MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2008
ROUND 1 ANALYTIC GEOMETRY: ANYTHING

ANSWERS

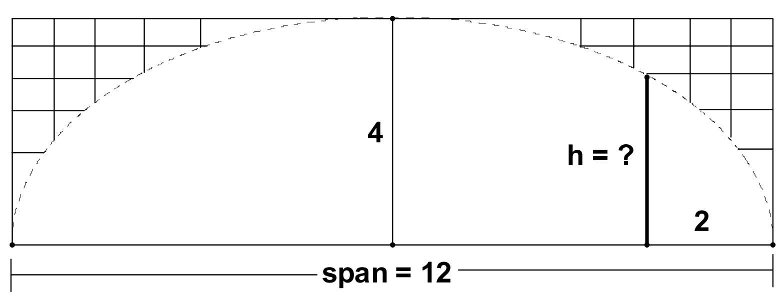
A) ______________________________

B) ______________________________

C) ______________________________

A) The equation of a circle of radius 4 is $(x^2 + 4x) + (y^2 - 2y) + F = 0$.
Determine the value of F.

B) The arch of a bridge is in the form of half an ellipse, with a horizontal major axis. The span of the bridge is 12 meters and the height of the arch above water is 4 meters at its center. How high (in meters) above the water is the arch at a point on the water 2 meters from the end of the arch? Your answer must be exact.



C) A parabola has a focal chord with endpoints at (2, 0) and (2, 6) and opens to the right.
The point $(2.5, y)$, where $y > 0$, lies on this parabola.
<u>Compute</u> all possible values of y.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2008
ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Find all values of <u>a</u> so the expression $4x^2 + 8ax + 25$ is a perfect trinomial square.

B) For some integer values of <u>a</u>, the expression $x^2 + ax - 15$ may be written as the product of two binomials with integer coefficients.
For which of these values of <u>a</u>, does the expression $ax^2 + 98$ have two distinct linear factors with integer coefficients?

Note: A linear factor has the form $mx + b$, where $m \neq 0$.

C) Find all real values of x for which $\dfrac{2x^2 + x - 1}{x^2 - x - 2} = 1 - 2x$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2008
ROUND 3 TRIG: EQUATIONS WITH A REASONABLE NUMBER OF SOLUTIONS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Solve for x over $0 \le x < 2\pi$: $2(\cos x - \sin x) = 1 - \tan x$

B) Solve for x over $0 \le x < 360°$. $\sin 140^\circ \cos 220^\circ = \dfrac{\cos x}{\sec 60^\circ}$

C) There are n values of x, where $0° \le x < 360°$ that satisfy: $\tan^2 x \cdot \sec^2 x + 1 = \tan^2 x + \sec^2 x$
Let S denote the sum of these solutions. Compute $S - n$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2008
ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) Equation: ______________________________

B) ______________________________________

C) ______________________________________

A) Find a quadratic equation of the form $x^2 + Bx + C = 0$, where B and C are integers, given that $2-i\sqrt{5}$ is one of its roots.

B) The sum of the squares of two positive real numbers L and W is 81. Twice the larger number is 9 more than the smaller number. Determine $|L - W|$.

C) $x^2 + Ax + B = 0$ and $x^2 + px + q = 0$ are different equations.
Each of the roots of the equation $x^2 + Ax + B = 0$ are 3 more than twice the corresponding roots of $x^2 + px + q = 0$. If $A : B = -2 : 3$, compute the ratio of $p : q$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2008
ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

ANSWERS

A) (__________ , __________)

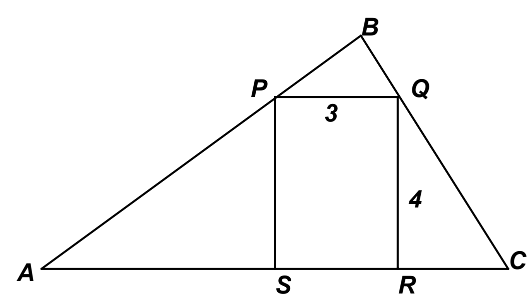
B) ________________________________

C) ________________________________

A) ΔABC is a right triangle with legs $AB = 3$ and $BC = 4$. $\Delta DEF \sim \Delta ABC$ and $DF = 6$. Determine the ordered pair (DE, EF).

B) A line parallel to the short sides of a 12 x 25 rectangle subdivides the rectangle into two similar noncongruent rectangles. Determine the area of the larger of these two rectangles.

C) If $PQRS$ is a 3 x 4 rectangle as illustrated, $\overline{AB} \perp \overline{BC}$ and $RC = 3$, compute the perimeter of ΔABC.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2008
ROUND 6 ALG 1: ANYTHING

ANSWERS

A) ______________________________

B) (________, ________)

C) (__________ , __________)

A) Earl is five years older than his favorite cousin. Thirteen years ago, he was twice his cousin's age. How old is Earl now?

B) Line L_1 has an x-intercept of 5 and a y-intercept of -2. Find the coordinates of the point on L_1 that is closest to $P(-1, 15)$.

C) Mixture #1 is 3 parts alcohol and 1 part water.
Mixture #2 is 2 parts alcohol and 1 part water.
x quarts of mixture #1 and y quarts of mixture #2 are combined to make at least 6 gallons of a mixture that is 5 parts alcohol and 2 parts water.
Determine the ordered pair (x, y) for which $x + y$ is a minimum.

Note: 4 quarts = 1 gallon

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2008
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ______________________________ D) ______________________________

B) ______________________________ E) ______________________________

C) ______________________________ F) ______________________________

A) In a plane, the locus of a curve is defined by the parametric equations

$$x = 9\sec(t) \text{ and } y = 7\tan(t), \text{ where } 90° < t < 180°$$

Express x directly as a simplified function of y.

B) Determine <u>all</u> ordered pairs (x, y) of positive integers, where

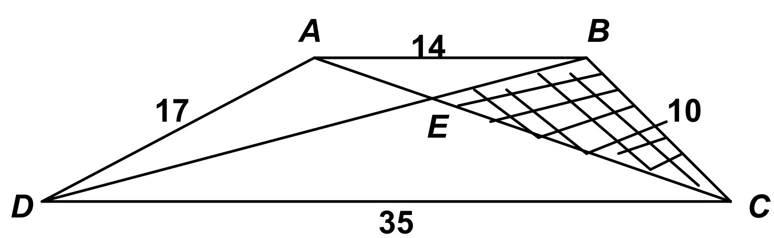
$$x > y \text{ and } x^3 - x^2y - xy^2 + y^3 = 1024.$$

C) Determine the <u>sum</u> of all values of x over $[0, 360°)$ for which

$$\cot^2(270° - 2x) - \csc(90° + 2x) - 1 = 0$$

D) Determine <u>all</u> ordered pairs of integers (n, x) for which $n > 3$ and $\sum_{k=3}^{k=n}(xk+3) = 45$.

E) In trapezoid $ABCD$ (with bases $\overline{AB}$ and $\overline{CD}$), $AB = 14$, $BC = 10$, $CD = 35$ and $AD = 17$.
<u>Compute</u> the area of ΔBEC.



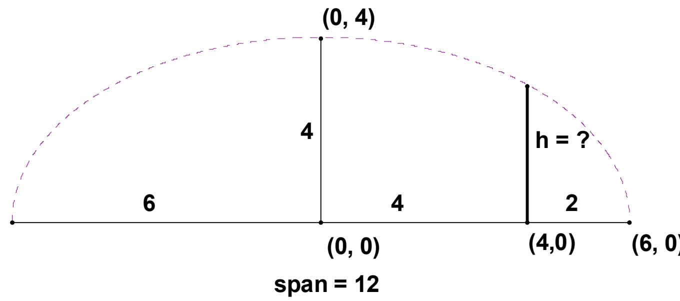
F) Let a, b and c be positive integers and a and b be consecutive.
If $a + b + c = 21$, determine the sum of all distinct products abc.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2008 SOLUTION KEY

Round 1

A) Completing the square, $(x^2 + 4x \underline{+ 4}) + (y^2 - 2y \underline{+ 1}) = -F + 4 + 1 = 5 - F = r^2 = 16 \rightarrow F = \underline{\mathbf{-11}}$

B) The equation of the ellipse is $\frac{x^2}{6^2} + \frac{y^2}{4} = 1$



$\rightarrow x^2 + 9y^2 = 36$
On the right side, 2 meters from the end of the arch is located at (4, 0). Substituting,
$y^2 = \frac{36-16}{9} \rightarrow y = \underline{\mathbf{\frac{2\sqrt{5}}{3}}}$

C) The focus of the parabola is located at (2, 3).
The focal width = $4|a| = 6 \rightarrow a = +3/2$.
Since the focal chord is vertical, the equation of the parabola has the form $(y-k)^2 = 4a(x-h)$, where (h, k) are the coordinates of the vertex.
$a = +3/2 \rightarrow$ the vertex is at (1/2, 3).

Thus, the equation of the parabola is $(y-3)^2 = +6(x - \frac{1}{2})$

Substituting $x = 2.5$, $(y-3)^2 = 12$. $y > 0 \rightarrow y = \underline{\mathbf{3 + 2\sqrt{3}}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2008 SOLUTION KEY

Round 2

A) $4x^2 + 8ax + 25 = (2x \pm 5)^2 = 4x^2 \pm 20x + 25 \rightarrow 8a = \pm 20 \rightarrow a = \underline{\pm\frac{5}{2}}$

B) -15 factors as (1)(-15), (-1)(15), (3)(-5), (-3)(5), $\rightarrow a = \pm 14$ or ± 2
The corresponding factorizations are: $14(x^2 + 7)$, $-14(x^2 - 7)$, $2(x^2 + 49)$ and $-2(x^2 - 49)$ and only the latter has two distinct linear factors over the integers. Thus, $a = \underline{\mathbf{-2}}$

C) $\dfrac{2x^2 + x - 1}{x^2 - x - 2} = 1 - 2x \rightarrow \dfrac{(2x-1)(x+1)}{(x-2)(x+1)} = 1 - 2x$

Clearly, $x = -1$ is not a solution. Canceling, $\dfrac{(2x-1)}{(x-2)} = 1 - 2x \rightarrow 2x - 1 = (x - 2)(1 - 2x)$

$2x - 1 = x - 2x^2 - 2 + 4x \rightarrow 2x^2 - 3x + 1 = (x - 1)(2x - 1) = 0 \rightarrow x = \underline{\mathbf{1, \frac{1}{2}}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2008 SOLUTION KEY

Round 3

A) Potential extraneous solutions: ($\cos x = 0$) $x \neq \pi/2 + n\pi$

$2(\cos x - \sin x) = 1 - \tan x = 1 - \dfrac{\sin x}{\cos x} \rightarrow 2\cos x(\cos x - \sin x) = \cos x - \sin x$

$(\cos x - \sin x)(2\cos x - 1) = 0$

$\rightarrow \cos x = \sin x \rightarrow x =$ **$\underline{\pi/4, 5\pi/4}$**

$\rightarrow \cos x = 1/2 \rightarrow x =$ **$\underline{\pi/3, 5\pi/3}$**

B) $\sin 140^\circ \cos 220^\circ = \dfrac{\cos x}{\sec 60^\circ} \rightarrow$

$\cos x = 2\sin 140^\circ \cos 220^\circ = \sin(A - B) + \sin(A + B) = 2\sin A \cos B$

$\rightarrow A = 140, B = 220$

Thus, $\cos x = \sin(-80) + \sin 360 = -\sin(80) = -\cos(10)$

Thus, x denotes a related value of 10° in quadrant 2 or 3 $\rightarrow x =$ **$\underline{170°, 190°}$**

C) $\tan^2 x \cdot \sec^2 x - \tan^2 x - \sec^2 x + 1 = 0 \rightarrow \tan^2 x(\sec^2 x - 1) - (\sec^2 x - 1) = (\tan^2 x - 1)(\sec^2 x - 1) = 0$

$\rightarrow \tan x = \pm 1 \rightarrow x =$ 45°, 135°, 225°, 315° or $\sec x = \pm 1 \rightarrow x =$ 0°, 180°

$\rightarrow 900 - 6 =$ **$\underline{894}$**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2008 SOLUTION KEY

Round 4

A) Integer coefficients → roots must occur in conjugate pairs.
Thus, the two roots are $2 \pm i\sqrt{5}$ → sum = 4 and product = 9 → $\underline{\mathbf{x^2 - 4x + 9 = 0}}$

B) Let L denote the larger of the positive numbers.

$$\begin{cases} L^2 + W^2 = 81 \\ 2L = 9 + W \end{cases}$$

→ $L^2 + (2L-9)^2 = 81$ → $5L^2 - 36L = L(5L - 36) = 0$ → $L = \dfrac{36}{5}$ and $W = \dfrac{27}{5}$ → $|L - W| = \underline{\mathbf{\dfrac{9}{5}}}$

C) Assume the roots of the original quadratic are r_1 and r_2 and the corresponding roots of the new equation are s_1 and s_2. Then $s_1 = 2r_1 + 3$ and $s_2 = 2r_2 + 3$

According to the root/coefficient relationship for quadratics, $p = -(r_1 + r_2)$ and $q = r_1r_2$.
Also $A = -(s_1 + s_2) = -(2(r_1 + r_2) + 6) = \underline{\mathbf{2p - 6}}$ or $\underline{\mathbf{2(p-3)}}$
$B = s_1s_2 = (2r_1 + 3)(2r_2 + 3) = 4r_2r_2 + 6(r_1 + r_2) + 9 = \underline{\mathbf{9 - 6p + 4q}}$
Continuing, $\dfrac{2p-6}{9-6p+4q} = \dfrac{-2}{3}$ → $6p - 18 = 18 - 12p + 8q$ → $6p = 8q$ → $\dfrac{p}{q} = \underline{\mathbf{4 : 3}}$

Note: If $A = 6$ and $B = 9$, then the first equation, $x^2 + 6x + 9 = 0$ has a double root of -3. Since 2(-3) + 3 = -3, the second equation would be identical.
In the above solution, $A = 6 - 2p = 6$ → $p = 0$ and $B = 9 - 6p + 4q = 9$ → $q = 0$
In this situation the ratio of $p : q$ would be indeterminant. Thus, it was necessary to require that the equations be different.

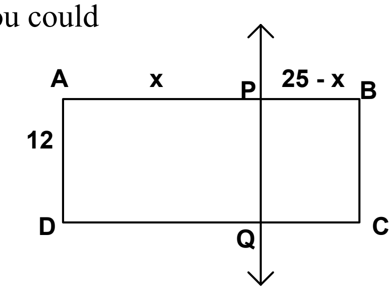
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2008 SOLUTION KEY

Round 5

A) $AC = 5$ → the scale factor is 6/5, the legs of ΔDEF are slightly longer than the legs in ΔABC.

Specifically, $\frac{6}{5}(3,4) = \underline{\left(\mathbf{\frac{18}{5}, \frac{24}{5}}\right)}$

B) If you don't want to experiment with various subdivisions of 25, you could approach the problem algebraically. Suppose the side of length 25 is divided into lengths of x and $(25 - x)$.



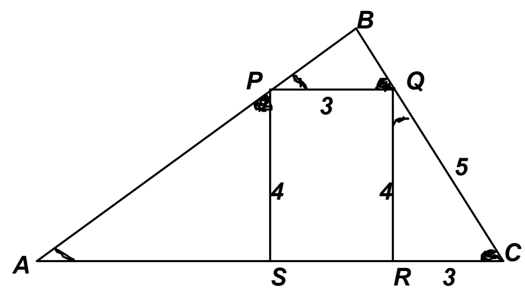
Then the ratio of corresponding sides (<u>short to long</u>) is:

$\frac{12}{x} = \frac{25-x}{12}$ → $x^2 - 25x + 144 = (x - 9)(x - 16) = 0$

→ $x = 9$ or 16 (Since x must be greater than 12, 9 is rejected.)

$x = 16$ → area = 16(12) = **<u>192</u>**.

C) QRC is a 3-4-5 right triangle.



$\Delta PBQ \sim \Delta QRC$ and the scale factor is $\frac{3}{5}$

→ $BQ = \frac{3}{5}(3) = \frac{9}{5}$ and $BP = \frac{3}{5}(4) = \frac{12}{5}$

$\Delta ASP \sim \Delta QRC$ and the scale factor is $\frac{4}{3}$

→ $AS = \frac{4}{3}(4) = \frac{16}{5}$ and $AP = \frac{4}{3}(5) = \frac{20}{3}$

Thus, the perimeter of ΔABC is $8 + \frac{21}{5} + \frac{36}{3} + 3$ = 23 + 4.2 = **<u>27.2</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2008 SOLUTION KEY

Round 6

		Now	Then(13yrs ago)	
A)	Earl	x	x -13	→ x – 13 = 2(x – 18) = 2x – 36 → x = **<u>23</u>**
	Cousin	$x-5$	x -18	

B) The equation of L_1 is 2x – 5y = 10. The point of L_1 closest to P(-1, 15) is the foot of the perpendicular drawn from P to L_1. Since perpendicular lines have negative reciprocal slopes, the equation of a perpendicular line to L_1 is of the form $5x + 2y = c$. Substituting x = -1 and y = 15, we can determine the value of c for which the perpendicular passes through point P.
Thus, c = 25. The solution of the system $\begin{cases} 2x-5y=10 \\ 5x+2y=25 \end{cases}$ is **<u>(5, 0)</u>**.

C) Alcohol: $\frac{3}{4}x+\frac{2}{3}y=\frac{5}{7}(x+y)$ and $x + y \geq 24$

Clearing fractions (LCM = 84), $63x + 56y = 60x + 60y$ → $3x = 4y$ or $y = \frac{3}{4}x$

$x + \frac{3}{4}x \geq 24$ → $7x \geq 96$ → $x > 13$ → $x = 16$ → **<u>(16, 12)</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2008 SOLUTION KEY

Team Round

A) $\dfrac{x^2}{81} = \sec^2 t$ and $\dfrac{y^2}{49} = \tan^2 t$

Since $1 + \tan^2 x = \sec^2 x$, $\dfrac{x^2}{81} = \dfrac{y^2}{49} + 1 \rightarrow x^2 = \dfrac{81}{49}\left(y^2 + 49\right) \rightarrow x = \pm\dfrac{9}{7}\sqrt{y^2+49}$

However, since $90° < t < 180°$, $\cos(t) < 0 \rightarrow \sec(t) < 0 \rightarrow x < 0 \rightarrow x =$ $\underline{\mathbf{-\frac{9}{7}\sqrt{y^2+49}}\text{ only}}$

B) $x^3 - x^2y - xy^2 + y^3 = x^2(x - y) - y^2(x - y) = (x^2 - y^2)(x - y) = (x - y)^2(x + y)$ and
Here's a list of factors of 1024, where the first factor is a perfect square.

1(1024), 4(256), 16(64), 64(16), 256(4), 1024(1)

Since $x + y$ and $x - y$ have the same parity (both even or both odd),
only the middle 4 case are considered.
Thus, $x - y = 2, 4, 8$ or 16 and the corresponding values of $x + y = 256, 64, 16$ or 4 respectively.
Adding, $2x = 258, 68, 24$ or $20 \rightarrow (x, y) =$ **<u>(129, 127), (34, 30), (12, 4)</u>**
[(10, -6) is rejected since both coordinates were required to be positive.]

C) The original equation is equivalent to: $\tan^2(2x) - \sec(2x) - 1 = 0$

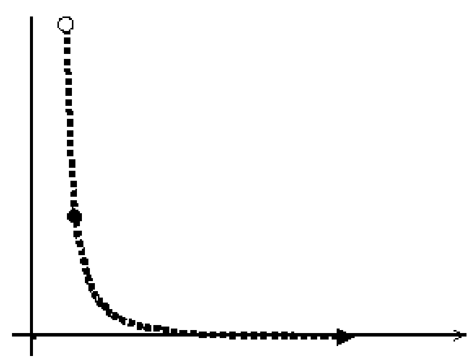
$\rightarrow \sec^2(2x) - 1 - \sec(2x) - 1 = \sec^2(x) - \sec(2x) - 2 = (\sec(2x) - 2)(\sec(2x) + 1) = 0$
$\sec(2x) = 2 \rightarrow \cos(2x) = ½ \rightarrow 2x = \pm 60° + 360n \rightarrow x = \pm 30 + 180n \rightarrow 30, 210, 150, 330$
$\sec(2x) = -1 \rightarrow \cos(2x) = -1 \rightarrow 2x = 180 + 360n \rightarrow x = 90 + 180n \rightarrow x = 90, 270$
The required sum is $30 + 90 + 150 + 210 + 270 + 330 =$ **<u>1080°</u>**

D) Expanding, $(3 + 4 + 5 + \ldots + n)x + 3(n - 3 + 1) = 45$

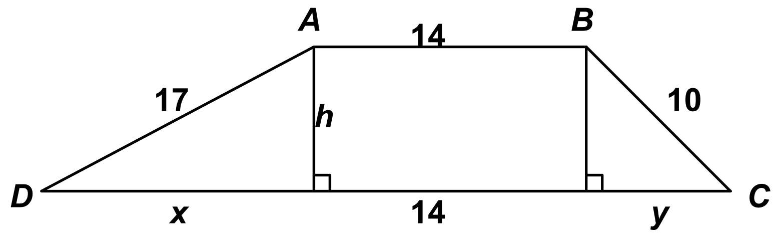
$\rightarrow \left(\dfrac{n(n+1)}{2} - 3\right)x + 3n - 6 = 45 \rightarrow \left(\dfrac{n^2+n-6}{2}\right)x = 51 - 3n \rightarrow (n + 3)(n - 2)x = 6(17 - n)$

$\rightarrow x = \dfrac{6(17-n)}{(n+3)(n-2)}$

A list provides us with integer solutions **<u>(5, 3)</u>** and **<u>(17, 0)</u>**.
Here is a graph of this function – the graph has an open point at (3, 14), intersects the horizontal axis at (17, 0), drops slightly below the axis and then becomes asymptotic to the axis for $n > 17$.



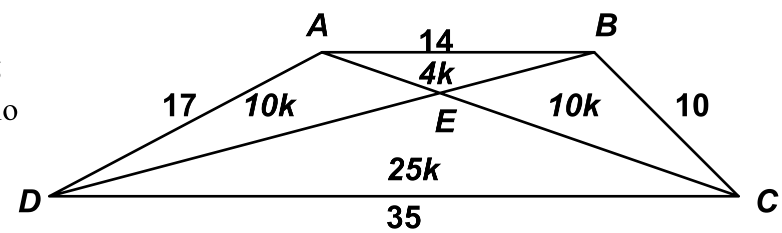
Team Round - continued



E) $\begin{cases} h^2 = 17^2 - x^2 = 10^2 - y^2 \\ x + y + 14 = 35 \end{cases} \rightarrow \begin{cases} x^2 - y^2 = 189 \\ x + y = 21 \end{cases}$

$\rightarrow 21(x - y) = 189 \rightarrow x - y = 9$
Adding, $2x = 30 \rightarrow (x, y) = (15, 6)$
$\rightarrow h = 8$

Thus, Area($ABCD$) = $\frac{1}{2} \cdot 8 \cdot (14 + 35)$ = 196



Now $\Delta ABE \sim \Delta CDE$ with sides in a 14 : 35 or 2 : 5 ratio $\rightarrow$ their areas are in a 4 : 25 ratio

Triangles ADE and ABE with the same altitude from A and bases (BE and DE) are in a 2 : 5 ratio must have areas in a 2 : 5 ratio.

A similar argument demonstrates that, although ΔBEC is not congruent to ΔAED, they do have the same area. Thus, the 4 triangles comprising the trapezoid have areas as indicated above.
$49k = 196 \rightarrow k = 4 \rightarrow$ area(ΔBEC) = **<u>40</u>**

F) Let $(a, b) = (x, x + 1)$. Then $x = \frac{20 - c}{2}$ and c must be even (and between 2 and 18 inclusive) to insure that a, b and c are all positive integers.
$\rightarrow (c, a, b)$ = (2, 9, 10), (4, 8, 9), (6, 7, 8), (8, 6, 7), (10, 5, 6), (12, 4, 5), (14, 3, 4), (16, 2, 3), (18, 1, 2)
$\rightarrow abc$ = 180, 288, 336, ~~336~~, 300, 240, 168, 96, 36
$\rightarrow$ sum = **<u>1644</u>**