MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2007
ROUND 1 ANALYTIC GEOMETRY: ANYTHING

ANSWERS

A) _______________________________

B) (________ , _________ , ________)

C) _____________________________ %

A) Find the distance between the parallel lines $3x + 4y = 5$ and $3x + 4y = 9$.

B) A quadratic function $y = ax^2 + bx + c$, has x-intercepts 1 and 5 and y-intercept 2. Determine the ordered triple (a, b, c).

C) The area of the ellipse $4(x-1)^2 + 2500(y+2)^2 = 10000$ is approximated by a rectangle whose sides are tangent to the ellipse at the endpoints of its major and minor axes. To the nearest tenth of a percent, what is the percent error in this overestimate?

Recall: The area of an ellipse is given by the formula πab, where a and b are the distances from the center to the endpoints of the major and minor axes, respectively.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2007
ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) 1741 is a prime number.
It does <u>not</u> factor as the product of two integers (except the trivial $(1 \cdot 1741)$.
Find the ordered pair of consecutive positive integers (a, b), where $a > b$, for which the product ab is closest to 1741.

B) Mersenne Numbers are numbers of the form $2^n - 1$, for integers $n > 2$. If n is even, this formula always generates numbers that are composite. If n is odd, this is not necessarily the case. Find the <u>sum</u> of all prime factors of the smallest composite Mersenne number generated by an <u>odd</u> value of n.
Note: 1 is neither prime nor composite.

C) Determine all values of x for which $\left(6\left(\frac{x-3}{x-7}\right)-4\right)^2 - 5\left(2-3\left(\frac{x-3}{x-7}\right)\right) = 21$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2007
ROUND 3 TRIG: EQUATIONS WITH A REASONABLE NUMBER OF SOLUTIONS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

******* NO CALCULATORS ON THIS ROUND ******

A) Solve for x, where $0 \le x < 360°$: $3\cos(x) + 3 = 2\sin^2(x)$

B) Solve for θ, where $0° \le \theta < 360°$: $2\sin\theta\tan\theta + \sqrt{3}\tan\theta = 2\sqrt{3}\sin\theta + 3$

C) How many solutions does the equation $4\sin^2(2007x) - 1 = 0$ have over the interval $0 \le x < \pi/4$?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2007
ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) (________ , ________)

B) a b c d e f

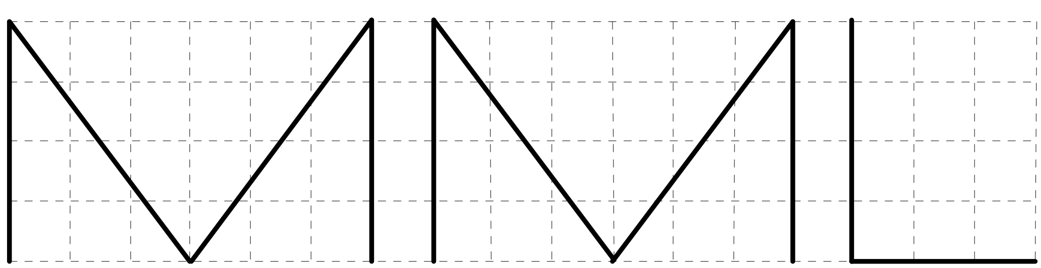
C) __________________________ oz.

A) The vertex of the parabola $y = (x - 1)(x - a) + b$ is located at (3, 7).
Determine the ordered pair (a, b).

B) Let a, b, and c, in some order, denote three consecutive positive integers.
Indicate which of the following orders of (a, b, c) guarantees that the sum of the roots of $ax^2 - \underline{bx} + c = 0$ is as <u>large</u> as possible.
(S, M and L denote the smallest, the middle and the largest integer.)

a) (S, M, L) b) (S, L, M) c) (M, S, L) d) (M, L, S) e) (L, M, S) f) (L, S, M)

C) 24 ounces of copper are drawn into a wire of uniform cross section. If the wire had been one foot longer (and still of the same uniform cross section), the wire would have weighed 0.1 oz. less per foot.
This wire is used to make 4" high letters, as shown below.
The grid squares are 1 inch on side.
If no extra wire is used to make the bends, find the weight of the wire used.
Give an exact answer or an approximation accurate to the nearest 0.01 ounce.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2007
ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

ANSWERS

A) _________________________ $units^2$

B) ___________________________feet

C) _____________ : ______________

A) A segment connects the midpoints of the legs of a right triangle with sides of length 3, 4 and 5, dividing the right triangle into a triangle and a trapezoid. What is the area of the trapezoid?

B) A building has a light mounted 15 feet above the ground. A person 6 feet tall is standing 10 feet from the base of the building.
Exactly how long is the person's shadow?
(Assume the person and the building are perpendicular to level ground.)

C) Given: $\Delta ABC \sim \Delta CAD$, $AB = 12$ and $CD = 27$. Determine the simplified ratio of the area of the circle inscribed in ΔABC to area of the circle inscribed in ΔCAD.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2007
ROUND 6 ALG 1: ANYTHING

ANSWERS

A) _______________________________ oz

B) _______________________________

C) _______________________________

A) A chemist needs to make 30 oz. of a 25% alcohol solution by mixing together a 15% alcohol solution and a 40% alcohol solution. How many ounces of the 40% solution should be used?

B) Determine the <u>sum</u> of all values of x for which $|x| = \dfrac{6}{|x+5|}$.

C) Find the <u>product</u> of all possible solutions over the reals:

$$(x^2 - 7x + 11)^{2x^2+11x-6} = 1$$

Consider 3 cases: $N^0 = 1$, provided $N \neq 0$
$1^N = 1$, for all real values of N
$(-1)^N = 1$ for all <u>even</u> values of N

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2007
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) h = ________________________

C) ____________________________ F) ______________________ years

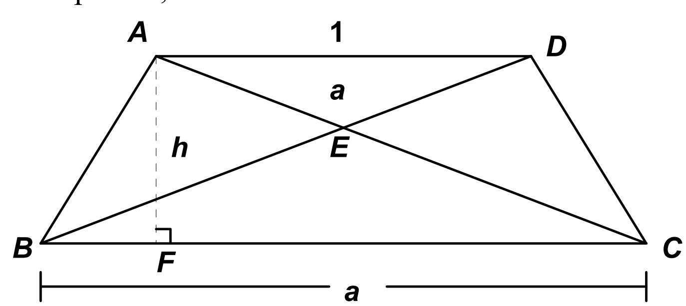
A) Find the equation of the line through $P(7, 2)$ perpendicular to the segment connecting the points of intersection of the line $2x - 5y + 3 = 0$ and the parabola $y = x^2$.
Express your answer as a simplified equation in the form $ax + by = c$, where a, b and c are integers and $a > 0$.

B) Find the sum of all integer values of k for which the quadratic $x^2 + kx + k + 11$ can be expressed as the product of two linear binomials with integer coefficients.

C) Find the maximum value of k for which $5\sin(x) + 12\cos(x) = k^2 - k + 1$ has a real solution. If necessary, express your answer in terms of a simplified radical.

D) Let $x = 2 + \cfrac{1}{a + \cfrac{1}{2 + \cfrac{1}{a + \ldots}}}$, where a is an integer.
Determine the integer value of a for which x is a rational number.

E) $ABCD$ is an isosceles trapezoid. If $AD = 1$ unit, $BC = a$ units and the area of ΔADE is a units^2, determine the value of h, the height of the trapezoid, in terms of a.



F) A man and his grandson have the same birthday. For the grandson's first six birthdays, his grandfather was an integral number of times as old as he was. How old was the grandfather on the grandson's sixth birthday? Assume the grandfather has not yet celebrated his 100th birthday.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2007 ANSWERS

Round 1 Analytic Geometry: Anything

A) 4/5 or 0.8	B) $\left(\frac{2}{5}, \frac{-12}{5}, 2\right)$	C) 27.3%

Round 2 Alg1: Factoring

A) (42, 41)	B) 80	C) 4, 23

Round 3 Trig: Equations

A) 120°, 240°, 180°	B) 60°, 240°, 300°	C) 1003

Round 4 Alg 2: Quadratic Equations

A) (5, 11)	B) b	C) $5\frac{11}{15}$ or 5.73 oz.

Round 5 Geometry: Similarity

A) 4.5	B) $6\frac{2}{3}$	C) 4 : 9

Round 6 Alg 1: Anything

A) 12	B) -10	C) -120

Team Round

A) $5x + 2y = 39$	B) 12	C) 4
D) -2	E) $h = 2a(a + 1)$	F) 66

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2007 SOLUTION KEY

Round 1

A) Method 1 is straightforward (but tedious).

Pick a point on one of the lines → P(3,0) is on the 2nd line $3x + 4y = 9$

Determine the slope of the perpendicular line through this point (negative reciprocal) → $\underline{+\frac{4}{3}}$

Determine the equation of this perpendicular line → $\underline{4x - 3y = 12}$

Determine point of intersection of this line and the first line → $Q\left(\frac{63}{25}, \frac{-16}{25}\right) = Q(2.52, -0.64)$

Using the distance formula, find the distance between P and Q

$\sqrt{(3-2.52)^2+(0+0.64)^2} = \sqrt{.48^2+.64^2} = \sqrt{.16^2(3^2+4^2)} = .16(5) =$ **0.8 or 4/5**

Method 2 uses the point to line distance formula .

The distance from point $P(x_1, y_1)$ to the line $Ax + By + C = 0$ is $\frac{|Ax_1+Bx_2+C|}{\sqrt{A^2+B^2}}$

Note that all terms in the equation must be on the same side.

Thus, using (3, 0) as the point and $3x + 4y - 5 = 0$ as the line we have

$$d = \frac{|3(3)+4(0)-5|}{\sqrt{3^2+4^2}} = \textbf{4/5 or 0.8}$$

Method 3: Determine the hypotenuse of a triangle with legs given by A and B, the coefficients of x and y. [(3, 4) → 5] Subtract this value from C, the constant term. [9 – 5 = 4] Divide this value by the hypotenuse [4/5] That's all folks!
The proof is left to you!

B) The coordinates (1, 0), (5, 0) and (0, 2) must satisfy the equation $y = ax^2 + bx + c$.
The last ordered pair implies $c = 2$.
(5, 0) → $25a + 5b + 2 = 0$
(1, 0) → $a + b + 2 = 0$ → $5a + 5b + 10 = 0$
Subtracting, we have $20a - 8 = 0$ → $a = 2/5$
Substituting, $2/5 + b + 2 = 0$ → $b = -12/5$

Alternately, x-intercepts of 1 and 5 → $y = k(x - 1)(x - 5)$

A y-intercept of 2 → constant terms $5k = 2$ → $k = 2/5$ Multiplying, $y = \frac{2}{5}x^2 - \frac{12}{5}x + 2$

Thus, $(a, b, c) = \underline{\left(\mathbf{\frac{2}{5}, \frac{-12}{5}, 2}\right)}$

C) $4(x-1)^2 + 2500(y+2)^2 = 10000 \rightarrow \frac{(x-1)^2}{2500} + \frac{(y+2)^2}{4} = 1 \rightarrow a = 50, b = 2$

Thus, the exact area is 100π and the overestimate is LW = 100(4) = 400
The percent error is $(400 - 100\pi)/100\pi = (4 - \pi)/\pi \approx 0.2732+$ → **27.3%**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2007 SOLUTION KEY

Round 2

A) Taking the square root of 1741 → 41.7^{+}
The product 40(41) = 1640 is obviously smaller then 1741, since both factors are smaller than the square root of 1741. Likewise 42(43) = 1806 is obviously larger then 1741, since both factors are larger than the square root of 1741. Thus, the product closest to 1741 is produced by the pair of integers that sandwich the square root, 41(42) = 1722. $a > b$ → (a, b) = **<u>(42, 41)</u>**.

B) n = 3, 5 and 7 produces 7, 31 and 127 respectively, all of which are primes.
$n = 9$ → 511 = 7(73)
7 + 73 = **<u>80</u>**

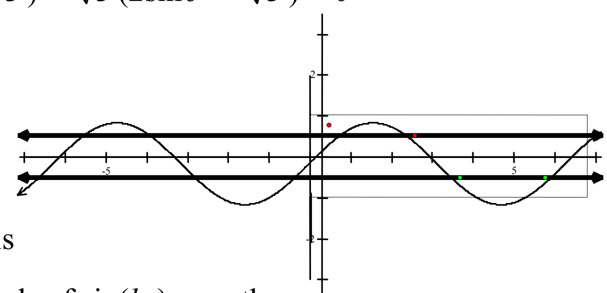
C) Let $A = \dfrac{x-3}{x-7}$. Then $\left(6\left(\dfrac{x-3}{x-7}\right)-4\right)^2 - 5\left(2-3\left(\dfrac{x-3}{x-7}\right)\right) = 21$ simplifies to
$(2(3A-2))^2 + 5(3A-2) - 21 = 0$
Letting $B = 3A - 2$, we have $4B^2 + 5B - 21 = (4B-7)(B+3) = 0$ or substituting back
$(4(3A-2)-7)(3A-2+3) = (12A-15)(3A+1) = 0$ → $A = 5/4$ or $-1/3$
Finally, substituting for A,
$\dfrac{x-3}{x-7} = \dfrac{5}{4}$ → $4x - 12 = 5x - 35$ → x = **<u>23</u>**

$\dfrac{x-3}{x-7} = \dfrac{-1}{3}$ → $3x - 9 = -x + 7$ → $4x = 16$ → x = **<u>4</u>**

Round 3

A) $3\cos(x) + 3 = 2(1-\cos^2(x))$ → $2\cos^2(x) + 3\cos(x) + 1 = (2\cos x + 1)(\cos x + 1) = 0$
→ $\cos x = -1/2$ → x = **<u>120°, 240°</u>**
→ $\cos x = -1$ → x = **<u>180°</u>**

B) $2\sin\theta\tan\theta + \sqrt{3}\tan\theta - 2\sqrt{3}\sin\theta + 3 = \tan\theta(2\sin\theta + \sqrt{3}) - \sqrt{3}(2\sin\theta + \sqrt{3}) = 0$
→ $(\tan\theta - \sqrt{3})(2\sin\theta + \sqrt{3}) = 0$
→ $\tan\theta = \sqrt{3}$ → θ = **<u>60°, 240°</u>**
→ $\sin\theta = -\dfrac{\sqrt{3}}{2}$ = → θ = 240°, **<u>300°</u>**



C) Let $k = 2007$. We need $\sin(kx) = \pm\dfrac{1}{2}$. The sine function is periodic with period 2π. This implies there will be k periods of $\sin(kx)$ over the given interval $[0, 2\pi)$ or $k/8$ periods over $[0, \pi/4)$. Both $y = +\dfrac{1}{2}$ and $y = -\dfrac{1}{2}$ cross one cycle of $\sin(kx)$ twice. Thus, there are $[k/2] = [2007/2]$ = **<u>1003</u>** points of intersection.
[2007/8 = 250.875 cycles → 4(250) + 3, the last three points of intersection occurring at the one quarter, halfway and three quarters points in the 251^{st} cycle]

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2007 SOLUTION KEY

Round 4

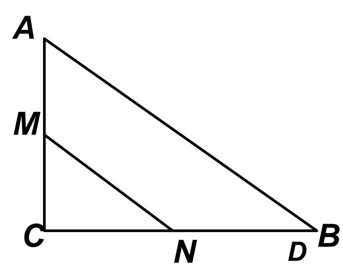
A) The minimum occurs at the vertex which lies on the axis of symmetry of this upward opening parabola. The axis of symmetry occurs at $x = \dfrac{1+a}{2} = 3 \rightarrow a = 5$
Substituting, $7 = (3 - 1)(3 - 5) + b \rightarrow 7 = -4 + b \rightarrow b = 11$
Thus $(a, b) =$ **(5, 11)**.

B) The sum of the roots is b/a. To maximize the value of this fraction you need to maximize the numerator and minimize the denominator. Thus, b = L, a = S and c = M $\rightarrow$ **b) (S, L, M)**

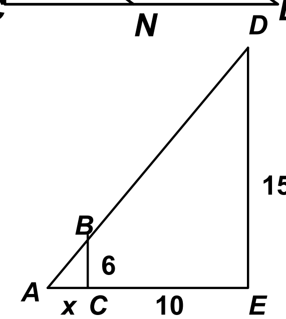
C) Assume the wire is x feet long. Thus, the weight per foot is $\dfrac{24}{x}$ and

$\dfrac{24}{x+1} = \dfrac{24}{x} - \dfrac{1}{10} \rightarrow 240x = 240(x + 1) - x(x + 1) \rightarrow 0 = 240 - x^2 - x$
$\rightarrow x^2 + x - 240 = (x + 16)(x - 15) = 0 \rightarrow x = 15$
Therefore, 15 feet of wire weighs 24 ounces $\rightarrow$ 2/15 oz per inch.
Wire needed is $5(4) + 4(5) + 3 = 43$ inches $\rightarrow 86/15 =$ **$5\dfrac{11}{15}$** or **5.73**

Round 5



A) The area of the 3-4-5 triangle is 6 units^2. The line connecting the midpoints of the legs is parallel to the hypotenuse and cuts off a triangle similar to the original 3-4-5. Since the ratio of their corresponding sides is 1 : 2, their areas are in a ratio of 1 : 4. Since the area of ΔMNC is ¼ the area of ΔABC, the area of the trapezoid is ¾ the area of ΔABC = **4.5**.



B) Since $\Delta ABC \sim \Delta ADE$, $\dfrac{x}{x+10} = \dfrac{6}{15} = \dfrac{2}{5} \rightarrow 5x = 2x + 20$
$\rightarrow 3x = 20 \rightarrow x = \boxed{6\dfrac{2}{3}}$

C) $\Delta ABC \sim \Delta CAD \rightarrow \dfrac{AB}{CA} = \dfrac{AC}{CD} \rightarrow AC^2 = AB(CD) = 12(27) = 18^2 \rightarrow AC = 18$.
Since the ratio of the radii of the inscribed circles is the same as the ratio of the corresponding sides,
$\dfrac{r_1}{r_2} = \dfrac{AB}{AC} = \dfrac{12}{18} = \dfrac{2}{3} \rightarrow \dfrac{A_1}{A_2} = \boxed{\dfrac{4}{9}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2007 SOLUTION KEY

Round 6

A) Assume x ounces of the 40% solution are required.
Then $0.40x + 0.15(30 - x) = 0.25(30) \rightarrow 40x + 15(30 - x) = 25\ (30)$
$\rightarrow 25x = 750 - 450 = 300 \rightarrow x = \underline{\mathbf{12}}$

B) Cross multiplying, $|x||x + 5| = 6 \rightarrow |x(x + 5)| = 6 \rightarrow x^2 + 5x = \pm 6$
$x^2 + 5x + 6 = (x + 3)(x + 2) = 0 \rightarrow x = -3, -2$
$x^2 + 5x - 6 = (x + 6)(x - 1) = 0 \rightarrow x = -6, 1 \rightarrow$ sum = $\underline{\mathbf{-10}}$.

C) $a^b = 1$ if and only if

Case 1: ($a = 1$) $x^2 - 7x + 11 = 1 \rightarrow x^2 - 7x + 10 = (x - 2)(x - 5) = 0 \rightarrow x = 2, 5$
Case 2: ($b = 0$ (and $a \neq 0$)) $2x^2 + 11x - 6 = 0 \rightarrow (2x - 1)(x + 6) = 0$
Since any real number raised to the zero power is 1, <u>except</u> 0,
$x = 1/2$ (checks since $(1/2)^2 - 7(1/2) + 11 \neq 0$) and
$x = -6$ (checks since $(-6)^2 - 7(-6) + 11 \neq 0$)
Case 3: ($a = -1$ and b is an even integer) $x^2 - 7x + 11 = -1 \rightarrow x^2 - 7x + 12 = (x - 3)(x - 4) = 0$
$x = 3$ (fails, since $2(3)^2 + 11(3) - 6 = 45$ which is not an even integer exponent)
$x = 4$ (checks, since $2(4)^2 + 11(4) - 6 = 70$ which is an even integer exponent)

Thus, the product of the solutions is $2(5)(1/2)(-6)(4) = \underline{\mathbf{-120}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2007 SOLUTION KEY

Team Round

A) To find the points of intersection, substitute x^2 for y: $2x - 5x^2 + 3 = 0$
→ $5x^2 - 2x - 3 = (5x + 3)(x - 1) = 0$ → $x = -3/5$ or $+1$.

Thus, the points of intersection are: $A\left(\frac{-3}{5}, \frac{9}{25}\right)$ and $B(1, 1)$.

The slope of the segment $\overline{AB}$ is $\dfrac{\frac{9}{25}-1}{\frac{-3}{5}-1} = \dfrac{-16/25}{-8/5} = \dfrac{16}{40} = \dfrac{2}{5}$.

The slope of the perpendicular is $+\dfrac{-5}{2}$.

The equation of the perpendicular line is of the form $5x + 2y = c$, for some constant c.
Since the point $P(7, 2)$ is on this line, its coordinates must satisfy the equation and we have
$5(7) + 2(2) = c$ → $c = 39$ → equation: **$5x + 2y = 39$**

B) Assume $x^2 + kx + k + 11$ factors to $(x + a)(x + b)$, where a and b are integers.

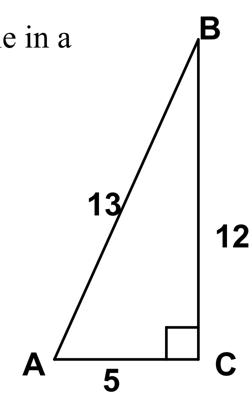
By the quadratic formula, $x = \dfrac{-k \pm \sqrt{k^2 - 4k - 44}}{2}$

To insure rational roots, $k^2 - 4k - 44 = (k - 2)^2 - 48$ must denote the perfect square of an integer, say t^2.
Thus, $(k - 2)^2 = t^2 + 48 = r^2$ → $r^2 - t^2 = (r + t)(r - t) = 48$

$(r + t)$	=	48	24	16	12	8
$(r - t)$	=	1	2	3	4	6
Adding/dividing by 2 → r =		impossimposs	13	imposs	8	7 → t = 11, 4 and 1

Thus, $(k - 2)^2 = 169$, 64 or 49 → $k = 2 \pm 13$, 2 ± 8 or 2 ± 7 → k = 15, -11, 10, -6, 9, -5
Adding these 6 possible values → **12**

C) $5\sin(x) + 12\cos(x) = 13\left(\frac{5}{13}\sin x + \frac{12}{13}\cos x\right)$. Let A denote the larger acute angle in a 5-12-13 triangle (see diagram). Using the $\sin(A + B)$ expansion, this simplifies to $13\sin(x + A)$ which has a maximum value of 13, when $x + A = 90°$ (or any coterminal value)



If $-13 \le k^2 - k + 1 \le 13$, there will be a solution.

To determine the maximum value of k, we need only solve
$k^2 - k + 1 \le 13$, since $-13 \le k^2 - k + 1$ → $k^2 - k + 14 \ge 0$ → $(k - ½)^2 + 55/4 \ge 0$ which is true for all real values of k.
$k^2 - k - 12 = (k + 3)(k - 4) \le 0$ → $-3 \le k \le 4$ and the maximum value of k is **4**.

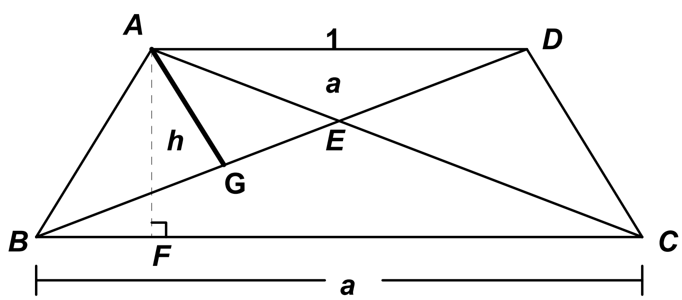
Team Round - continued

D) $x = 2+\cfrac{1}{a+\cfrac{1}{\mathbf{2}+\cfrac{\mathbf{1}}{\mathbf{a}+\ldots}}}$ → $x = 2+\cfrac{1}{a+\cfrac{1}{x}}$ → $x-2 = \dfrac{x}{ax+1}$

→ $ax^2 - 2ax - 2 = 0$ → $x = \dfrac{2a \pm \sqrt{4a^2+8a}}{2a}$ → $x = 1 \pm \dfrac{\sqrt{a^2+2a}}{a}$

To be rational, the radicand $a^2 + 2a$ must be a perfect square.
Thus, $a^2 + 2a = k^2$ for some integer k.
Then $a^2 + 2a + 1 = (a + 1)^2 = k^2 + 1$ or $(a + 1)^2 - k^2 = 1$
The only integer perfect squares that differ by 1 are 0 and 1.
$a = -1$ → $k^2 = -1$ which contradicts the condition that k is an integer.
$a = 0$ causes division by 0
$a = -2$ → $k = 0$ → $x = 1$
Therefore, the only value of a for which the given continued fraction is rational is **-2**.

E) $A_{trap} = \dfrac{h}{2}(1+a)$ Since $\Delta ADE \sim \Delta CBE$,



$BE : DE = a : 1$ and

$\dfrac{a}{\text{Area}(\Delta CBE)} = \left(\dfrac{1}{a}\right)^2$ → Area(ΔCBE) = a^3

Since ΔADE and ΔABE share a common altitude from point A, namely $\overline{AG}$, their areas are in the same ratio as their bases.

Thus, $\dfrac{\text{Area}(\Delta ABE)}{a} = \dfrac{a}{1}$ → Area(ΔABE) = Area(ΔDCE) = a^2

and $\dfrac{h}{2}(1+a) = a + 2a^2 + a^3 = a(a+1)^2$ → **$\underline{h = 2a(a+1)}$**

Alternate: Draw altitude $\overline{HI}$ through E(H on $\overline{AD}$ and I on $\overline{BC}$), and let $HE = x$ and $EI = h - x$

Then $\dfrac{x}{h-x} = \dfrac{1}{a}$ → $x = \dfrac{h}{a+1}$ and the area of $\Delta AED = a = \dfrac{1}{2}\cdot 1\cdot \dfrac{h}{a+1}$ → **$\underline{h = 2a(a+1)}$**

F) Assume the grandfather (GF) was x years old on his grandson's first birthday.
Next birthday: GF = $(x + 1)$ must be divisible by 2 → x is odd.
3rd birthday: GF = $(x + 2)$ must be divisible by 3.
(The only possibilities are: $x = 3n$, $3n + 1$ or $3n + 2$) Only $x = 3n + 1$ works→ GF = $3n + 3$
4th birthday: GF = $3n + 4$ which must be divisible by 4 and, in turn, this implies n must be divisible by 4. n must be of the form $4k$ → GF = $12k + 4$
5th birthday: GF = $12k + 5$ which must be divisible by 5 → $k = 5j$ → GF = $60j + 5$
6th birthday $60j + 6$. Only j =1 produces a possible age for the grandfather that is less than 100, namely **66**.

Addendum

No revisions were made to this contest.