the right. $\sin(\theta) = \dfrac{3\sqrt{2}}{3\sqrt{3}} = \boxed{\dfrac{\sqrt{6}}{3}}$









Round 6

A) Since ΔCED is an isosceles triangle with a base angle of 69°, its vertex angle CED has a measure of 42°. Thus, $\overline{OC}$ is both parallel and congruent to $\overline{FE}$, forcing $OCEF$ to be a parallelogram. Since opposite angles in a parallelogram are congruent, $m\angle OFE = 42°$
$\rightarrow m\angle AFO = 93 - 42 = 51°$. Finally, in ΔFAB, $x = m\angle FAB = 180 - (20 + 51) =$ **<u>109</u>**

B) $4(m\angle 1 + m\angle 1) = 180 \rightarrow m\angle C = 135°$. Draw a line thru point C parallel to n.
Since alternate interior angles of parallel lines are congruent, $x + y = 135 \rightarrow y =$ **<u>135 - *x*</u>**

C) Since $\Delta ACE \cong \Delta AFB$ (by SAA), $AC = AF$ and ΔACF is isosceles w/base CF.
$BC = 6 \rightarrow EF = 6 \rightarrow AE = 2 \rightarrow AC = 8$
Thus, $8 = CF/2 + 6 \rightarrow CF =$ **<u>4</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006 SOLUTION KEY

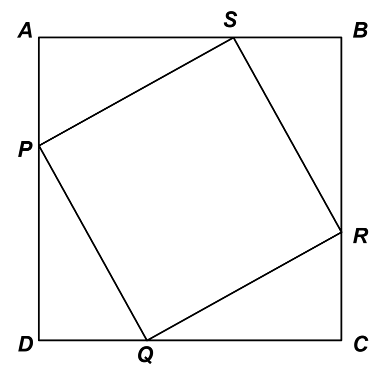
Team Round

A) $(3-4i)+\dfrac{3-4i}{25} = 3.12 - 4.16i = \sqrt{z} + c$

To determine $\sqrt{z}$, $\sqrt{3+4i} = \sqrt{(a+bi)^2} = \sqrt{(a^2-b^2)+2abi}$ → $a^2 - b^2 = 3$ and $ab = 2$
→ $(a, b) = (2, 1)$ → $3.12 - 4.16i = 2 + i + c$ → $c =$ **$\underline{1.12 - 5.16i}$**

B) Let (k, c) denote the rowing rate of the kayaker in still water and the current of the stream.
$(k - c) = (4/5)(k + c)$ → $5k - 5c = 4k + 4c$ → $k = 9c$ or $k : c = 9:1$
$R_{up}: R_{down} = (9x - x) + (9x + x) = 8x : 10x$ → $T_{up} : T_{down} = 10x : 8x$
$T_{up} + T_{down} = 18x = 3$ → $x = 1/6$ → $T_{up} = 5/3$ hour and $T_{down} = 4/3$ hour
Upstream rate: $k - c = 6/(5/3) = 18/5$ mph Downstream rate: $k + c = 6/(4/3) = 18/4$ mph
Solving simultaneously, $2k = 81/10$ → $k = 81/20 =$ **$\underline{4.05}$** mph.

C) Let $SB = x$ → $BR = 12 - x$
With no loss of generality, assume $SB < SA$ as appears to be the case in the given diagram.
$x^2 + (12 - x)^2 = 128$ → $x^2 - 12x + 8 = 0$ → $SB = 6 - 2\sqrt{7}$
→ $SB^2 = 64 - 24\sqrt{7}$
$SD^2 = SA^2 + AD^2 = (6 + 2\sqrt{7})^2 + 12^2 = 64 + 24\sqrt{7} + 144 =$
$208 + 24\sqrt{7}$



It is true, in general, for any point S in the plane of a square $ABCD$, that

$$SA^2 + SC^2 = SB^2 + SD^2$$

Thus, $SA^2 + SB^2 + SC^2 + SD^2 = 2(SB^2 + SD^2) = 2(64 - 24\sqrt{7} + 208 + 24\sqrt{7}) =$ **$\underline{544}$**

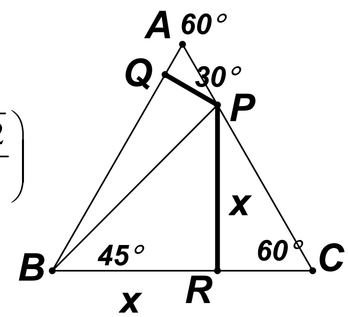
D) $-x^{10} + x^4 + x - x^7 = x(-x^9 + x^3 + 1 - x^6) = x(x^3(1 - x^6) + (1 - x^6)) = x(1 + x^3)(1 - x^6)$
$= x(1 + x^3)^2(1 - x^3) = x[(1 + x)(1 - x + x^2)]^2(1 - x)(1 + x + x^2)$
$=$ **$\underline{x(1 + x)^2(1 - x)(1 - x + x^2)^2 (1 + x + x^2)}$** – or equivalent

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006 SOLUTION KEY

Team Round – continued

E) Method 1: Let $BR = PR = x$. Then $RC = \frac{x}{\sqrt{3}} \rightarrow PC = \frac{2x}{\sqrt{3}} \rightarrow AP = 6 - \frac{2x}{\sqrt{3}}$

$\rightarrow PQ = \frac{1}{2}(6 - \frac{2x}{\sqrt{3}})\sqrt{3} = 3\sqrt{3} - x$ Thus, $PQ + PR = 3\sqrt{3} - x + x =$ $\underline{\mathbf{3\sqrt{3}}}$



Method 2: Using the law of sine on ΔBQP, $PQ = x\sqrt{2}\sin 15^\circ = x\sqrt{2}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$

$= \left(\frac{\sqrt{3}-1}{2}\right)x \rightarrow PQ + PR = \left(\frac{\sqrt{3}+1}{2}\right)x$.

$x + \frac{x}{\sqrt{3}} = 6 \rightarrow x = 3(3-\sqrt{3}) \rightarrow \left(\frac{\sqrt{3}+1}{2}\right) \cdot 3(3-\sqrt{3}) = \frac{3}{2}(\sqrt{3}+1)(3-\sqrt{3}) =$ $\underline{\mathbf{3\sqrt{3}}}$

Method 3: Using the theorem

"From any point on or in the interior of an equilateral triangle, **the sum of the altitudes to each side is equal to the length of an altitude of the equilateral triangle**."

$PQ + PR = \frac{1}{2} \cdot 6 \cdot \sqrt{3} =$ $\underline{\mathbf{3\sqrt{3}}}$

F) At each vertex of the enclosed regular polygon, you must have three angles, two from the surrounding regular polygons and one the enclosed regular polygon. Thus, for the square-octagon, (360 – 90)/2 = 135 gives the measure of the interior angle of the surrounding regular polygons and the exterior angles are 45°. Since 360 is divisible by 45, we have a solution! $360/n = 45 \rightarrow n = 8$ (i.e. the surrounding regular polygons are octagons). The results of repeating this scenario for different values of m are shown in the following chart. There are only 3 other possibilities.

enclosed polygon			**surrounding polygons**		
sides	interior	interior	exterior	# sides	
	angle	(360 - *int*)/2	180 - θ	360/*E*	
m	*int*	θ	*E*	***n***	**(*m*, *n*)**
3	60	150	30	12	**(3, 12)**
4	90	135	45	8	(4, 8)
5	108	126	54	*reject*	
6	120	120	60	6	**(6, 6)**
7	***REJECT***				
8	135	112.5	67.5	*reject*	
9	140	110	70	*reject*	
10	144	108	72	5	**(10, 5)**

How do we know there are no more ordered pairs awaiting discovery?

The interior angle in an m-sided regular polygon measures $\frac{180(m-2)}{m}$°.

Algebraically representing the technique described above to find n given m we have:

$$n = \frac{360}{E} = \frac{360}{180 - \left(\frac{360 - \frac{180(m-2)}{m}}{2}\right)}$$

Carefully simplifying this expression,

$$\frac{360}{180 - \left(\frac{360 - \frac{180(m-2)}{m}}{2}\right)} = \frac{2}{1 - \left(\frac{2 - \frac{(m-2)}{m}}{2}\right)} = \frac{2}{1 - (1 - \frac{m-2}{2m})} = \frac{4m}{m-2}$$

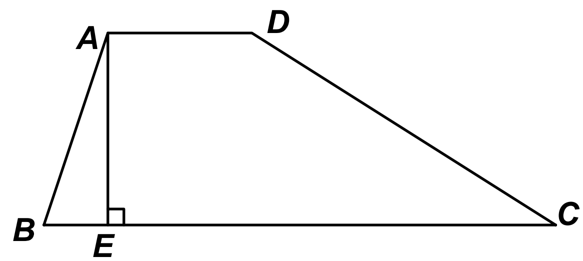
Using long division, we have $n = 4 + \frac{8}{m-2}$ and, for $m > 10$,

$(m - 2)$ will never be a divisor of 8 and the search stops.

Addendum:

The original statement of question A in round 3 was problematic.

Find the exact area of trapezoid *ABCD*, given $AB = 25$, $BC = DC = 40$, $AE = 24$
The diagram is not necessarily drawn to scale.



Answers of 492 and 1260 were accepted.