MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 – OCTOBER 2007
ROUND 1 VOLUME & SURFACES

ANSWERS

A) ______________________________

B) ______________________________

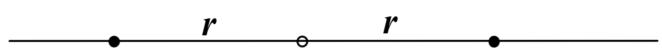
C) ______________________________

A) The radius of a right circular cylinder is 10 inches and its height is 4 inches. Determine the value of x (> 0) which, when added to either the radius or the height, produces two different right circular cylinders of equal volume.

B) **Definition**: A locus of points refers to a set of all points (and only those points) that satisfy a given condition.

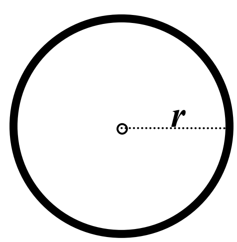
Example: The locus of points equidistant from a given point

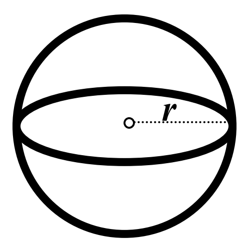
on a line: 2 points



in a plane: a circle

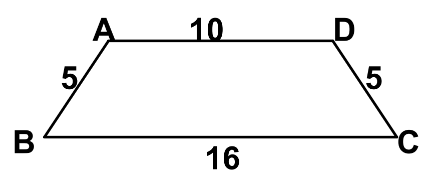
in space: a sphere





Compute the total area of the region(s) bounded by the locus of points in space 4 cm from a given plane and 6 cm from a fixed point in that plane.

C) The following diagram is the cross-section of the frustum of a right circular cone. Compute the volume of the frustum.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007
ROUND 2 PYTHAGOREAN RELATIONS IN RECTILINEAR FIGURES

ANSWERS

A) ______________________________

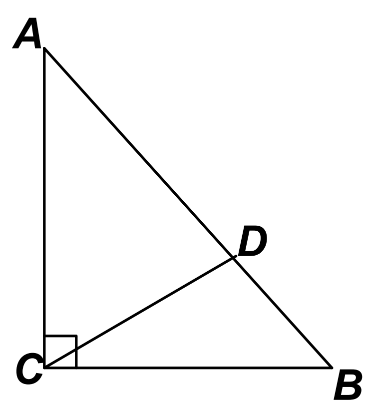
B) ______________________________

C) ______________________________

A) The hypotenuse and long leg of a right triangle have lengths $11\sqrt{3}$ and $7\sqrt{7}$ respectively. Compute the length of the short leg.

B) Compute the distance between the bisectors of a pair of opposite angles in a 5 x 12 rectangle.

C) In right ΔABC,
$AC = 28$, $BC = 21$,
$\overline{AC} \perp \overline{BC}$ and
$\dfrac{\text{Area}(\Delta BCD)}{\text{Area}(\Delta ACD)} = \dfrac{3}{4}$.
Compute CD.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Find a simplified expression for the value of x in terms of a and b, given $a + b \neq 0$:
$a(a - 2x) = b(b + 2x)$

B) For how many ordered pairs of <u>positive</u> integers does $x - \frac{11-y}{3} = 18$

C) What is the original cost of a dozen eggs, if buying an additional 4 eggs for 32¢ lowers the cost per dozen by 4¢?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

ANSWERS

A) _______________________________

B) _______________________________

C) _______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Compute: $$\frac{2\frac{7}{16} - 3\frac{3}{4}}{5\frac{5}{8} + 7\frac{1}{2}}$$

B) Solve for x. $$\frac{x-3}{x+1} - \frac{2x-8}{x^2-1} = \frac{3}{x-1}$$

C) If the integer $n \geq 1$ and $\dfrac{(n+1)!(2n-2)!}{(n-1)!(2n)!} = \dfrac{2}{7}$, compute n.

Recall: $n! = n(n-1)(n-2) \ldots \cdot 2 \cdot 1$ and $0! = 1$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Find the area of the region defined by $\begin{cases} y \le |x| \\ x \le 2 \\ x \ge -1 \\ y \ge 0 \end{cases}$

B) Solve for x over the reals. $\dfrac{|x-3|(x-4)}{(x+5)^3} \ge 0$

C) Determine the set of values of x (over the reals) for which the following inequality is satisfied:

$$\frac{1}{x} \le \frac{1}{x-1} - \frac{1}{2}$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007
ROUND 6 ALG 1: EVALUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) (____________ , _____________)

A) 22/7 and 3.14 are often used as approximations of π.
When 22/7 is approximated as 3.14, the error is $\frac{1}{n}$, where n is a positive integer?
Compute n.

B) Given: $\frac{3A+5B}{4B-2A}=2$ Compute: $\frac{2A-3B}{A}$

C) An ad cost 25¢ per line to run on weekdays, 20¢ per line on Saturdays and 50¢ on Sundays. Of course, since you are competing in an MML meet today, it is the first Thursday of the month. The cost for running a one line ad for 30 days, starting on the first of a month will vary slightly depending on what day of the week the first day of the month falls. Determine the ordered pair of integers (A, B), where A denotes the date of the first Thursday of the month in which the cost of running the ad is the maximum B (in cents).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) ____________________________

C) (___________ , ___________) F) ____________________________

A) An equilateral triangle with side $\underline{s}$ and its inscribed circle are rotated around one of the axes of symmetry of the equilateral triangle generating a cone and a sphere. Determine a simplified expression in terms of $\underline{s}$ for the volume of the region inside the cone and outside the sphere.

B) In a right triangle with sides of integer length, the hypotenuse is 1 unit longer than the longer leg. Determine the lengths of the sides of such a right triangle if the perimeter is less than 1000, but as large as possible.

C) A number of pencils were purchased for 30¢. If the price per dozen were lowered by 27¢, then 3 more pencils could have been bought for 30¢. If Q denotes the number of pencils that could be bought for 30¢ at the lower unit price P (in cents), determine the ordered pair (Q, P).

D) Let the solution of the equation $\dfrac{1}{1+\dfrac{2}{x+3}} = 4 - 0.\overline{5}$ be $x = \dfrac{A}{B}$, a reduced fraction, where A and B denote integers and $A < B$. Determine the smallest positive integer n such that $\dfrac{A+n}{B}$ is an integer.

E) The region satisfying the inequality $|x - 2007| + |y + 2008| < A$ contains 1985 lattice points, i.e. points with integer coordinates. Determine the integer A for which this statement is true.

F) Four students enter a room. Each student has a cell phone. The students put all their cell phones into the same pile. When the students leave, each student is given a cell phone. What is the probability that none of the students will receive their own cell phone?

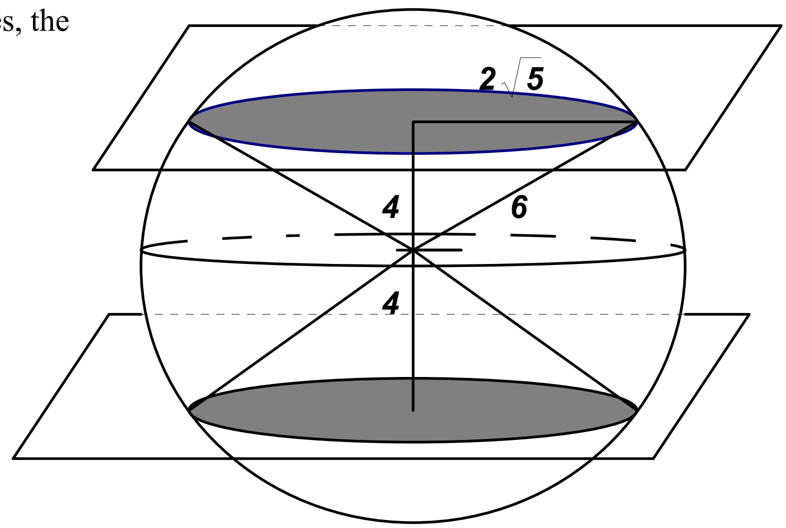
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007 SOLUTION KEY

Round 1

A) Let x denote the increase in radius or height. Then $\pi(10 + x)^2(4) = \pi(10)^2(4 + x)$
$\rightarrow 4(100 + 20x + x^2) = 400 + 100x \rightarrow 4x^2 - 20x = 4x(x - 5) = 0 \rightarrow x = \underline{\mathbf{5}}$

B) The regions consist of two congruent circles, the intersection of a sphere and two parallel planes.

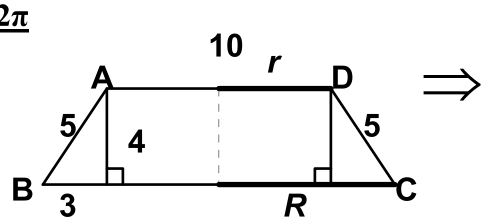
Area = $2\pi(2\sqrt{5})^2 = \underline{\mathbf{40\pi}}$

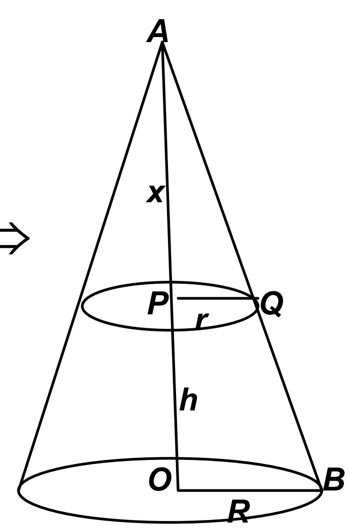


C) Method #1 [using V(frustum) = $\dfrac{\pi h}{3}\left(R^2 + Rr + r^2\right)$]

$r = 5$, $R = 8$ and $h = 4$

Thus, V = $\dfrac{1}{3}\pi(4)[25 + 40 + 64] = \underline{\mathbf{172\pi}}$





Method #2 [V(cone$_1$) – V(cone$_2$)]

(Note: $\Delta APQ \sim \Delta AOB \rightarrow \dfrac{x}{x+h} = \dfrac{r}{R} \rightarrow x = \dfrac{hr}{R-r}$)

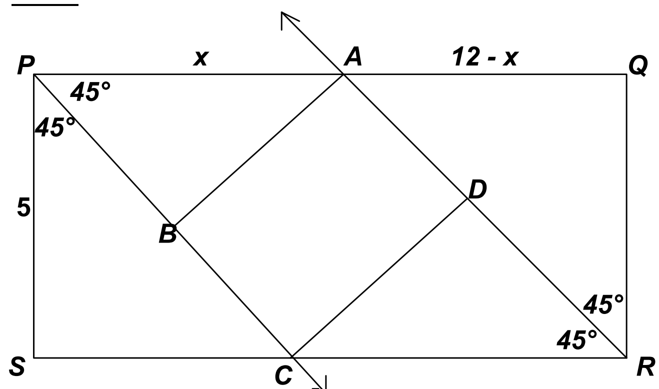
$\dfrac{x}{x+4} = \dfrac{5}{8} \rightarrow x = \dfrac{20}{3}$

V(cone$_1$) = $\dfrac{1}{3}\pi(8)^2\left(4 + \dfrac{20}{3}\right) = \dfrac{2048\pi}{9}$ V(cone$_2$) = $\dfrac{1}{3}\pi(5)^2\left(\dfrac{20}{3}\right) = \dfrac{500\pi}{9}$

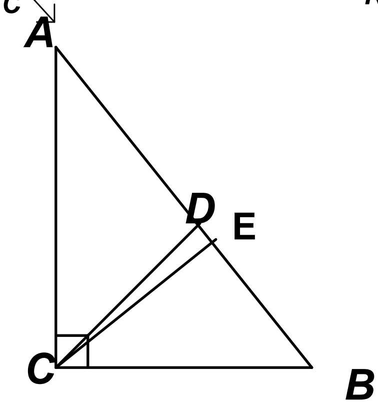
Therefore, V(frustum) = $\dfrac{1548\pi}{9} = \underline{\mathbf{172\pi}}$

Round 2

A) $b^2 = (11\sqrt{3})^2 - (7\sqrt{7})^2 = 363 - 343 = 20 \rightarrow b = \underline{\mathbf{2\sqrt{5}}}$



B) Let $\overline{AB}$ (drawn $\perp \overrightarrow{PC}$) represent the distance between the angle bisectors.
Clearly, Δs PSC and RQA are both isosceles and congruent. Thus, x must be 7 and since $\overline{AB}$ is the leg in an isosceles right triangle with hypotenuse 7, $AB = CD = \underline{\mathbf{\dfrac{7\sqrt{2}}{2}}}$

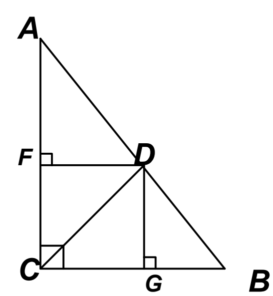


C) Since ΔCDA and ΔCDB have a common altitude and areas in a 3 : 4 ratio, their bases must be in a 3 : 4 ratio also.
Apply the Pythagorean theorem to ΔABC or note
$(CB, CA, AB) = (21, 28, ?) = 7\underline{(3, 4, ?)}$
Since the missing number in the triple is 5, $AB = 35$
$\rightarrow AD = 20$ and $BD = 15$

Drop a perpendicular from C to $\overline{AB}$ (Call it $\overline{CE}$.)

Then $(BC)(AC) = (AB)(CE)$, since each gives twice the area of ΔABC.
Thus, $21(28) = 35(CE) \rightarrow CE = 16.8$

Applying the Pythagorean Theorem to right triangle ACE, $AE = 22.4 \rightarrow DE = 2.4$
and to right triangle CDE, $CD^2 = 16.8^2 + 2.4^2 = 288 \rightarrow CD = \underline{\mathbf{12\sqrt{2}}}$



Alternative: Drop perpendiculars from D to $\overline{AC}$ and $\overline{BC}$.
Let $AD = 4x$ and $DB = 3x$. $AB = 7x = 35 \rightarrow x = 5$, $BD = 15$, $AD = 20$
Since $\Delta DGB \sim \Delta AFD \sim \Delta ACB \sim$ 3-4-5 right triangle,
$(DG, GB) = (12, 9)$ and $(DF, FA) = (12, 16)$ and $FDGC$ is a <u>12 x 12 square</u>!
Therefore, $CD = \underline{\mathbf{12\sqrt{2}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007 SOLUTION KEY

Round 3

A) $a(a-2x) = b(b+2x) \rightarrow a^2 - b^2 = 2bx + 2ax \rightarrow (a+b)(a-b) = 2x(a+b) \rightarrow x = \underline{\boldsymbol{\dfrac{a-b}{2}}}$

B) Solving for y in terms of $x \rightarrow y = 65 - 3x$. Clearly, for $x = 1 \ldots 21$, y will be a positive integer. Thus, there are **<u>21</u>** solutions.

C) Assume 12 eggs cost x¢ and 16 cost $(x + 32)$¢ or $\dfrac{3}{4}(x+32)$ ¢/dozen

Then $\dfrac{3}{4}(x+32) = x - 4 \rightarrow 3x + 96 = 4x - 16 \rightarrow x =$ **<u>112 or $1.12</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007 SOLUTION KEY

Round 4

A) $$\frac{2\frac{7}{16} - 3\frac{3}{4}}{5\frac{5}{8} + 7\frac{1}{2}} = \frac{\left(\frac{39}{16}-\frac{15}{4}\right)16}{\left(\frac{45}{8}+\frac{15}{2}\right)16} = \frac{39-60}{90+120} = \frac{-21}{210} = \mathbf{-\underline{\frac{1}{10}}} \text{ or } -\underline{\mathbf{0.1}}$$

B) Multiplying by the LCD = $(x + 1)(x - 1)$, $(x - 3)(x - 1) - 2x + 8 - 3x - 3 = 0$
→ $x^2 - 4x + 3 - 5x + 5 = 0$ → $x^2 - 9x + 8 = (x - 1)(x - 8) = 0$ → $x = \underline{\mathbf{8}}$ ($x = 1$ is extraneous)

C) $$\frac{(n+1)!(2n-2)!}{(n-1)!(2n)!} = \frac{(n+1)n\cancel{(n-1)!}\,\cancel{(2n-2)!}}{\cancel{(n-1)!}(2n)(2n-1)\cancel{(2n-2)!}} = \frac{(n+1)n}{(2n)(2n-1)} = \frac{2}{7} \rightarrow 7n^2 + 7n = 8n^2 - 4n$$
→ $n^2 - 11n = n(n - 11) = 0$ → $n = \underline{\mathbf{11}}$

Here's the original question C (nixed by the proofreaders):
For extremely large positive values of n, the following fraction approaches a fixed value L. Compute L.

$$\frac{(n+1)!(2n-2)!}{(n-1)!(2n)!}$$

You might want to try your hand at solving this question before peeking at the solution below.

$$\frac{(n+1)!(2n-2)!}{(n-1)!(2n)!} = \frac{(n+1)n\cancel{(n-1)!}\,\cancel{(2n-2)!}}{\cancel{(n-1)!}(2n)(2n-1)\cancel{(2n-2)!}} = \frac{n^2+n}{4n^2-2n}$$

Dividing numerator and denominator by n^2, we have $\dfrac{1+\frac{1}{n}}{4+\frac{2}{n}}$

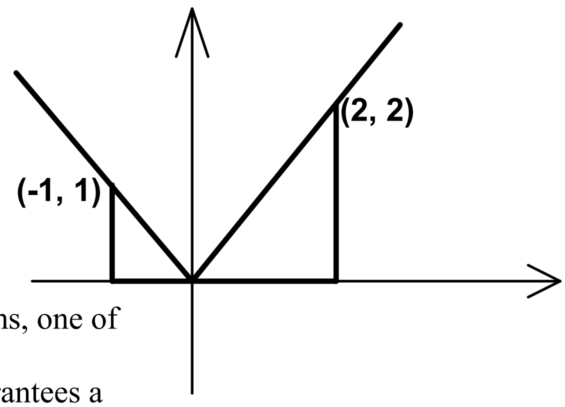
As n takes on extremely large positive values (i.e. approaches infinity), the fractional terms in the numerator and denominator each approach zero and the overall fraction approaches **<u>1/4</u>** or **<u>0.25</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007 SOLUTION KEY

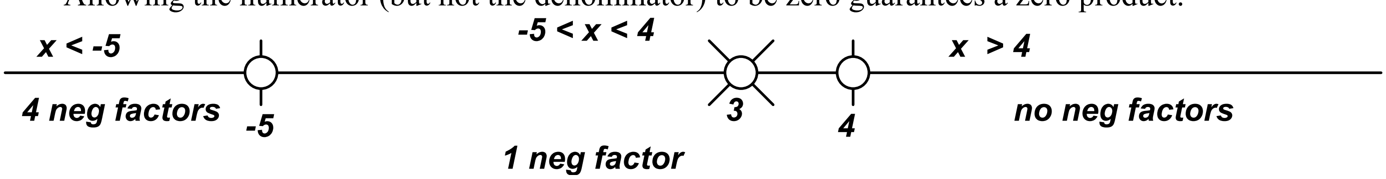
Round 5

A) The region consists of 2 right triangles

Area = $\frac{1}{2}\cdot 1\cdot 1+\frac{1}{2}\cdot 2\cdot 2$ = **<u>2.5</u>**



B) The quotient on the left-hand side is comprised of 5 terms, one of which is never negative.
Having an <u>even</u> number of factors that are negative guarantees a positive product.
Allowing the numerator (but not the denominator) to be zero guarantees a zero product.

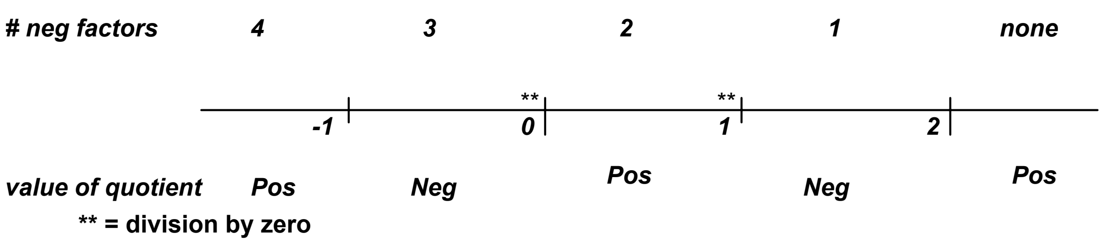


→ <u>**$x < -5$ or $x \geq 4$ or $x = 3$**</u>

C) $\frac{1}{x} \leq \frac{1}{x-1} - \frac{1}{2} \rightarrow \frac{1}{x-1} - \frac{1}{x} - \frac{1}{2} \geq 0 \rightarrow \frac{2x-2(x-1)-x(x-1)}{2x(x-1)} \geq 0 \rightarrow \frac{2+x-x^2}{2x(x-1)} \geq 0$

$\rightarrow \frac{(2-x)(1+x)}{2x(x-1)} \geq 0 \rightarrow \frac{(x-2)(x+1)}{2x(x-1)} \leq 0$

The critical values for this quotient are -1, 0, 1 and 2.



Thus, the solution intervals are: <u>$-1 \leq x < 0$ or $1 < x \leq 2$</u>.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007 SOLUTION KEY

Round 6

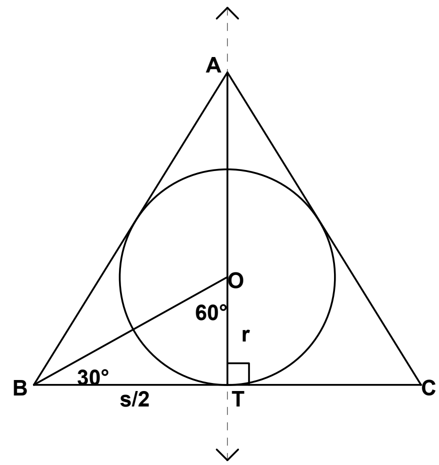
A) $\frac{22}{7}-\frac{314}{100} = \frac{22(100)-314(7)}{7(100)} = \frac{2200-2198}{700} = \frac{1}{350}$ → n = **350**

B) $\frac{3A+5B}{4B-2A}=2$ → $8B-4A=3A+5B$ → $\frac{B}{A}=\frac{7}{3}$ Then $\frac{2A-3B}{A}= 2-3\left(\frac{B}{A}\right)=2-7=$ **-5**

C) The delivery period of 30 days consists of 4 full weeks and 2 days.
The day of the week on which the first of the month falls is the same as the 29th day.
To maximize the cost, the 29th day must fall on a Sunday, adding a cost of 75¢ for the last two days. (The other possible costs for the last two days are: 45¢, 50¢ and 70¢)
Thus, the maximum cost will be 4[5(25) + 20 + 50] + 75 = **855**
If the first day of the month falls on a Sunday, then the first Thursday is the **5**th. → **(5, 855)**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007 SOLUTION KEY

Team Round



A) Consider the diagram at the right.

The volume of the cone is given by $\frac{1}{3}\pi\left(\frac{s}{2}\right)^2\left(\frac{s}{2}\sqrt{3}\right) = \frac{\sqrt{3}\pi s^3}{24}$

The volume of the sphere is given by $\frac{4}{3}\pi\left(\frac{s\sqrt{3}}{6}\right)^3 = \frac{\sqrt{3}\pi s^3}{54}$

The difference is $5\pi\sqrt{3}s^3\left(\frac{1}{24}-\frac{1}{54}\right) = 5\pi\sqrt{3}s^3\left(\frac{54-24}{24(54)}\right) = \underline{\mathbf{\frac{5\pi\sqrt{3}}{216}s^3}}$

B) Examining right triangles in which the lengths of the hypotenuse and long leg differ by 1:

a	*b*	*c*	Per	Factors
3	4	5	12	3(4)
5	12	13	30	5(6)
7	24	25	56	7(8)
9	40	41	90	9(10)

Apparently, the perimeter is given by $a(a + 1)$
We want a as large as possible with $a(a + 1) \le 1000$ $a = 31 \rightarrow 992$, $a = 32 \rightarrow 1056$
Thus, $a = 31$ and $31^2 + b^2 = (b + 1)^2 \rightarrow 961 = 2b + 1 \rightarrow b = 480$ and the sides are **(31, 480, 481)**

C) Let n denote the number of pencils originally bought for 30¢.
Let p denote the cost of a single pencil(in cents) .
Then $np = 30$ and $(n + 3)(p - 27/12) = 30$.
$(n + 3)(p - 2.25) = np - 2.25n + 3p - 6.75 = 30$
Cancelling, $-2.25n + 3p - 6.75 = 0 \rightarrow -9n + 12p - 27 = 0 \rightarrow -3n + 4p - 9 = 0 \rightarrow p = \frac{9+3n}{4}$

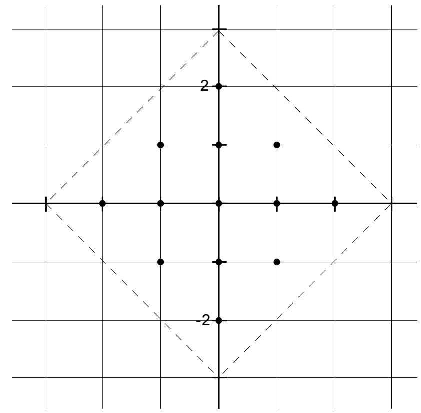
Substituting, $np = n\cdot\frac{9+3n}{4} = 30 \rightarrow 3n^2 + 9n - 120 = 0 \rightarrow (3n + 24)(n - 5) = 0$
$\rightarrow n = 5$ and $p = 6$
which means 8 pencils could be bought for the lower price of 6 – 9/4 = 3.75 cents→ **(8, 3.75)**

D) $\dfrac{1}{1+\dfrac{2}{x+3}} = 4-0.\overline{5} \rightarrow \dfrac{x+3}{x+5} = 4-\dfrac{5}{9} = \dfrac{31}{9} \rightarrow 9x + 27 = 31x + 155 \rightarrow x = \dfrac{-128}{22} = \dfrac{-64}{11}$

Note: $\frac{64}{-11}$ is disallowed since $A < B$. If $\frac{-64+n}{11}$ must be an integer and $n > 0$, the minimum possible value of n is **9**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007 SOLUTION KEY

Team Round – continued



E) The regions bounded by $|x - 2007| + |y + 2008| = A$ and $|x| + |y| = A$ contain the same number of lattice points.
The latter is a square (diamond) with vertices at $(\pm A, 0)$ and $(0, \pm A)$.
The diagram at the right illustrates the lattice points for $A = 3$.
The number of lattice points is $2(1 + 3) + 5 = 13$.

Method #1: Pattern appears to be:
Considering the first A consecutive odd integers, the number of lattice points is found by doubling the sum of the first $A - 1$ odd integers, and then adding the Ath.

$2[1 + 3 + 5 + \ldots + (2A - 3)] + (2A - 1) = 1985$
Applying sum formulas for arithmetic sequences,

$$2\left[\frac{A-1}{2}(1+2k-3)\right]+(2A-1)=1985 \rightarrow (A-1)(2A-2)+(2A-1)$$

$$\rightarrow 2(A-1)^2 + 2A - 1 = 2A^2 - 2A - 1984 = 0 \rightarrow A^2 - A - 992 = 0 \rightarrow A(A-1) = 992 \rightarrow A = \underline{\mathbf{32}}$$

Method #2
Construct a chart for small values of A and look for a pattern.
Note that the number of lattice pts $= 2A^2 - 2A + 1$.

A	***#LPs***	
1	1	2(1)(0)+1
2	5	2(2)(1)+1
3	13	2(3)(2)+1
4	25	2(4)(3)+1
5	41	2(5)(4)+1
	2A(A-1) + 1	

F) With 4 people (A, B, C, D) and 4 phones (a, b, c, d),
examine the $4! = 24$ permutations of *abcd*.
Any that start with *a* are eliminated.
Starting with *b*, ~~*ba**cd***~~, *badc*, ~~*bca**d***~~, *bcda*, bdac, ~~*bd**c**a*~~ → 3 total mismatches
Starting with *c*, ~~*cab**d***~~, *cadb*~~, *c**bad**, c**b**da*~~, *cdab*, *cdba* → 3 total mismatches
Starting with d, *dabc*, ~~*da**c**b, d**b**ac, d**bc**a*~~, *dcab*, *dcba* → 3 total mismatches
Thus, for 4 phones, there are $3(3) = 9$ total mismatches → P(total mismatch) = $9/24 = \underline{\mathbf{\frac{3}{8}}}$