MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 – DECEMBER 2006
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINE AND COSINE

ANSWERS

A) ________________________________

B) ________________________________

C) (________ , ________)

A) An equilateral triangle has sides of length 6.
Points A, B, C, D, E and F are trisection points of the sides.
What is the exact length of a segment that

- connects two of these points not on the same side of the triangle and
- is not parallel to any sides of the triangle?

Express your answer as an exact value in simplified form.

B) In ΔABC, $m\angle B = 150°$, $a = BC = 10$ and $b = AC = 15$.
Determine the exact value of $\sin(B + C)$.

C) The perimeter of a regular n-sided polygon is p. A simplified expression for the apothem of the polygon in terms of p and n may be written in the form $\dfrac{p\cot(\frac{X}{n})}{Yn}$, where $\dfrac{X}{n}$ is the degree-measure of an angle whose vertex is at the center of the regular polygon.
Determine the ordered pair (X, Y).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006
ROUND 2 ARITHMETIC/ ELEMENTARY NUMBER THEORY

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) W is the units digit of a base 10 integer N. When N is raised to a positive integer power, the units digit may equal exactly two distinct values. Find all values of W for which this is true.

B) A positive integer has exactly 8 positive factors. Two of them are 77 and 119. Find this integer.

C) The 4-digit base 10 positive integer $ABBA$ (where $A > 0$) is divisible by 12. A and B are distinct digits. Find the sum of all integers satisfying this condition.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006
ROUND 3 ANALYTIC GEOMETRY OF THE STRAIGHT LINE

ANSWERS

A) _______________ : _______________

B) ________________________________

C) $x =$ __________ , $y =$ ____________

A) The segment connecting $A(1, 8)$ and $B(6, -2)$ crosses the x-axis at point P. Determine the ratio BP: AP.

B) Given: $A(0, 2006)$ and $B(4250, 0)$
The point $C(p, q)$ is the point on $\overline{AB}$ with integer coordinates that is closest to, but different from, point A.
The point $D(r, s)$ is the point on $\overline{AB}$ with integer coordinates that is closest to, but different from, point B.

Find $p + q + r + s$

C) ΔPQR has vertices at $P(-12, 0)$, $Q(14, 0)$ and $R(2, 42)$. There is a single point $S(x, y)$ in the interior of ΔPQR that is equidistant from points P, Q and R. Find the numerical values of x and y.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

***** NO CALCULATORS ON THIS ROUND ****

A) If A and B are the roots of $3x^2 - 22x + 27 = 0$, then what is the exact value of $\log_3 A + \log_3 B$?

B) If $a = \log_8 45$ and $b = \log_2 7.5$ and $\log_3 2 = \dfrac{1}{c}$, then find a simplified expression for b in terms of a and c.

C) Determine the domain of the real-valued function f: $f(x) = \log_3\left(\dfrac{(x^2 + 3x - 4}{(2x-3)^2}\right)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) The ratio of the sum of three consecutive positive odd integers to the sum of the next three larger consecutive even integers is 3 : 4. The smallest even integer is 1 more than the largest odd integer. Determine the sum of these 6 integers.

B) On a certain test, the average grade of those who passed was 84%, while the average grade of those who failed was 54%. If the overall average of the group was 78%, what part of the group passed? Express your answer as a simplified proper fraction.

C) x varies jointly as y and z and inversely as the square of w.
When $(w,\ y,\ z) = (3, 12, 15)$, $x = 100$.
Find wx^2 when $w : x : y : z = 1 : 2 : 3 : 4$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

ANSWERS

A) _______________ : _______________

B) ________________________________

C) ________________________________

A) In rhombus $ABCD$, $BD = 40$ and $AC = 42$ and E is the point of intersection of the diagonals. Determine the ratio of the numerical value of the area of ΔDEC to the numerical value of the perimeter of $ABCD$.

B) In a regular polygon with k sides and consecutive vertices $A_1, A_2, \ldots, A_k$, for some value of i, $\overline{A_iA_{i+3}} \parallel \overline{A_{i+1}A_{i+2}}$ forming an isosceles trapezoid with a pair of $18°$ base angles. Determine the value of k.

C) A regular octagon is formed by cutting off the corners of a square whose sides have length k. Determine the exact positive difference between the perimeter of the square and the perimeter of the regular octagon in terms of k, expressed as a simplified radical expression.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006
ROUND 7 TEAM QUESTIONS

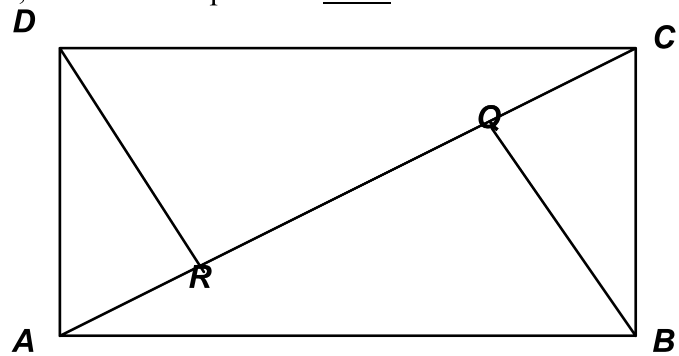
ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) ____________________________

C) (___________ , ___________) F) ____________ : ____________

A) Rectangle $ABCD$ has an area of 300 units^2, a perimeter of P units, $\overline{DR} \perp \overline{AC}$, $\overline{BQ} \perp \overline{AC}$ and A, C, R and Q are collinear. If $RQ = 7$, determine all possible exact values of P.



B) What is the largest power of 12 which is a factor of 732! ?

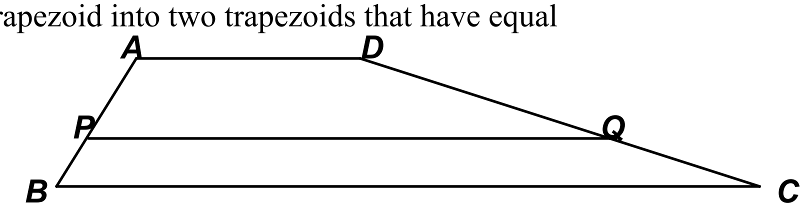
C) Find the point of intersection of the system of intersecting lines represented by the equation

$$2x^2 + xy - 6y^2 + 7y - 2 = 0$$

D) Find all values of a for which $\log_{10} \dfrac{2a-1}{2-a} \le 0$

E) In a 20 km race, four runners, A, B, C and D each run at different, but uniform rates of speed. A beats B by 2 km, A beats C by 5 km and A beats D by k km, where $k > 5$. Determine the value of k, if C beats D by 1 km.

F) The bases of trapezoid $ABCD$ are 6 and 15 and the nonparallel sides are 4 and 8. $\overline{PQ}$, a segment parallel to the bases, divides the trapezoid into two trapezoids that have equal perimeters. Determine the ratio $PB : PA$.



CONTEST 3 - DECEMBER 2006 ANSWERS

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

A) $2\sqrt{3}$ B) $\frac{1}{3}$ C) (180, 2) $\left[a = \dfrac{p\cot(\frac{180}{n})}{2n} \right]$

Round 2 Arithmetic/Elementary Number Theory

A) 4 and 9 only B) 1309 [$7^1 \cdot 11^1 \cdot 17^1$] C) 35772

Round 3 Analytic Geometry of the Straight Line

A) 1 : 4 B) 6256 [C(125, 1947) D(4125, 59)] C) $x = 1$, $y = 19$

Round 4 Alg 2: Log and Exponential Functions

A) 2 B) $3a - 1 - c$ C) $x < -4$ or $1 < x < 3/2$ or $x > 3/2$ or equivalent

Round 5 Alg 1: Ratio, Proportion or Variation

A) 105 [(13+15+17):(18+20+22)→ 45:60] B) 4/5 C) 108,000

Round 6 Plane Geometry: Polygons (no areas)

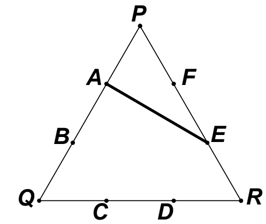
A) 105 : 58 B) 20 C) $4k(3 - 2\sqrt{2})$

Team Round

A) 70 B) 363 C) (-1/7, 4/7)

D) $½ < a \leq 1$ E) 5.75 (or 23/4) F) 1 : 7

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006 SOLUTION KEY

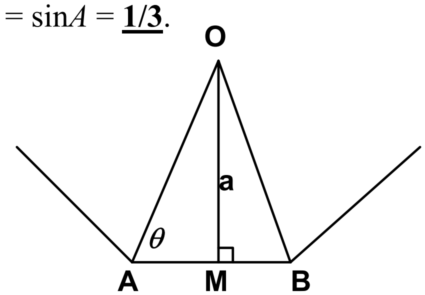


Round 1

A) Using the law of cosine, $AE^2 = 2^2 + 4^2 - 2(2)(4)\cos 60°$
$= 20 - 16(1/2) = 12 \rightarrow PQ = \underline{\mathbf{2\sqrt{3}}}$.

B) Using the law of sine, $\dfrac{\sin A}{10} = \dfrac{\sin 150°}{15} \rightarrow \sin A = \dfrac{2}{3}\cdot\dfrac{1}{2} = \dfrac{1}{3}$

The given information (2 sides and the non-included angle) is the ambiguous case, but since $\angle B$ is obtuse, there is exactly one triangle satisfying the given conditions.
Since $A + B + C = 180°$, $B + C = 180 - A$ and $\sin(B + C) = \sin(180 - A) = \sin A = \underline{\mathbf{1/3}}$.



C) $m\angle BOA = (360/n)° \rightarrow \theta = 90 - 180/n$ and $AM = ½(p/n) = p/(2n)$
$\tan(\theta) = (OM)/(AM) = (2na)/p \rightarrow a = p\tan(\theta)/(2n)$
Replacing the angle θ by its complement and the trig function by its cofunction, $\rightarrow \dfrac{p\cot(\frac{180}{n})}{2n} \rightarrow (X, Y) = \underline{\mathbf{(180, 2)}}$.

Round 2

A) The rightmost digit of positive integer powers of 4 are alternately 4 and 6.
The rightmost digit of positive integer powers of 9 are alternately 9 and 1
2 → 2,4,8,6 ***3*** → 3,9,7,1 *7* → 7,9,3,1 ***8*** → 8,4,2,6 ***0*** → 0 ***1*** → 1 ***5*** → 5 ***6*** → 6
Thus, d may be **4 or 9**.

B) $77 = 7 \cdot 11$ and $119 = 7 \cdot 17$

If $N = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdot \ldots \cdot p_k^{e_k}$, then the number of factors of N is given by $(e_1 + 1)(e_2 + 1)\cdot\ldots\cdot(e_k + 1)$.
Note: The number of positive factors of N does not depend on what its prime factors are, only how many of each there are.

Thus, $7 \cdot 11 \cdot 17 = 7^1 \cdot 11^1 \cdot 17^1 = \underline{\mathbf{1309}}$ has $(1 + 1)(1 + 1)(1 + 1) = 8$ positive factors.
(The factors are: 1, 7, 11, 17, $7 \cdot 11 = 77$, $7 \cdot 17 = 119$, $11 \cdot 17 = 187$, $7 \cdot 11 \cdot 17 = 1309$)

C) Since $12 = 3 \cdot 4$, a number divisible by 12 is divisible by 3 and 4 and vice versa.
Divisibility Rules
÷ by 3: check the sum of the digits – it must be divisible by 3
÷ by 4: check the number formed by the rightmost 2 digits – it must be divisible by 4

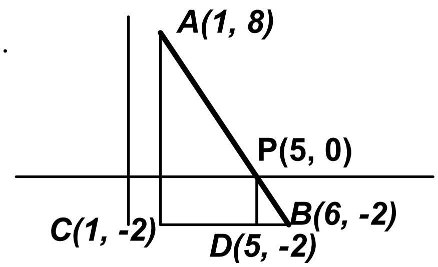
The digit sum $2(A + B)$ must be divisible by 3 → $(A + B)$ must be divisible by 3
Thus, for some integer k, $A + B = 3k$ or $B = 3k - A$

The positive two-digit number $10B + A = 10(3k - A) + A = 30k - 9A = 3(10k - 3A)$ must be a multiple of 4, so $(10k - 3A)$ must be a multiple of 4 and $k \geq 1$.
Remember A and B denote digits in base 10 and, therefore, are restricted to 0, 1, …, 9.
$k = 1 \rightarrow B = 3 - A$ and $10 - 3A = 4j \rightarrow A = 2, B = 1 \rightarrow$ **2112**
$k = 2 \rightarrow B = 6 - A$ and $20 - 3A = 4j \rightarrow A = 4, B = 2 \rightarrow$ **4224**
$k = 3 \rightarrow B = 9 - A$ and $30 - 3A = 4j \rightarrow A = 2$ or 6 and $B = 7$ or $3 \rightarrow$ **2772** or **6336**
$k = 4 \rightarrow B = 12 - A$ and $40 - 3A = 4j \rightarrow A = 4$ or 8 and $B = 8$ or $4 \rightarrow$ **4884** or **8448**
$k = 5 \rightarrow B = 15 - A$ and $50 - 3A = 4j \rightarrow A = 2$ or 6
and only $A = 6$ produces a legal value for $B \rightarrow$ **6996**
$k = 6 \rightarrow B = 18 - A$ and $60 - 3A = 4j \rightarrow A = 4$ or 8 and neither produces a legal value for B
and the search stops. 2112 + 2772 + 4224 + 4884 + 6336 + 6996 + 8448 = **35772**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006 SOLUTION KEY

Round 3

A) Method 1:



Since the equation of $\overleftrightarrow{AB}$ is $y = -2x + 10$, the x-intercept is at (5, 0). From the diagram, it is clear that $\overline{PD} \parallel \overline{AC}$ and the required ratio is the same as $BD : CD =$ **<u>1 : 4</u>**.

Method 2:
Alternately, after finding the x-intercept P, using the distance formula, you could compute the distance between P and B ($\sqrt{5}$) and the distance between A and P ($\sqrt{80} = 4\sqrt{5}$) → **<u>1 : 4</u>**.

Method 3 (**<u>without</u>** finding the equation or x-intercept of $\overleftrightarrow{AB}$):
Let X denote the x-intercept of the <u>vertical</u> line $\overleftrightarrow{AC}$. Clearly, the coordinates of X are (1, 0) and $CX : AX = 1 : 4$ → $BP : AP =$ **<u>1 : 4</u>** (since $\Delta BPD \sim \Delta BAC$).

B) The slope of $\overline{AB}$ is $\dfrac{-2006}{4250} = \dfrac{-2(17)(59)}{2(17)(125)} = \dfrac{-59}{125}$

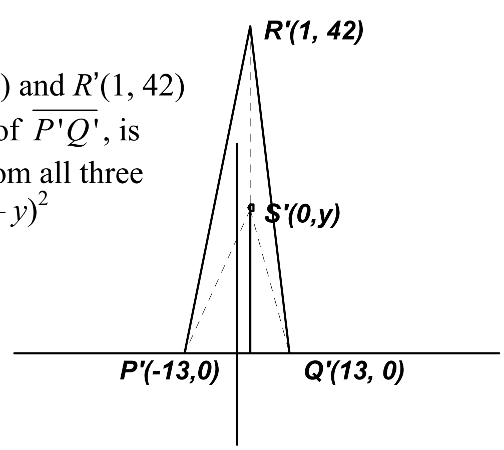
Points with integer coordinates (i.e. lattice points), may be determined by starting at A and increasing the x-coordinate by 125 and decreasing the y-coordinate by 59 or alternately, starting at B and decreasing the x-coordinate by 125 and increasing the y-coordinate by 59.

Both strategies produce: (0, 2006) **(125, 1947)**, (250, 1888) … **(4125, 59)**, (4250, 0)
125 + 1947 + 4125 + 59 = **<u>6256</u>**.

In fact, suppose the slope of $\overline{AB}$ were $\dfrac{-a}{b}$, where a and b are positive integers.
Then $C(b, 2006 - a)$ and $D(4250 - b, a)$ → $p + q + r + s = 4250 + 2006 = 6256$ and it wasn't even necessary to find the slope of $\overline{AB}$. In the worst case scenario, if the slope fraction could not be reduced, point C would coincide with point B and D would coincide with point A.

C) Method 1:
Point S is the intersection of the perpendicular bisectors of the sides of ΔPQR.
The perpendicular bisector of $\overline{PQ}$ is the vertical line **<u>$x = 1$</u>**.
The perpendicular bisector of $\overline{PR}$ is $x + 3y = 58$.
→ $3y = 57$ → **<u>$y = 19$</u>**.

Method 2:
Shifting each vertex of ΔPQR left 1 unit. $P'(-13, 0)$, $Q'(13, 0)$ and $R'(1, 42)$
Clearly, point $S'(0, y)$, a point on the perpendicular bisector of $\overline{P'Q'}$, is equidistant from P' and Q'. To insure that it is equidistant from all three vertices, we require $(S'Q')^2 = (S'R')^2$ → $13^2 + y^2 = 1^2 + (42 - y)^2$
→ $169 + y^2 = 1 + 1764 - 84y + y^2$ → $84y = 1596$ → $y = 19$
→ $S'(0, 19)$ → $S(1, 19)$ → $x =$ **<u>1</u>**, $y =$ **<u>19</u>**.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006 SOLUTION KEY

Round 4

A) Note that A and B, the roots of the quadratic equation, are each positive numbers
$\left(\frac{22 \pm \sqrt{22^2 - 12(27)}}{6}\right)$ and we can let $x = \log_3 A + \log_3 B = \log_3(AB)$
But AB, the product of the roots of the quadratic, is given by the constant term divided by the lead coefficient → 27/3 = 9
Thus, $x = \log_3(9) =$ **2**.

B) $8^a = 2^{3a} = 45$ and $2^b = 7.5$ or $2^{b+1} = 15$. Dividing, $2^{3a-(b+1)} = 3$
→ $3a - b - 1 = \log_2 3 = \dfrac{1}{\log_3 2} = c$ → $b =$ **$3a - 1 - c$**

C) $(2x-3)^2 > 0$ for all x except 3/2. The critical points in the numerator of the argument $(x-1)(x+4)$ are 1 and -4. The product is positive for $x < -4$ or $x > 1$ and negative in between. Since the log of zero or negative values is undefined, the domain is restricted to $x < -4$ or $x > 1$ (excluding $x = 3/2$).

Round 5

A) Let x, $x + 2$ and $x + 4$ denote the three consecutive odd integers. Then the next three larger consecutive even integers are $x + 5$, $x + 7$ and $x + 9$.
$(3x + 6) : (3x + 21) = 3 : 4$ → $(x + 2) : (x + 7) = 3 : 4$ → $4x + 8 = 3x + 21$ → $x = 13$
$(13 + 15 + 17) + (18 + 20 + 22) = 45 + 60 =$ **105**

B) $(.84P + .54F)/(P + F) = .78$ → $6P = 24F$ → $P = 4F$
Part of group that passed $= P/(P + F) = 4F/(4F + F) =$ **4/5**

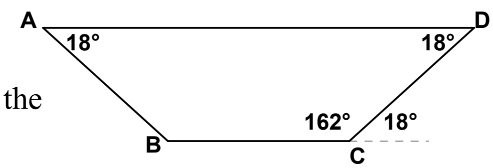
C) Substituting for x, y, z and w in $x = kyz/(w^2)$ → $k = 5$.
Let $w = n$, $x = 2n$, $y = 3n$ and $z = 4n$ → $(2n)(n^2) = 5(3n)(4n)$ → $n = 30$ and $wx^2 = 4n^3$
→ $4 \cdot 30^3 = 4(27000) =$ **108,000**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006 SOLUTION KEY

Round 6

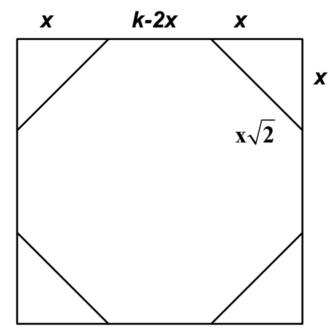
A) Since the diagonals of a rhombus are perpendicular and bisect each other, ΔDEC is a right triangle with legs of length 20 and 21. Using the Pythagorean Theorem (or a common Pythagorean triple), the side of the rhombus is 29. Thus, the required ratio is $\frac{1}{2}\cdot 20\cdot 21 : 4\cdot 29 \rightarrow$ **<u>105 : 58</u>**.

B) Refer to the 4 consecutive vertices A_i, A_{i+1}, A_{i+2} and A_{i+3} as A, B, C and D respectively. Since the other pair of 162° base angles are each interior angles of the regular polygon, the exterior angles measure 18°.



Thus, $\frac{360}{k} = 18 \rightarrow k =$ **<u>20</u>.**

C) Method 1:



$k - 2x = x\sqrt{2} \rightarrow x(2 + \sqrt{2}) = k \rightarrow x = \frac{k}{2+\sqrt{2}} = \frac{k(2-\sqrt{2})}{2}$

Thus, per(square) = $4k$ and per(octagon) = $8x\sqrt{2} = 4k(2-\sqrt{2})\sqrt{2}$
$= 8k(\sqrt{2}-1)$ and the positive difference is $4k - (8k(\sqrt{2}-1))$
$= 12k - 8k\sqrt{2} = \boxed{\mathbf{4k(3-2\sqrt{2})}}$.

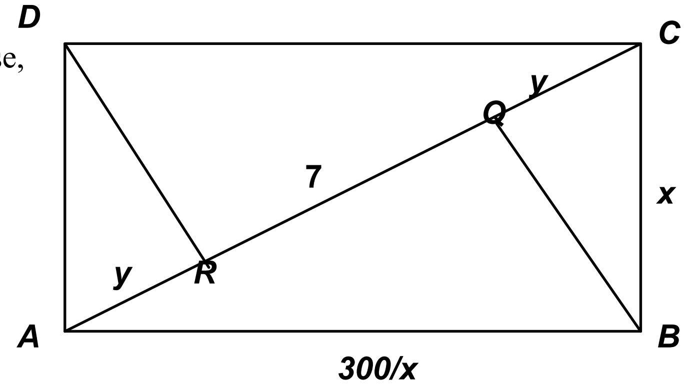
Method 2:
The triangles in the 4 corners are 45 – 45 – 90 triangles. If the sides of these triangles were 1, 1 and , $\sqrt{2}$, then the side of the square would be $2 + \sqrt{2}$, the perimeter of the square would be $4(2 + \sqrt{2})$ and the perimeter of the octagon would be $8\sqrt{2}$. Since the square has the larger perimeter, the positive difference is $8 - 4\sqrt{2}$. Applying a scale factor of $\frac{k}{2+\sqrt{2}}$ makes the side of the square k. Thus, the positive difference is $(8 - 4\sqrt{2})\cdot\frac{k}{2+\sqrt{2}} =$

$$4(2-\sqrt{2})\cdot\frac{k}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{4k(2-\sqrt{2})^2}{2} = 2k(4-4\sqrt{2}+2) = \boxed{\mathbf{4k(3-2\sqrt{2})}}.$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006 SOLUTION KEY

Team Round

A) In right ΔABC, $x^2 + (300/x)^2 = (2y + 7)^2$ and since $\overline{BQ}$ is an altitude to the hypotenuse, $x^2 = y(2y + 7)$.



Substituting in the 1st equation for x^2,

$y(2y + 7) + \dfrac{300^2}{y(2y+7)} = (2y + 7)^2$

$y^2(2y + 7)^2 + 300^2 = y(2y + 7)^3$

$= 4y^4 + 56y^3 + 245y^2 + 343y - 90000$

$= (y - 9)(4y^3 + 92y^2 + 1073y + 10000)$

Clearly $y = 9$ is a solution and since the coefficients of the cubic factor are all positive, there are no additional positive roots. Substituting in the 1st equation, $x^2 = 9(25) \rightarrow x = 15$. Thus, the only possible perimeter of rectangle $ABCD$ is $2(15 + 20) =$ **70**.

Alternative:

Let $BQ = z$. Then $\begin{cases} z^2 = y(y+7) \\ z(2y+7) = 300 \end{cases}$. Solving for z in the 2nd equation and substituting in the 1st, $y(y + 7)(2y + 7)^2 = 300^2 = 3^2 \cdot 2^4 \cdot 5^4$. By inspection, if $y = 9$, $(y + 7) = 16 = 2^4$ and $(2y + 7)^2 = 5^4$. Thus, $y = 9$ is a positive solution and, for $y > 9$, the left hand side $> 300^2$ and, for $0 < y < 9$, the left hand side is $< 300^2$. $y = 9 \rightarrow z = 12$, $x = 15$ and finally, $P =$ **70**.

B) We must count the total number of factors of 2 and of 3 in the product of the 732 consecutive integers denoted by 732!, since these are the only prime factors of 12.
In 732!, every 2nd integer is a multiple of 2, every 4th a multiple of 4, every 8th a multiple of 8, etc. Some multiples of 2 need to be counted once (e.g. 2, 6, 10, …), some twice (e.g. 4, 12, 20, …), some three times (e.g. 8, 24, 40, …) etc.
The multiples of 2: 2, 4, 6, 8, 10, 12, 14, 16, …., 732 [= 2(366)] - a total of 366 numbers
The multiples of 4: 4, 8, 12, 16, …, 732 [= 4(183)] - a total of 183 numbers
The multiples of 8: 8, 16, 16, …, 728 [= 8(91)] – a total of 91 numbers
The number of factors of 2 is equal to the following sum:

$$366 + 183 + 91 + 45 + 22 + 11 + 5 + 2 + 1 + 0 = 726$$

Start by dividing 732 by 2 and record the quotient. Continue dividing by 2 and recording the quotient (disregarding any remainder), until a quotient of zero is obtained. In the above sum,
5 → exactly 5 multiples of 128 (2^7) are less than 732, namely 128, 256, 384, 512 and 640.
2 → exactly 2 multiples of 256 (2^8) are less than 732, namely 256 and 512.
1 → only 1 multiple of 512 (2^9) is less than 732.
0 → no multiples of 1024 (2^{10}) are less than 732.

The powers of 3 can be counted similarly as

$$244 + 81 + 27 + 9 + 3 + 1 + 0 \rightarrow 365$$

Thus, $732! = 2^{726} \cdot 3^{365} \cdot$(a bunch of other primes raised to various powers)
Since $12 = 2^2 3^1$, twice as many 2s are needed as 3s to form factors of 12.
$2^{726}3^{365}$ […] $= 2^{2(363)}3^{363}3^2$ […] $= (2^2 3)^{363} \cdot 9 \cdot$ […] $= 12^{363} \cdot 9 \cdot$ […] → **363** factors of 12.

Team Round - continued

C) Method 1: Hammer and Tongs (Brute Force)
Consider $2x^2 + xy - 6y^2 + 7y - 2 = 0$ a quadratic equation in x, namely
$Ax^2 + Bx + C = 0 \leftrightarrow 2x^2 + yx + (-6y^2 + 7y - 2) = 0 \rightarrow A = 2, B = y$ and $C = -6y^2 + 7y - 2$
Applying the quadratic formula, $x = \dfrac{-y \pm \sqrt{y^2 - 4(2)(-6y^2 + 7y - 2)}}{4} = \dfrac{-y \pm \sqrt{49y^2 - 56y + 16}}{4}$
$= \dfrac{-y \pm \sqrt{(7y-4)^2}}{4} = \dfrac{-y \pm (7y-4)}{4}$
$\rightarrow x = \dfrac{6y-4}{4} = \dfrac{3y-2}{2} \rightarrow 2x - 3y + 2 = 0$ or
$x = \dfrac{-8y+4}{4} = -2y + 1 \rightarrow x + 2y - 1 = 0$

Thus, the equation in factored form is: $(2x - 3y + 2)(x + 2y - 1) = 0$
Solving $\begin{cases} 2x - 3y + 2 = 0 \\ x + 2y - 1 = 0 \end{cases} \rightarrow \begin{cases} 2x - 3y + 2 = 0 \\ -2x - 4y + 2 = 0 \end{cases} \rightarrow -7y + 4 = 0 \rightarrow (x, y) =$ **<u>(-1/7, 4/7)</u>.**

Method 2: Indeterminant Coefficients (Guess and Check)

Suppose $2x^2 + xy - 6y^2 + 7y - 2$ factors to $(2x + ay + 2)(x + by - 1)$ for some constants a and b.
Multiplying out the trinomials leads to the linear equations $-a + 2b = 7$ and $a + 2b = 1$
and $(a, b) = (-3, 2)$ producing the factors $(2x - 3y + 2)(x + 2y - 1)$, as above.

But what would have happened if we assumed a factorization of $(2x + ay - 2)(x + by + 1)$?
Multiplying out these trinomials leads to the linear equations $a - 2b = 7$ and $a + 2b = 1$
and $(a, b) = (4, -1.5)$ producing the factors $(2x + 4y - 2)(x - 1.5y + 1)$. At first glance this appears to be a different factorization; however, taking out a factor of 2 from the first factor and distributing it through the second produces the same factors as before.

Note: The sum of the coefficients in the original polynomial is $2 + 1 - 6 + 7 - 2 = +2$
Compare this with the product of the sum of the coefficients in each factor!
$(2 - 3 + 2)(1 + 2 - 1) = (1)(2) = +2$
This is always true! Check it out.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006 SOLUTION KEY

Team Round - continued

D) Appealing to the graph of the common log function (i.e. base 10), $\log_{10}\frac{2a-1}{2-a} \le 0$

is equivalent to : $0 < \frac{2a-1}{2-a} \le 1$

$\frac{2a-1}{2-a} \le 1 \rightarrow \frac{2a-1}{2-a} - 1 \le 0 \rightarrow \frac{2a-1-2+a}{2-a} \le 0 \rightarrow \frac{3a-3}{2-a} \le 0 \rightarrow \frac{a-1}{2-a} \le 0 \rightarrow a \le 1$ or $a > 2$

$0 < \frac{2a-1}{2-a} \rightarrow$ ½ < a < 2 Taking the overlap, we have $\boxed{\mathbf{\frac{1}{2} < a \le 1}}$.

E) Let a, b, c and d denote the rates of each of the 4 runners and T, the elapsed time when the leader (runner A) crosses the finish line.

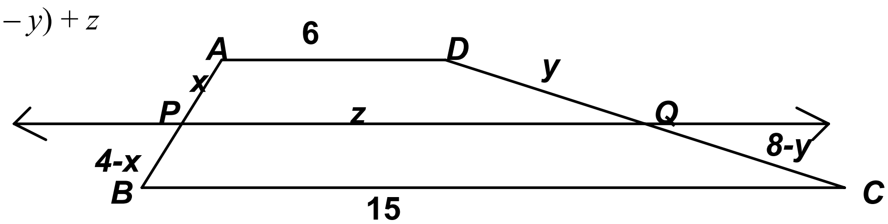
D	C	B	A
(20-k)	**15**	**18**	**20**

Since *distance* = *rate x time*, $T = \frac{20}{a} = \frac{18}{b} = \frac{15}{c} = \frac{20-k}{d}$

When runner C reaches the finish line, runner D is 1 km behind and, therefore, $\frac{20}{c} = \frac{19}{d}$.

Thus, $\frac{d}{c} = \frac{20-k}{15} = \frac{19}{20} \rightarrow 400 - 20k = 19 \cdot 15 = 285 \rightarrow k = 115/20 =$ **<u>23/4</u>** = **<u>5.75</u>.**

F) $6 + x + y + z = 15 + (4 - x) + (8 - y) + z$
$\rightarrow 2x + 2y = 21$
But $\overline{PQ} \parallel \overline{AD} \parallel \overline{BC}$
$\rightarrow \frac{x}{4-x} = \frac{y}{8-y}$
$\rightarrow 8x - xy = 4y - xy$
$\rightarrow y = 2x$
Thus, $6x = 21 \rightarrow x = 7/2 \rightarrow PB : PA = 0.5 : 3.5 =$ **<u>1 : 7</u>**.



Addendum

The only change was a last minute re-wording of question 3 in round 1.
Any printed copies with the original wording used at the meets should be discarded.