MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2007
ROUND 1 ALG 2: COMPLEX NUMBERS (No Trig)

ANSWERS

A) (__________) + (__________)i

B) ______________________________

C) ______________________________

A) Solve over the complex numbers, expressing your answer in simplified $a + bi$ form.
(Note: $\overline{z} = a - bi$ and denotes the conjugate of z.)

$$z + 6\overline{z} = 7 + 3i$$

B) Find all possible solutions of $z^2 = 75 + 100i$.
Leave your answer(s) in $a + bi$ form.

C) Solve for x.

$$|-3+4i|\,x^2 - |12+16i|\,x = |7-24i|$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007
ROUND 2 ALGEBRA 1: ANYTHING

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________ minutes

A) The mean of a set of numbers is 20. If one of the numbers is increased by 300, the mean increases to 35. How many numbers are in the set?

B) Solve for k. $$\frac{k^2+9}{k^2-3k+2}-\frac{6}{k-1}=\frac{2k}{k-2}$$

C) Each family member can clean the family room alone in the times specified below:

Mother – 12 minutes
Father – 27 minutes
Son – $2x$ minutes
Daughter – $3x$ minutes

The family room can be cleaned by this foursome, if they work together for 6 minutes.
How long would it take the daughter to clean the family room by herself?
Assume that they each work at the individual rates implied by the above time table and since they work so well together as a family that no time is lost as they coordinate their efforts.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

ANSWERS

A) ______________________________

B) ______________________________

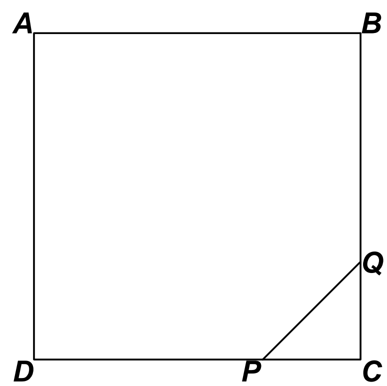
C) ______________________________

A) A right triangle has an area of 60 and its legs have lengths in a 2 : 3 ratio.
<u>Compute</u> the length of the hypotenuse.

B) In square $ABCD$, $AB = 6$, $PC = QC$ and

$$\frac{\text{Area}(PQC)}{\text{Area}(PDABQ)} = \frac{1}{5}$$

P and Q lie on $\overline{DC}$ and $\overline{BC}$ respectively.
<u>Compute</u> PQ.



C) <u>Compute</u> the area of the region bounded by $\begin{cases} y = |x-1| + |x-2| + |x-4| \\ x = 0 \\ x = 8 \\ y = 0 \end{cases}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007
ROUND 4 ALG 1: FACTORING AND ITS APPLICATIONS

ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) ______________________________

A) Solve for x. $$\frac{3}{(x-1)^2}+\frac{4}{(x-1)}+1=0$$

B) The difference $\sqrt{3}-17$ may be expressed as the product $\left(\sqrt{3}-A\right)\left(\sqrt{3}-B\right)$, where A and B are integers and $A > B$. Determine the ordered pair (A, B)

C) Solve for x. $$\frac{2x}{x^2+2x-8}+\frac{4}{x^2+5x+4}-\frac{x+3}{x^2-x-2}=0$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007
ROUND 5 TRIG: FUNCTIONS OF SPECIAL ANGLES

ANSWERS

A) ________________________________

B) ____________________ (in degrees)

C) (________ , ________ , ________)

***** NO CALCULATORS ON THIS ROUND *****

A) Compute: $\sin 450° \cot 315° \tan^2 225° \sin^3\left(-\frac{7\pi}{4}\right)$

B) Solve for θ over $0° \leq \theta < 360°$: $\sin^2(-600°) = \tan(945°) \cdot \cot(405°) - \sin^2(x)$

C) Solve for x over $0° \leq x < 360°$: $\dfrac{\tan x + \sqrt{3}}{1 + \sqrt{3}\cot x} = 1$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007
ROUND 6 PLANE GEOMETRY: ANGLES, TRIANGLES AND PARALLELS

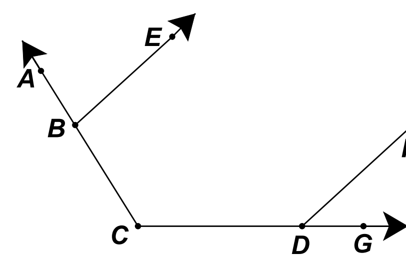
ANSWERS

A) ________________________________ °

B) ________________________________

C) ________________________________

A) Given: $\overrightarrow{BE} \| \overrightarrow{DF}$, m$\angle GDF = 24°$ and m$\angle ABE = 88°$
Find m$\angle BCD$.



B) In equilateral ΔABC, altitude $\overline{AD}$ intersects angle bisector $\overrightarrow{CE}$ at point P.
D lies on side $\overline{BC}$ and E lies on side $\overline{AB}$.
<u>Compute</u> the length of a side of ΔABC, if $AP = 12$.

C) Given a regular 15-gon, if <u>k</u> more sides were added (producing a regular polygon with $15 + k$ sides), then the measure of each interior angle would increase by $(k + 1)$ degrees.
Find <u>all</u> possible values of k.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007
ROUND 7 TEAM QUESTIONS

ANSWERS

A) (____________ , ____________) D) ____________________________

B) ____________________________ E) ____________________________

C) ____________________________ F) ____________________________

A) The equation $x^8 - 81 = 0$ has 8 roots which may be expressed the forms:

$$\pm A, \quad \pm Ai, \quad \pm(B + Bi), \quad \pm(B - Bi)$$

where A and B are <u>positive</u> reals. <u>Compute</u> the ordered pair (A, B).

B) Find <u>all</u> ordered triples (a, b, c) of positive integers that satisfy the following conditions:

- b is prime
- $a + c$ is composite
- $\begin{cases} a+3b+5c=50 \\ 5a+3b+c=70 \end{cases}$

C) An equilateral triangle is inscribed in a square whose side has length 1 unit. One vertex of this equilateral triangle coincides with a vertex of the square. <u>Compute</u> the area of this equilateral triangle.

D) Factor completely. $32a^{7x} - 240a^{6x} + 720a^{5x} - 1080a^{4x} + 810a^{3x} - 243a^{2x}$

E) Given: $\sin(4x) = a\sin x\cos x + b\sin^3 x\cos x$
Find the numerical value of $a + 3b$.

F) In regular polygon P, the reduced ratio of the <u>interior</u> angle to the <u>exterior</u> angle is $a : b$.
In regular polygon Q, the reduced ratio of the <u>exterior</u> angle to the <u>interior</u> angle is $c : d$
If a, b, c and d are positive integers, $a + b = 15$ and $c + d = 12$,
<u>compute</u> all possible ratios of the interior angle of P to the exterior angle of Q.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007 SOLUTION KEY

Round 1

A) $\bar{z} = a - bi$ Thus, $z + \bar{z} = 7a - 5bi = 7 + 3i \rightarrow 7a = 7$ and $-5b = 3 \rightarrow (a, b) = \left(1, -\frac{3}{5}\right)$

$\rightarrow$ $\underline{\mathbf{1 + \left(-\frac{3}{5}\right)\mathit{i}}}$ or $\underline{\mathbf{1 + (-0.6)\mathit{i}}}$

B) Let $z = a + bi$. Then $(a + bi)^2 = 25(3+ 4i) \rightarrow a^2 - b^2 = 3$ and $2ab = 4 \rightarrow (a, b) = (2,1)$ or $(-2, -1)$
$\rightarrow z = 5(2 + i)$ or $5(-2 - i) \rightarrow$ $\underline{\mathbf{10 + 5\mathit{i}, -10 - 5\mathit{i}}}$

C) $|-3+4i| = \sqrt{(-3)^2 + 4^2} = 5, |12+16i| = \sqrt{12^2 + 16^2} = 20, |7-24i| = \sqrt{7^2 + (-24)^2} = 25$
$5x^2 - 20x - 25 = 5(x^2 - 4x - 5) = 5(x - 5)(x + 1) = 0 \rightarrow x =$ $\underline{\mathbf{5, -1}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007 SOLUTION KEY

Round 2

A) Let n denote the number of numbers in the set. Then $20n + 300 = 35n \rightarrow n =$ **<u>20</u>**

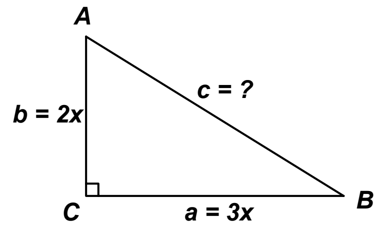
B) Multiplying by $(k-2)(k-1)$, $k^2 + 9 - 6(k-2) = 2k(k-1)$
$\rightarrow k^2 + 9 - 6k + 12 = 2k^2 - 2k \rightarrow k^2 + 4k - 21 = (k+7)(k-3) = 0 \rightarrow k =$ **<u>-7, 3</u>**

C) $\left(\frac{1}{12}+\frac{1}{27}+\frac{1}{2x}+\frac{1}{3x}\right)6 = 1 \rightarrow \frac{5}{6x} = \frac{1}{6} - \frac{1}{12} - \frac{1}{27} \rightarrow \frac{5}{2x} = \frac{1}{2} - \frac{1}{4} - \frac{1}{9} = \frac{5}{36} \rightarrow x = 18$
$\rightarrow$ daughter would take **<u>54 minutes</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007 SOLUTION KEY

Round 3

A) Area $= \frac{1}{2}bh = \frac{1}{2}\cdot 2x \cdot 3x = 60 \rightarrow x = 2\sqrt{5}$



and $(2x)^2 + (3x)^2 = c^2$

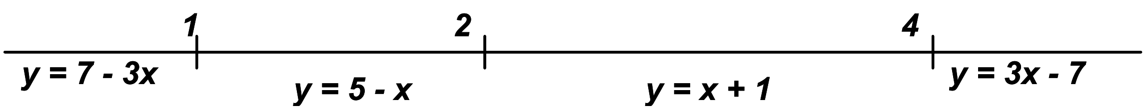
$\rightarrow$ hypotenuse $c = x\sqrt{13} = 2\sqrt{5}\cdot\sqrt{13} = \underline{\mathbf{2\sqrt{65}}}$

B) Let $PC = QC = x$. Then $PQ = x\sqrt{2}$ and

$\frac{1}{2}x^2 : (36 - \frac{1}{2}x^2) = 1:5 \rightarrow \frac{x^2}{72 - x^2} = \frac{1}{5} \rightarrow 6x^2 = 72$

$\rightarrow x = 2\sqrt{3}$ and $PQ = \underline{\mathbf{2\sqrt{6}}}$

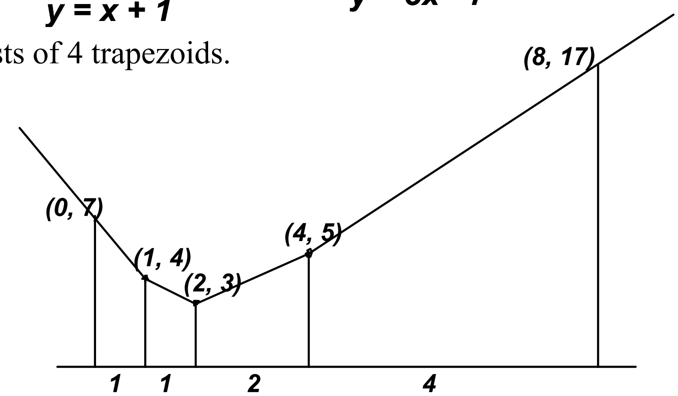
C) The critical points occur at $x = 1$, 2 and 4.
The first equation may be expressed without absolute value over restricted domains as follows:



Thus, the region bounded by this system consists of 4 trapezoids.

$$A = \frac{1}{2}\left(1(4+7) + 1(4+3) + 2(3+5) + 4(5+17)\right)$$

$$= \frac{1}{2}(11 + 7 + 16 + 88) = \frac{122}{2} = \underline{\mathbf{61}}$$



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007 SOLUTION KEY

Round 4

A) Clearing fractions, $3 + 4(x - 1) + (x - 1)^2 = x^2 + 2x = x(x + 2) = 0 \rightarrow x =$ **<u>0, -2</u>**

B) Multiplying out the product we have $(3 + AB) - (A + B)\sqrt{3}$.
Equating coefficients, $3 + AB = -17$ and $A + B = -1$
Substituting for B in the first equation, $3 + A(-A - 1) = -17 \rightarrow A^2 + A - 20 = (A + 5)(A - 4) = 0$
$\rightarrow A = -5$ and $B = 4$ or $A = 4$ and $B = -5$
Since $A > B$, $(A, \mathrm{B}) =$ **<u>(4, -5)</u>**

C) LCD $= (x - 2)(x + 4)(x + 1) \rightarrow 2x(x + 1) + 4(x - 2) - (x + 3)(x + 4) = 0$
$\rightarrow 2^{x2} + 2x + 4x - 8 - x^2 - 7x - 12 = x^2 - x - 20 = (x - 5)(x + 4) = 0$
$\rightarrow$ ***x* = <u>5</u>** (-4 causes division by zero and is extraneous)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007 SOLUTION KEY

Round 5

A) $= 1(-1)(1)^2\left(\frac{\sqrt{2}}{2}\right)^3 = \underline{\mathbf{-\frac{\sqrt{2}}{4}}}$

B) $\sin^2(120) = \tan(225)\cdot\cot(45) - \sin^2(x) \rightarrow \frac{3}{4} = 1 - \sin^2 x \rightarrow \sin x = \pm\frac{1}{2} \rightarrow$ **<u>30°, 150°, 210°, 330°</u>**

C) $\frac{\tan x + \sqrt{3}}{1 + \sqrt{3}\cot x} = 1 \rightarrow \tan x - 1 + \sqrt{3}(1 - \cot x) = 0 \rightarrow \tan^2 x - \tan x - \sqrt{3}(\tan x - 1) = 0$

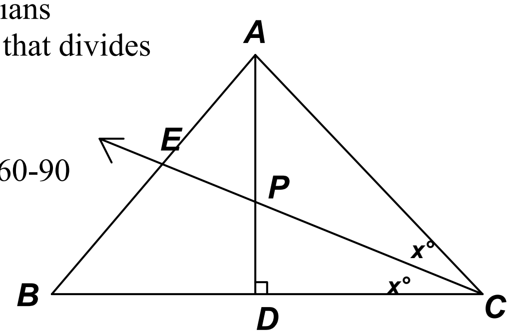
$\rightarrow (\tan x - 1)(\tan x + \sqrt{3}) = 0 \rightarrow x =$ **<u>45°, 225°</u>** [120°, 300° are extraneous]

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007 SOLUTION KEY

Round 6

A) Draw a line through C parallel to $\overrightarrow{BE}$. Since interior angles on the same side of a transversal crossing parallel lines are supplementary, $m\angle BCD = 24 + 88 =$ **<u>112°</u>**.

B) In an equilateral triangle, altitudes, angle bisectors and medians are one and the same! Medians always intersect at a point that divides the median into segments in a 2 : 1 ratio.
$AP = 12 \rightarrow PD = 6$. Thus, the altitude $\overline{AD}$ has length 18.
Since $\overline{AD}$ is also the side opposite the 60° angle in the 30-60-90 ΔBAD, $AB = \frac{18}{\sqrt{3}} \cdot 2 = \underline{\mathbf{12\sqrt{3}}}$



C) The measure of the interior angle of a 15-gon is $\frac{180(15-2)}{15} = 156°$.

The measure of the interior angle of a (15 + k)-gon is $\frac{180(15+k-2)}{15+k} = 180\left(1-\frac{2}{15+k}\right)$

Thus, $180\left(1-\frac{2}{15+k}\right)-156 = k+1 \rightarrow 24-\frac{360}{15+k} = k+1$

$360 + 24k - 360 = k^2 + 16k + 15 \rightarrow k^2 - 8k + 15 = (k-3)(k-5) = 0 \rightarrow k =$ **<u>3, 5</u>**

Check:
$k = 3 \rightarrow$ 15-gon and 18-gon $\rightarrow$ interior angles: 156° and 160°, a 4° difference
$k = 5 \rightarrow$ 15-gon and 20-gon $\rightarrow$ interior angles: 156° and 162°, a 6° difference

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007 SOLUTION KEY

Team Round

A) $x^8 - 81 = (x^4 - 9)(x^4 + 9) = (x^2 - 3)(x^2 + 3)(x^2 - 3i)(x^2 + 3i) = 0$
→ roots: $\pm\sqrt{3}$, $\pm\sqrt{3}i$ $\pm\sqrt{3i}$ and $\pm\sqrt{-3i}$

The last two pairs must be further simplified.

Let $\sqrt{3i} = a + bi$. Squaring, $(a^2 - b^2) + 2abi = 0 + 3i$.

Thus, $\begin{cases} a^2 - b^2 = 0 \\ 2ab = 3 \end{cases}$ → $b = \dfrac{3}{2a}$ and $a^2 - \dfrac{9}{4a^2} = 0$

Substituting, $4a^4 - 9 = (2a^2 + 3)(2a^2 - 3) = 0$ → $a^2 = 3/2$ → $a = \dfrac{\sqrt{6}}{2}$ and $b = \dfrac{3}{\sqrt{6}} = \dfrac{\sqrt{6}}{2}$

and $\sqrt{3i} = \pm\left(\dfrac{\sqrt{6}}{2} + \dfrac{\sqrt{6}}{2}i\right)$. You should verify that $\sqrt{-3i} = \pm\left(\dfrac{\sqrt{6}}{2} - \dfrac{\sqrt{6}}{2}i\right)$.

Therefore, $(A, B) = \underline{\left(\boldsymbol{\sqrt{3}, \dfrac{\sqrt{6}}{2}}\right)}$.

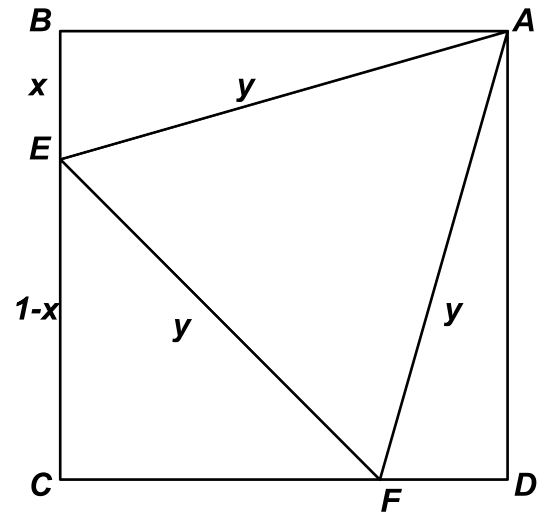
B) Subtracting the second equation from the first, $4a - 4c = 20$ → $a = c + 5$
Substituting in the first equation, $(c + 5) + 3b + 5c = 50$ → $6c + 3b = 45$ → $b = 15 - 2c$

Since all variables must be positive integers, $c \geq 1$ → $a > 6$ and $b = 13, 11, 9, \ldots, 1$
b prime → $b = 13, 11, 7, 5$ or 3

Substituting in the two equations, $\begin{cases} \quad\;\; b = 13\;\; 11\;\;\; 7\;\;\; 5\;\;\; 3 \\ a + 5c = 11\;\; 17\;\; 29\;\; 35\;\; 41 \\ 5a + c = 31\;\; 37\;\; 49\;\; 55\;\; 61 \end{cases}$

Adding $6(a + c) = 42, 54, 78, 90$ or 102 → $a + c = 7, 9, 13, 15$ or 17
Only 9 and 15 are composite $a = c + 5$, 9 → $(a, c) = (7, 2)$ and 15 → $(a, c) = (10, 5)$
→ **<u>(7, 11, 2)</u>** and **<u>(10, 5, 5)</u>**

Team Round - continued



C) $x^2 + 1 = y^2$

$(1-x)^2 + (1-x)^2 = y^2$

→ $2(1-x)^2 = x^2 + 1$

→ $x^2 - 4x + 1 = 0$

→ $x = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 - \sqrt{3}$ ($2+\sqrt{3}$ is rejected)

Thus, $x^2 = 7 - 4\sqrt{3}$ → $y^2 = 8 - 4\sqrt{3}$

Since the area of an equilateral triangles is given by $\dfrac{\text{side}^2\sqrt{3}}{4}$, we have $\dfrac{(8-4\sqrt{3})\sqrt{3}}{4} = \boxed{\mathbf{2\sqrt{3} - 3}}$

Alternate solution (finding y is not necessary):

Let K denote the area of equilateral ΔAEF.

$m\angle BAE = 15°$ → $x = \tan(15°) = 2 - \sqrt{3}$

$2\text{Area}(\Delta ABE) + \text{Area}(\Delta ECF) + K = 1$

→ $2\left(\dfrac{1}{2}\cdot x \cdot 1\right) + \dfrac{1}{2}(1-x)^2 + K = 1$ → $K = (1-x) - \dfrac{1}{2}(1-x)^2 = \dfrac{1}{2}(1-x^2)$

$= \dfrac{1}{2}\left(1 - \left(2-\sqrt{3}\right)^2\right) = \dfrac{1}{2}\left(1 - 4 + 4\sqrt{3} - 3\right) = \underline{\mathbf{2\sqrt{3} - 3}}$

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Team Round – continued

D) Factoring out the common factor of a^{2x} and noticing that there are 6 terms with alternating signs and the first and last terms are perfect fifth powers, we check to see if the expansion is the fifth power of a binomial.

$$a^{2x}\left(\left(2a^{x}\right)^{5}-240a^{4x}+720a^{3x}-1080a^{2x}+810a^{x}-3^{5}\right)$$

$$a^{2x}\left(\left(2a^{x}\right)^{5}-(5)(2a^{x})^{4}(3)+(10)(2a^{x})^{3}(9)-(10)(2a^{x})^{2}(27)+(5)(2a^{x})(81)-3^{5}\right)$$

$$= \underline{\boldsymbol{a^{2x}\left(2a^{x}-3\right)^{5}}}$$

E) $\sin(4x) = \sin(2(2x)) = 2\sin(2x)\cos(2x) = 2[2\sin(x)\cos(x)][1 - 2\sin^2(x)]$
$= 4\sin(x)\cos(x) - 8\sin^3(x)\cos(x) \rightarrow a = 4$ and $b = -8 \rightarrow a + 3b =$ **<u>-20</u>**

or let $x = \pi/4 \rightarrow 0 = 2a + b$ Then let $x = \pi/6 \rightarrow 4a + b = 8 \rightarrow (a, b) = (4, -8) \rightarrow a + 3b =$ **<u>-20</u>**

Note: $(X+Y)^5 = X^5 + \binom{5}{1}X^4Y^1 + \binom{5}{2}X^3Y^2 + \binom{5}{3}X^2Y^3 + \binom{5}{4}X^1Y^4 + Y^5$

where $\binom{n}{r}$ denotes a combination of n items taken r at a time and is evaluated by $\dfrac{n!}{r!\cdot(n-r)!}$.

F) Since the interior and exterior angles in a regular polygon with n sides are given by $\dfrac{180(n-2)}{n}$ and $\dfrac{360}{n}$ respectively, the ratio of the interior angle to the exterior angle is $(n-2) : 2$.

Let P and Q have n and m sides respectively.
$\dfrac{n-2}{2}=\dfrac{a}{b}\rightarrow bn-2b=2a\rightarrow n=\dfrac{2(a+b)}{b}=\dfrac{30}{b}$ and b must be a factor of 30 (and $b \leq 15$)
Thus, $(a, b) =$ (14, 1) corresponding to $n = 30$ or (13, 2) corresponding to $n = 15$.
All other ordered pairs (12, 3), (10, 5), (9, 6) and (10, 5) correspond to <u>un</u>reduced ratios.
Possible interior angles of P are: 168° (for a 30-gon) or 156° (for a 24-gon)

$\dfrac{2}{m-2}=\dfrac{c}{d}\rightarrow cm-2c=2d\rightarrow m=\dfrac{2(c+d)}{c}=\dfrac{24}{c}$ and c must be a factor of 24 (and $c \leq 12$)
Thus, $(c, d) = (1,11)$ corresponding to $m = 24$.
All other ordered pairs (2, 10), (3, 9), (4, 8), (56, 6) and (8, 4) correspond to <u>un</u>reduced ratios.
The only possible exterior angle for Q: 15°
Therefore, possible ratios are: **<u>56 : 5</u>** and **<u>52 : 5</u>**