MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 – DECEMBER 2007
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

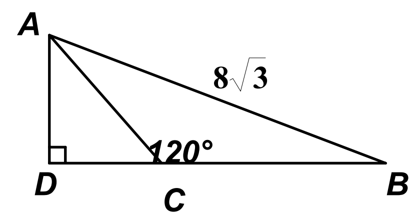
ANSWERS

A) ______________________________

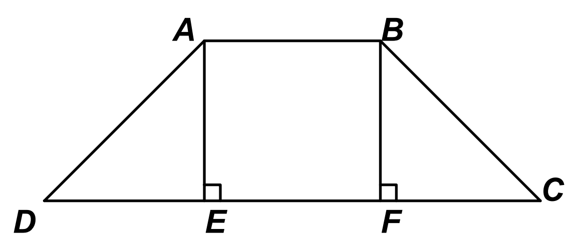
B) ______________ : ______________

C) ______________________________

A) Given: ΔABC is isosceles, $m\angle ACB = 120°$ $AB = 8\sqrt{3}$
<u>Compute</u> AD.



B) The area of an isosceles trapezoid is 840 square units.
Altitudes $\overline{AE}$ and $\overline{BF}$ divide the longer base into three segments of equal length. If the length of an altitude of the trapezoid is
1 unit less than the length of the longer base, what is the ratio of the perimeter of the trapezoid to the altitude of the trapezoid?



C) In ΔABC, $AB = 4$, $AC = 6$, $m\angle C = 30°$.
Determine all possible values for the exact length of $\overline{BC}$ in simplified radical form.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007
ROUND 2 ARITHMETIC/ ELEMENTARY NUMBER THEORY

ANSWERS

A) _______________________________

B) _______________________________

C) _______________________________

A) A magic integer is defined to be a positive integer that is both a perfect square and a perfect cube. Determine the sum of all magic integers less than 100,000.

B) Given the following pattern:

	1	4	4	1	
	1	5	8	5	1
1	6	13	13	6	1

The sum of the entries in row 1 is 10.
Each row has one more entry than the previous row.
Each row begins and ends with 1 and the in-between entries are the sum of the entries immediately to the right and left in the previous row.
What is the sum of the entries in the 16^{th} row?

C) Determine a simplified factored expression, in terms of the positive integer x, for the number of even factors of the following expression:

$$\left(12^{x+1}\right)\cdot\left(18^{x-1}\right)\cdot\left(75^{3}\right)$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007
ROUND 3 ANALYTIC GEOMETRY OF THE STRAIGHT LINE

ANSWERS

A) ______________________________

B) (________ , ________)

C) ______________________________

A) L_1 is a line with equation $2x - ay = 7$.
L_2 is a line with equation $ax - 4y - 12 = 0$.
L_1 intersects L_2 at the point $P(8, a)$. Find <u>all</u> possible values of a.

B) A line segment has endpoints at $A(-6, 10)$ and $B(29, -18)$.
Find the coordinates of the point P that is 5/7 of the way from A to B.

C) The line perpendicular to $3x + 2y - 13 = 0$ at $(1, 5)$ passes through the points $P(a, b)$ and $Q(b, a)$.
<u>Compute</u> the distance between P and Q.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

ANSWERS

A) ________________________________

B) A = __________ B = ____________

C) ________________________________

***** NO CALCULATORS ON THIS ROUND *****

A) Determine all possible values of x for which $\left(\log_3 x\right)^2 - \frac{7}{3}\log_5 125 - 6\log_3 x = 0$

B) Given: $a = \log_{36}(8)$
In terms of a, find a simplified expression for $\log_{216}(48)$.

C) Compute all possible values of x for which $3^{3\log_3 x + 1} - 2^{2\log_2 x} = 2x^4$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

ANSWERS

A) $______________________

B) (________ , ________)

C) ______________________

A) The ratio of the price of an item to the sales tax charged is 20 : 1. If the total cost, including tax, for an item is \$3.99, what is the price of the item?

B) The first of two numbers exceeds the second by x. The sum of the two numbers is y. In terms of x and y, what is the simplified ratio of the first number to the second number?

C) Dick purchased a new sailboat. He was able to determine that the pressure on the sail varies jointly as the area of the sail and the square of the wind velocity in miles per hour. He computed the pressure to be 3 pounds per square foot when the wind velocity was 18 miles per hour. What would the pressure per square foot be if the area of the sail was doubled and the wind velocity was $\frac{1}{2}$ mile per minute?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

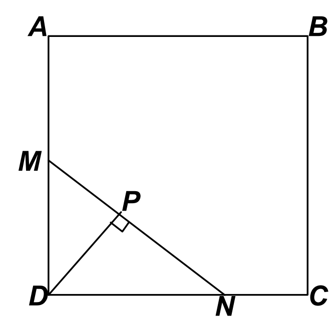
ANSWERS

A) ________________________________

B) ________________________________

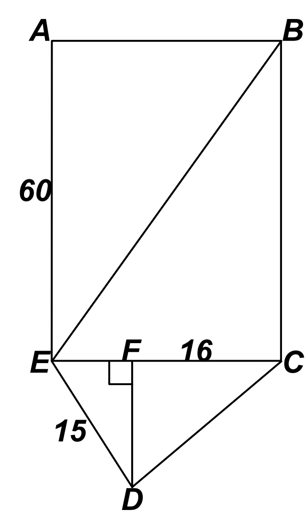
C) ________________________________

A) *ABCD* is a square with side 12. *M* is a midpoint of $\overline{AD}$ and *N* is the trisection point of $\overline{CD}$, closest to *C*. Compute *DP*.



B) If a regular polygon had one more side it would have 23 more diagonals. How many degrees in one of original polygon's interior angles?

C) Pentagon *ABCDE* is the union of rectangle *ABCE* and right triangle *CDE*. $\overline{DF} \perp \overline{CE}$, $AE = 60$, $DE = 15$ and $FC = 16$. Compute *BE*.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ____________________________ D) (___________ , __________)

B) ____________________________ E) ____________________________

C) ____________________________ F) ____________________________

A) In acute ΔABC, all sides a, b and c have integer length. If $a = 7$ and $b = 13$, determine all possible values of c.

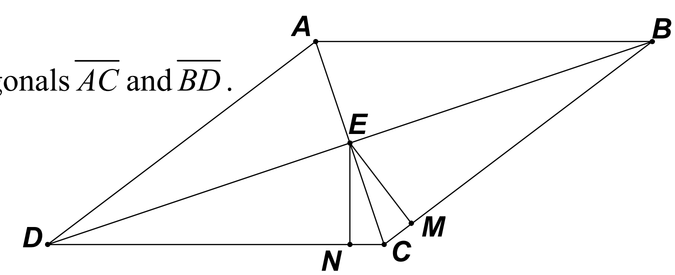
B) The two-digit numeral ab, where $a \neq 0$, represents a prime number in base 8, 10 and 12. Find all possible ordered pairs (a, b).

C) Given: $A(1, 1)$ and $B(3, 6)$.
Point P is the point on the x-axis that minimizes the sum $AP + PB$.
Point Q is the point on the y-axis that minimizes the sum $AQ + QB$.
Let P_x denote the x-coordinate of point P and let Q_y denote the y-coordinate of point Q.
Compute the sum $P_x + Q_y$.

D) A student who knew nothing about logarithms volunteered to be put in this category since his team had no other mathletes who could handle this topic.
He was supposed to evaluate an expression of the form $\dfrac{\log A}{\log B}$ and he simply cancelled the 'log's and arrived at an answer of 2/3 which proved to be correct!!!
Determine the numerical values of ordered pair (A, B).

E) At the end of July, the Red Sox had a record of 85 wins and 47 losses, for a winning percentage of 0.644. Later in the season, after winning W games and losing L games, their winning percentage is greater than or equal to 0.700. There are a maximum of 162 games in a season and all games need not be played. For how many ordered pairs (W, L) is this true?
Note: The winning percentage is always rounded off to three decimal places.

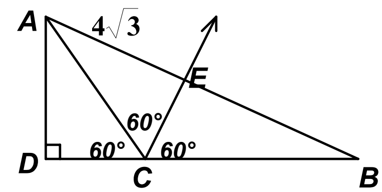
F) $ABCD$ is a rhombus with perimeter 240.
$AC = 72$ and E is the point of intersection of the diagonals $\overline{AC}$ and $\overline{BD}$.
$\overline{EN} \perp \overline{DC}$ and $\overline{EM} \perp \overline{BC}$
Compute the perimeter of $MENC$.



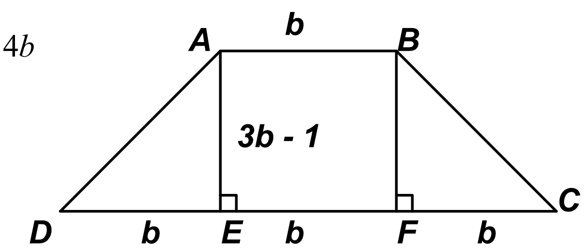
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007 SOLUTION KEY

Round 1

A) ΔABC is a 30-60-90 triangle. Draw altitude $\overline{CE}$ from C to $\overline{AB}$.
$\overline{AB}$ must be the base in isosceles triangle ABC.
Therefore, E must also be a midpoint of $\overline{AB}$ and ΔACE must also be a 30-60-90 triangle congruent to ΔACD → $AD = \underline{\mathbf{4\sqrt{3}}}$
($ADCE$ is a kite)



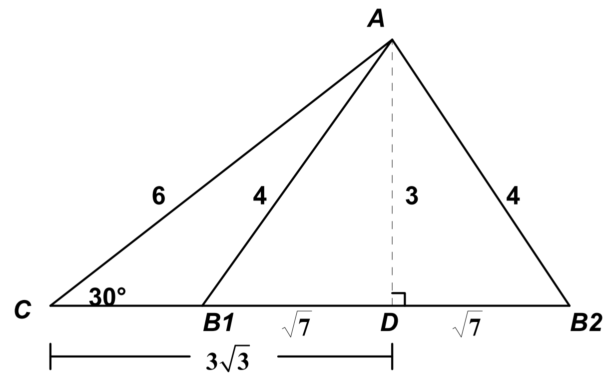
B) $A = 840 = \frac{h}{2}(b_1 + b2) = (3b-1)(4b)/2$ → $1680 = 12b^2 - 4b$
→ $3b^2 - b - 420 = (3b+35)(b-12) = 0$ → b = 12
ΔADE, ΔBCF are 12 – 35 – 37 right triangles
→ $Per = 74 + 48 = 122$, $AE = 35$ → **<u>122 : 35</u>**



C) This is the ambiguous case, where we have information about two sides and the <u>non</u>-included angle. In general, there could be 0, 1 or 2 possible solutions. In this problem there are two solutions.

Using the law of sine, $\frac{AB}{\sin C} = \frac{AC}{\sin B}$ → $\frac{4}{.5} = \frac{6}{\sin B}$

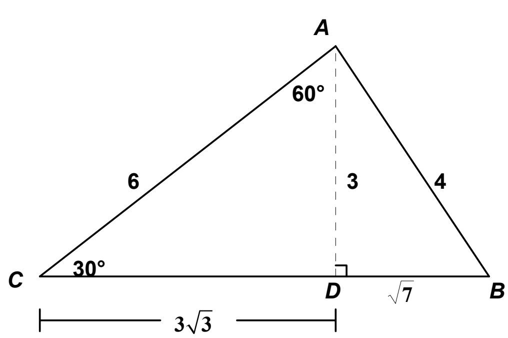
→ $\sin B = ¾$ → $\cos B = \pm\frac{\sqrt{7}}{4}$

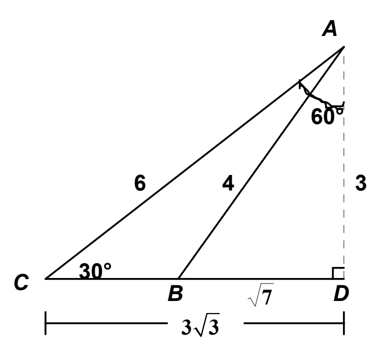


Dropping an altitude from A creates a 30-60-90 triangle ACD → $\overline{AD}$, the side opposite 30°, must have length 3 and $\overline{CD}$, the side opposite 60°, must have length $3\sqrt{3}$.

Clearly, referring to the diagram, the negative cosine value is associated with an obtuse angle ($\angle B$ in ΔACB_1) and the positive cosine value is associated with the acute angle ($\angle B$ in ΔACB_2)
Thus, $BC = \underline{\mathbf{\sqrt{7} + 3\sqrt{3}} \text{ or } \mathbf{3\sqrt{3} - \sqrt{7}}}$.

Alternate solution: Using only 30 – 60 – 90 right triangles (2 diagrams are possible)





MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007 SOLUTION KEY

Round 2

A) Since magic integers are 6^{th} powers of positive integers, we have

$1^6 = 1$, $2^6 = 64$, $3^6 = 729$ $4^6 = 2^{12} = 1024(4) = 4096$

$5^6 = 625(25) = \dfrac{625(100)}{4} = 15615$ $6^6 = 2^6 3^6 = 64(729) = 46656$

$1 + 64 + 729 + 4096 + 15625 + 46656 =$ **<u>67171</u>**

Note in the expansion of 5^6, instead of multiplying by 25, a two-digit number, we multiply by 100 and divide by 4, i.e. append two zeros and divide by a 1-digit multiplier, 4.

B) Summing the entries in the first three rows we have: 10, 20 and 40
The sum of the entries is apparently doubling from one row to the next; or, looking from another perspective, is given by the formula $10(2^{\text{row}-1})$.
$\rightarrow S_{16} = 10(2^{15}) =$ **<u>327680</u>**

$2^{10} = 1024 \rightarrow 2048 \rightarrow 4096 \rightarrow 8192 \rightarrow 16384 \rightarrow 32768$

C) $\left(12^{x+1}\right)\cdot\left(18^{x-1}\right)\cdot\left(75^3\right) = \left(2^2\cdot 3\right)^{x+1}\cdot\left(2\cdot 3^2\right)^{x-1}\cdot\left(5^2\cdot 3\right)^3 = 2^{2x+2}\cdot 3^{x+1}\cdot 2^{x-1}\cdot 3^{2x-2}\cdot 5^6\cdot 3^3$

$= 2^{3x+1}\cdot 3^{3x+2}\cdot 5^6$

Since the factor must be even, there are $(3x + 1)$ choices for the exponent of 2, namely $1, 2, \ldots, 3x+ 1$.
However, since 0 is an allowable exponent for 3 and 5, there are $(3x + 3)$ and 7 choices for the exponents of 3 and 5 respectively.

Thus, the number of even factors is $\mathbf{(3x + 1)(3x + 3)(7) =}$ **<u>$3\cdot 7(x + 1)(3x + 1)$</u>** or **<u>$21(x + 1)(3x + 1)$</u>**

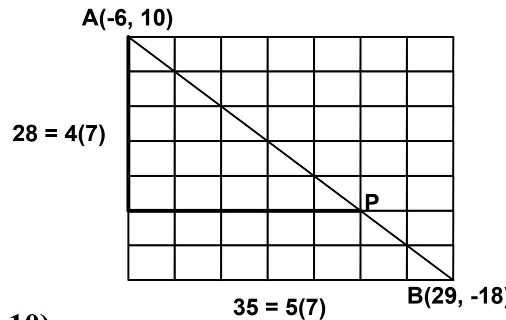
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007 SOLUTION KEY

Round 3

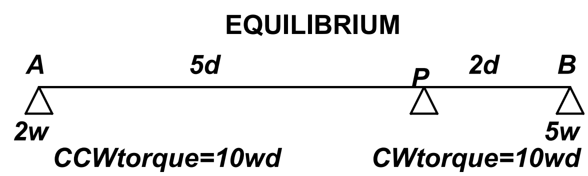
A) The coordinates of point P must satisfy both equations.
Thus, both $2\cdot 8 - a^2 = 7$ and $8a - 4a - 12 = 0$ must be true.
$a^2 = 9$ is satisfied by both ± 3, but the second equation is only satisfied by a = **3**.

B) Each block in the vertical direction is 4 units and each block in the horizontal direction is 5 units.
$(-6 + 5 \cdot 5, 10 - 5 \cdot 4) \rightarrow$ **(19, -10)**
Alternate solution:
$5/7 \rightarrow AP : PB = 5 : 2$ Thus, the coordinates of P are determined by 'weighting' the coordinates.
Since P is closer to B than A its 'influence" is greater.
In fact, B will be counted 5 times and A only twice.

$$P\left(\frac{2(-6)+5(29)}{2+5}, \frac{2(10)+5(-18)}{2+5}\right) = \left(\frac{145-12}{7}, \frac{20-90}{7}\right) = \mathbf{(19, -10)}$$



The diagram at the right illustrates this weighting of x- and y-coordinates in terms of a balancing act where the force producing a clockwise turn around point P equals the force producing a counterclockwise turn around point P; hence, the term equilibrium.



C) Since perpendicular lines have negative reciprocal slopes, the perpendicular to $3x + 2y - 13 = 0$ has the form $2x - 3y + c = 0$.
Since this line must also pass through (1, 5), we can find c by substituting for x and y.
$2(1) - 3(5) + c = 0 \rightarrow c = 13$ and the required line is $2x - 3y + 13 = 0$.

Substituting the coordinates of the points that lie on this line, $\begin{cases} P)\ 2a - 3b + 13 = 0 \\ Q)\ 2b - 3a + 13 = 0 \end{cases}$

Subtracting, $5a - 5b = 0 \rightarrow a = b \rightarrow P$ and Q are the same point $\rightarrow PQ$ = **0**
Aside #1:
Since the slope of $\overline{PQ}$, given $P(a, b)$ and $Q(b, a)$ is $\dfrac{b-a}{a-b}$ = -1 and the slope of the given line $\neq$ -1, the only way both P and Q could be on the line is for P and Q to be the same point!

Aside #2:
Suppose both (h, k) and (k, h) lie on a line $Ax + By + C = 0$. Then

$\begin{cases} Ah + Bk + C = 0 \\ Ak + Bh + C = 0 \end{cases}$ Subtracting, $A(h-k) + B(k-h) = 0 \rightarrow A(h-k) = -B(k-h) = B(h-k)$

$\therefore A = B$ or $h = k \rightarrow$ if x- and y- coefficients are unequal, (h, k) and (k, h) must be the same point.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007 SOLUTION KEY

Round 4

A) $\left(\log_3 x\right)^2 - 6\log_3 x - 7 = \left(\log_3 x + 1\right)\left(\log_3 x - 7\right) = 0 \rightarrow \log_3 x = -1, +7 \rightarrow x = \underline{\mathbf{\frac{1}{3}, 2187}}$

B) $a = \log_{36}(8) \rightarrow 2a = \log_6(8)$

If $N = \log_{216}(48)$, then $3N = \log_{\sqrt[3]{216}}(48) = \log_6(48) = \log_6(6\cdot 8) = 1 + \log_6(8) = 1 + 2a$

Thus, $N = \underline{\mathbf{\frac{2a+1}{3}}}$

C) $3^{3\log_3 x\ +\ 1} - 2^{2\log_2 x} = 2x^4 \rightarrow 3x^3 - x^2 = 2x^4 \rightarrow x^2(2x^2 - 3x + 1) = x^2(x - 1)(2x - 1) = 0$

$\rightarrow x = 0$ (extraneous), $\underline{\mathbf{1, \frac{1}{2}}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007 SOLUTION KEY

Round 5

A) Let x denote the sales tax (in cents).

$\frac{20}{1} = \frac{399 - x}{x} \rightarrow 20x = 399 - x \rightarrow 21x = 399 \rightarrow x = 19 \rightarrow$ price = **$3.80**

B) Let A, B denote the 1st and 2nd number respectively. Then $\begin{cases} A = B + x \\ A + B = y \end{cases}$

$\rightarrow 2B + x = y \rightarrow B = \frac{y - x}{2}$ and $A = \frac{y - x}{2} + x = \frac{y + x}{2}$

Thus, $A : B =$ $\underline{\mathbf{(y + x) : (y - x)}}$

C) We were given that the pressure varies according to the formula $P = kAv^2$, where k is a proportionality constant to be determined.

Substituting for the first set of conditions, $3 = kA18^2 \rightarrow k = \frac{3}{A \cdot 18^2}$

Substituting for the second set of conditions, $P = \frac{3}{A \cdot 18^2}(2A)(\mathbf{30})^2 = \frac{6 \cdot 30^2}{18^2} = \frac{6 \cdot 5^2}{3^2} = \underline{\mathbf{\frac{50}{3}}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007 SOLUTION KEY

Round 6

A) $MD = 6$ and $DN = 8$ → $MN = \sqrt{6^2 + 8^2} = 10$.

The area of ΔMDN is given by $\frac{1}{2} \cdot MD \cdot DN$ and $\frac{1}{2} \cdot MN \cdot DP$. Thus, $\frac{1}{2} \cdot 6 \cdot 8 = \frac{1}{2} \cdot 10 \cdot DP$

→ $DP =$ **<u>4.8</u>**

B) Let n denote the number of sides in the original polygon.

Then $\frac{(n+1)(n-2)}{2} - \frac{n(n-3)}{2} = 23$ → $-n - 2 + 3n = 46$ → $n = 24$ and $\frac{180(22)}{24} =$ **<u>165</u>**.

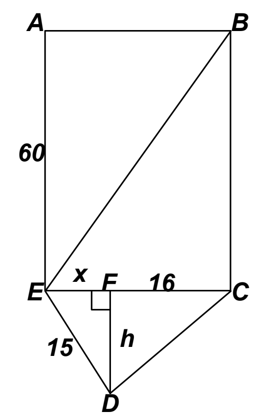
C) Let $FE = x$ and $DE = h$ Then:

$x^2 + h^2 = 15^2 = 225$ and $h^2 = 16x$

→ $x^2 + 16x = 225$ → $x^2 + 16x - 225 = (x - 9)(x - 25) = 0$

→ $x = 9$ → $AB = CE = 25$

Applying the Pythagorean Theorem or using Pythagorean Triples, $BE =$ **<u>65</u>**



$BE^2 = 25^2 + 60^2 = 4225$ and $\sqrt{4225} =$ **<u>65</u>**

(25, 60, __) = 5(5, 12, *13*) → **<u>65</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007 SOLUTION KEY

Team Round

A) An acute triangle must have 3 acute angles.
Therefore, cos$A$, cos$B$ and cos$C$ must each be positive.

The triangle inequality requires that $a + c > b \rightarrow 7 + c > 13 \rightarrow c \geq 7$.
Using the Law of Cosines,

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{218-c^2}{2(7)(13)} \geq 0 \rightarrow c \leq 14$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{c^2-120}{2(7)(c)} \geq 0 \rightarrow c \geq 11$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{120+c^2}{2(13)(c)} \geq 0 \text{ (true for all values of } c \text{ in consideration)}$$

Thus, for $11 \leq c \leq 14$, ΔABC is acute $\rightarrow$ **<u>11, 12, 13, and 14</u>**

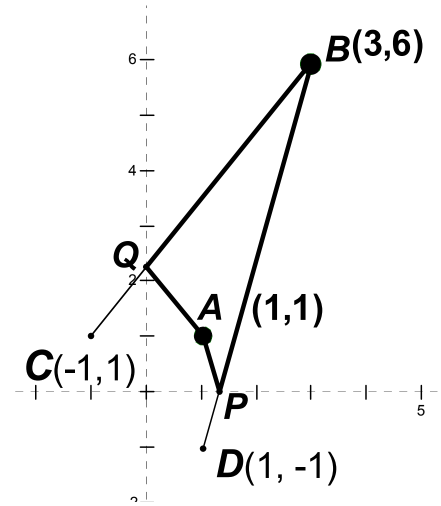
B) $\begin{cases} 8a+b \\ 10a+b \\ 12a+b \end{cases}$ each represent a prime number.

The possible values of a and b are limited to the digits 0.. 7 (allowable digits in all 3 bases).
b can't be even (otherwise each expression would generate a nonprime)
$b \neq 5$ (otherwise $10a + b$ would not be prime)
$b \neq 3$ (otherwise $12a + b$ would not be prime)
Thus, we exam only those cases where $b = 1$ or 7.

		b = 1			b=7	
a	***8a+b***	***10a+b***	***12a+b***	***8a+b***	***10a+b***	***12a+b***
1	9	x	x	15	x	x
2	17	21	x	23	27	x
3	25	x	x	**<u>31</u>**	**<u>37</u>**	**<u>43</u>**
4	33	x	x	39	x	x
5	41	51	x	47	57	x
6	49	x	x	55	x	x
7	57	x	x	63	x	x

Thus, the only ordered pair producing 3 primes is **<u>(3, 7)</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007 SOLUTION KEY



C) Let $C(-1, 1)$ be the reflection of point A across the y-axis.

$$\begin{cases} QA = QC \\ BC = BQ + QC \end{cases} \rightarrow BC = BQ + QA$$

Thus, Q is the point on the y-axis that minimizes $AQ + QB$.

$\overleftrightarrow{BC}$: slope = 5/4 equation: $5x - 4y + 9 = 0 \rightarrow b = 9/4$

Similarly, let $D(1, -1)$ be the reflection of point A across the x-axis.

$\overleftrightarrow{BD}$: slope = 7/2 equation: $7x - 2y - 9 = 0 \rightarrow a = 9/7$

$$\frac{9}{4} + \frac{9}{7} = \frac{63+36}{28} = \underline{\mathbf{\frac{99}{28}}}$$

D) $$\frac{\log A}{\log B} = \frac{\cancel{\log} A}{\cancel{\log} B} = \frac{A}{B} = \frac{2}{3} \rightarrow B = \frac{3A}{2} \text{ Then } \frac{\log A}{\log\left(\frac{3A}{2}\right)} = \frac{2}{3}$$

$$\rightarrow 3\log A = 2\log\left(\frac{3A}{2}\right) = 2(\log 3 + \log A - \log 2)$$

$$\rightarrow \log A = 2(\log 3 - \log 2) = 2\log\left(\frac{3}{2}\right) = \log\left(\left(\frac{3}{2}\right)^2\right) = \log\left(\frac{9}{4}\right)$$

Since the log function is a one-to-one function, $A = \frac{9}{4}$ and $B = \frac{3 \cdot \frac{9}{4}}{2} = \frac{27}{8} \rightarrow \underline{\left(\mathbf{\frac{9}{4}, \frac{27}{8}}\right)}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007 SOLUTION KEY

Team Round - continued

E) $\dfrac{85+W}{132+W+L} \geq 0.700$ (rounded to 3 dec. pl.)

$850 + 10W \geq 924 + 7W + 7L \rightarrow L \leq \dfrac{3W-74}{7}$

The minimum value of W is determined by letting $L = 0$.

$\dfrac{85+W}{132+W} \geq 0.700 \rightarrow 3W > 74 \rightarrow W \geq 25$

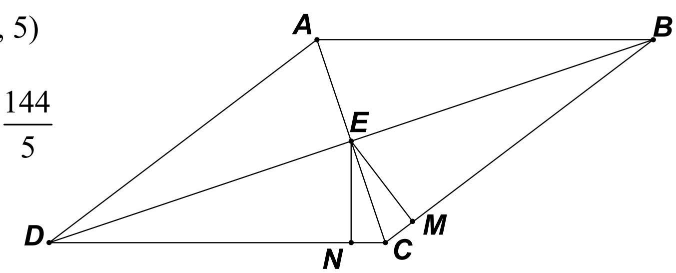
There are a maximum of 30 games remaining in the schedule.

The ordered pair (25, 0) works (0.701), as well as $(W, 0)$ for $26 \leq W \leq 30$ $\rightarrow$ 6 pairs
(26,1) fails (0.698)
(27,1) passes (0.700), as do (28, 1) and (29, 1)
(27, 2) fails (0.696)
(28, 2) fails (0.6975)
Thus, there are **9** pairs.

F) $DC = 60$, $EC = 36$

ΔDEC is a right triangle $\rightarrow$ (36, ? , 60) = 12(3, x, 5)

$\rightarrow x = 4 \rightarrow DE = 48$



Area(ΔDEC) = $\dfrac{1}{2}\cdot 36\cdot 48 = \dfrac{1}{2}\cdot 60\cdot NE \rightarrow NE = \dfrac{144}{5}$

In right ΔNEC, $(EN, NC, EC) = \left(\dfrac{144}{5}, ?, 36\right)$

$= \dfrac{1}{5}(144, ?, 180) = \dfrac{36}{5}(4, x, 5)$

$x = 3 \rightarrow NC = \dfrac{108}{5}$

Since $MENC$ is a kite, its perimeter is $2\left(\dfrac{144+108}{5}\right) = \mathbf{\dfrac{504}{5}}$ or **100.8**