MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 – OCTOBER 2008
ROUND 1 VOLUME & SURFACES

ANSWERS

A) ______________________________

B) ___________________________ cm

C) ______________________________

A) Determine the exact length of the interior diagonal of a cube if its total surface area is 1.

B) A hollow spherical metal ball has a 3 cm thick wall. If the total volume of metal is 684π cm^3, compute the outer diameter of the spherical ball.

Note: The outer diameter refers to the longest segment between two points on the outer surface of the ball.

C) A cube has edges of length 2. A plane determined by the midpoints of three edges of the cube that intersect at a common vertex divides the cube into two regions whose volumes are in the simplified ratio of $a : b$, where $a > b$. Determine $a - b$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008
ROUND 2 PYTHAGOREAN RELATIONS IN RECTILINEAR FIGURES

ANSWERS

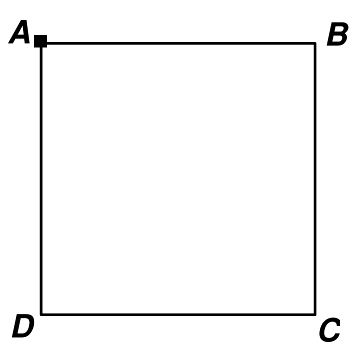
A) ____________________________ rungs

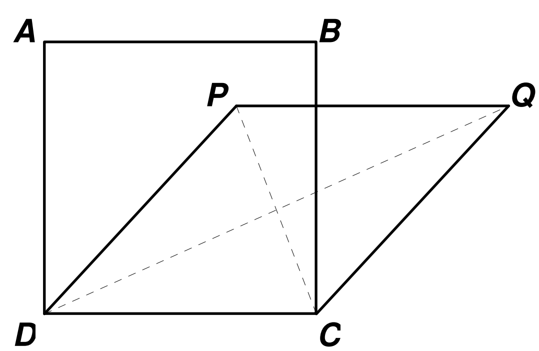
B) ____________________________

C) ____________________________

A) A ladder has rungs 1 foot apart, starting 1 foot from each end.
The bottom of the ladder is 15 feet from the base of a wall when the top of the ladder reaches a point on the wall 36 feet above the ground. How many rungs on this ladder?

B) Four straws of equal length are held together by a string running through them. The ends of the string are tied together at A to form the square $ABCD$. Applying a little pressure at point A, while holding $\overline{DC}$ fixed, the rhombus $PQCD$ is formed. If $AB = 50$ and the diagonals of the rhombus have <u>integer</u> length. Compute the <u>minimum</u> value of $PC + QD$.





C) Find <u>two</u> values of x so that a triangle with sides of length $x - 1$, $2x - 6$ and $x + 3$ will contain a right angle.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) $x =$ ____________________________

B) ______________________________

C) (__________ , _________)

A) Solve the following equation for x. $\frac{1}{3} + 0.1\overline{x} = 0.5$

Note: $0.1\overline{x}$ represents a repeating decimal, where x is the repeating digit.

B) In a club with 120 members, there were $66\frac{2}{3}$ % more members present than members who were absent. If two-thirds of the members constitute a quorum, how many additional members would have been necessary to attain a quorum?

C) All points (x, y) that satisfy $\begin{cases} x = 1 - 2t \\ y = \frac{t}{2} + 1 \end{cases}$ lie on a straight line which can be written in the form $x + ny = c$ for some integer constants n and c. Compute the ordered pair (n, c)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

******* NO CALCULATORS ON THIS ROUND ******

ANSWERS

A) increase decrease by a factor of ____

B) ______ AM PM on _______

C) ________________________________

A) Given: $t = \pi\sqrt{\dfrac{L}{g}}$

If L is multiplied by 100 and g is divided by 4,
what is the effect on t? Circle the correct word and fill in the blank.

B) Four bells A, B, C and D ring regularly with the following schedules:

A every 24 minutes
B every 30 minutes
C every hour and 40 minutes
D every 6 hours and 15 minutes.

At 3:30 PM on Thursday, these four bells rang simultaneously.
Specify the day of the week and the time of day when these bells will next ring simultaneously. Circle AM or PM and fill in the blanks.

C) Given: $1 + \dfrac{rR}{f} = \dfrac{r}{F}$

If $r = 48$ and $\dfrac{f}{F} = \dfrac{12}{5}$, determine all <u>positive</u> integer values of F for which R will also be a positive integer.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) If $|2x - 5| \leq 13$, compute the average of the <u>largest positive</u> and <u>smallest negative</u> solutions.

B) Determine the sum of all integers which do **<u>not</u>** satisfy $|\, 101 - 8x \,| > 27$

C) Solve for x (over the real numbers): $$\frac{\left(x^2+7x\right)^3\left(x^2+\frac{x}{2}\right)}{x^3-9x^2+27x-27} \leq 0$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008
ROUND 6 ALG 1: EVALUATIONS

ANSWERS

A) JAN FEB MAR APR MAY JUN

JUL AUG SEP OCT NOV DEC

B) ______________________________

C) ______________________________

A) The first Thursday in 2025 is on New Year's Day, 1/1. Specify in what month the first Thursday falls later than the 5^{th} for the first time. Circle the correct month above.
The numbers of days in each month for a non-leap year are:

31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 30 and 31.

B) The proper factors of 20 are 1, 2, 4, 5 and 10.
Since the sum of these factors is greater than 20, the integer 20 is called an ***abundant*** number.
Determine the largest abundant number N strictly between 20 and 30, that is $20 < N < 30$.

The following is a formal definition of an abundant number:

An positive integer N is abundant if and only if the sum of its proper factors is greater than N itself.

C) Consider the following binary operation: $a \blacklozenge b = (a + 1)(2 - b)$
Let S be a set of 7 ordered pairs, specifically $S = \{(x, x^2) \mid -3 \leq x \leq 3\}$.
For how many ordered pairs in S – call them (a, b) – does $a \blacklozenge b = b \blacklozenge a$?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008
ROUND 7 TEAM QUESTIONS

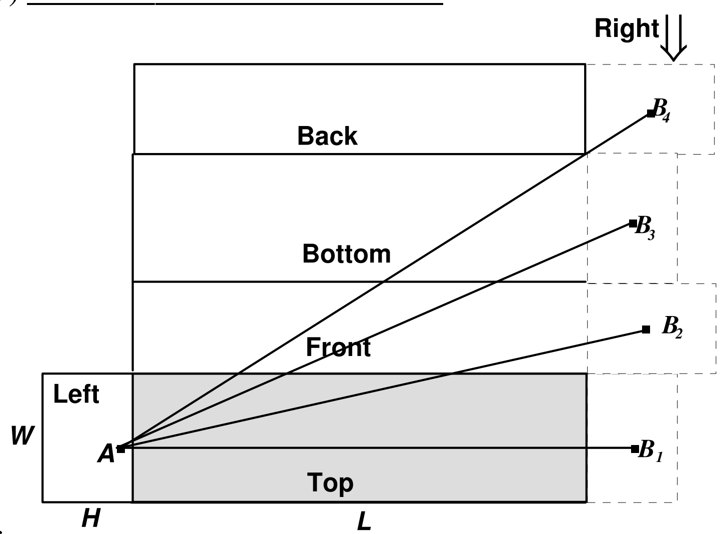
ANSWERS

A) ____________________units D) __________________________

B) __________________________ E) __________________________

C) __________________feet/min F) __________________________

A) The diagram at the right illustrates possible templates for a rectangular solid (a box) with dimensions $L = 58$, $W = 20$ and $H = 12$. The six faces of the box are marked accordingly. Point A is located on the left face 1 unit from the top and halfway between the front and back. Point B should be located on the right face 1 unit from the bottom and halfway between the front and back. The right face in this template could be attached in any one of four possible positions, as indicated. Position B appropriately on the possible right faces and compute the shortest possible length of $\overline{AB}$.

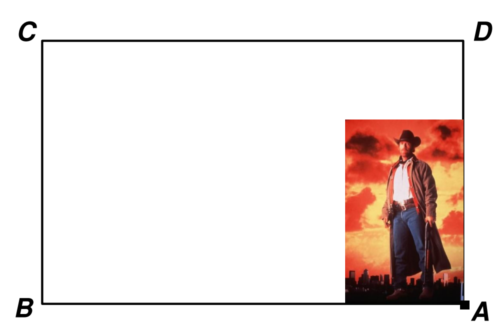


B) The hypotenuse c in right ΔABC has length 10. $\overline{CN}$ is an altitude and $\overline{CM}$ is a median. Let P and Q denote the areas of ΔACB and ΔCNM respectively and let $h = CN$. If h is an integer, compute all possible rational values of $\frac{P}{Q}$.

C) Walker Texas Ranger (Chuck Norris) travels in a rectangular path, *clockwise* starting at A, completing a distance of 1320 feet in 12 minutes.

We are also given the following facts:

$2AB = 3BC$
His velocity v is uniform between any two consecutive vertices and $v_{AB} : v_{BC} : v_{CD} : v_{DA} = 1 : 2 : 4 : 40$
Compute Walker's velocity between D and A in feet per minute.



D) A basketball team plays 82 regular season games. A tie is not possible.
Suppose a team has won 45 games and lost 13 games. If this team wins A games out of every B games for the remainder of the season, they will have won better than 4 out of every 5 games. If the GCD of A and B is 1 and $B > A$, <u>how many</u> ordered pairs (A, B) are possible?

E) For some ordered pair (x, y) that satisfies $\begin{cases} x \geq 0 \\ x - 2y \leq 0 \\ x + 2y - 24 \leq 0 \\ 7x - 2y - 24 \leq 0 \end{cases}$ the expression $2008 + 5x - 2y$ assumes a maximum value. Compute this <u>maximum</u> value.

F) S is a set of all positive reduced fractions $\frac{p}{q}$, where $p < q$, with denominators less than or equal to 10. If the elements of S are listed from smallest to largest, the smallest fraction would be 1/10. What is the <u>seventeenth</u> smallest fraction in this list?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008 ANSWERS

Round 1 Geometry Volumes and Surfaces

A) $\frac{\sqrt{2}}{2}$ (or $\frac{1}{\sqrt{2}}$) B) 18 C) 46 [47 : 1]

Round 2 Pythagorean Relations

A) 38 B) 124 C) 7, 13

Round 3 Linear Equations

A) 6 B) 5 C) (4, 5)

Round 4 Fraction & Mixed numbers

A) incr. by factor of 20 B) 5:30 PM on SAT C) 8, 28

Round 5 Absolute value & Inequalities

A) 2.5 B) 91 C) $x \le -7$ or $-\frac{1}{2} \le x < 3$

Round 6 Evaluations

A) MAY B) 24 C) 2 [(0,0), (1,1)]

Team Round

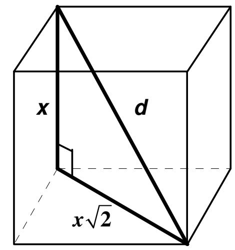
A) 68 D) 3

B) $\frac{5}{2}, \frac{10}{3}$ E) 2024

C) 2112 F) $\frac{5}{9}$

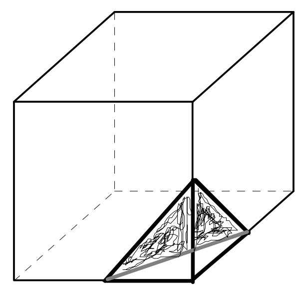
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008 SOLUTION KEY

Round 1



A) Let x and d denote the edge and diagonal respectively of the cube.
Notice from the diagram at the right that $d = x\sqrt{3}$.
The TSA = $6x^2 = 1$. Substituting, $6\left(\dfrac{d}{\sqrt{3}}\right)^2 = 1 \rightarrow 2d^2 = 1 \rightarrow d = \underline{\dfrac{\sqrt{2}}{2}}$.

B) Let r denote the <u>inner</u> radius of the ball.
$684\pi = \dfrac{4}{3}\pi(r+3)^3 - \dfrac{4}{3}\pi r^3$
Canceling the common factor of π and multiplying by $\dfrac{3}{4}$,
$3(171) = 513 = (r+3)^3 - r^3 = 9r^2 + 27r + 27 \rightarrow 9r^2 + 27r - 486 = 0$
$\rightarrow 9(r^2 + 3r - 54) = 9(r+9)(r-6) = 0 \rightarrow r = 6 \rightarrow$ outside diameter = 2(6 + 3) = <u>**18**</u> cm



C) The region bounded by the plane and the three faces intersecting in the common vertex is a pyramid with a right triangular base.
V(cube) : V(Pyramid) = 8 : (1/3)(1/2)(1)
$\rightarrow$ the required ratio (8 – 1/6) : 1/6 = 47 : 1 $\rightarrow$ <u>**46**</u>

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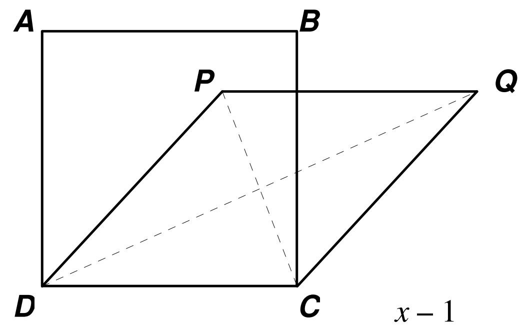
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008 SOLUTION KEY

Round 2

A) Let L denote the length of the ladder (the hypotenuse of the right triangle).
Then: $(a, b, c) = (15, 36, L) = 3(5, 12, ?)$
From special right triangles, the '?' must denote 13 and $L = 39$.
Looking for a pattern, a 3 foot ladder would have 2 rungs (__.__.__) and a 4 foot ladder 3 rungs (__.__.__.__). Clearly, the 39 foot ladder would have **38** rungs.

B) The diagonals of any rhombus are perpendicular and bisect each other. Let $PC = 2a$ and $QD = 2b$.
The diagonals intersect and form 4 right triangles with legs a, b and hypotenuse 50.
Either $(a, b, 50) = 10(3, 4, 5)$ or $2(7, 24, 25)$
→ $(PC, QD) = (60, 80)$ or $(28, 96)$
→ minimum sum = **124**



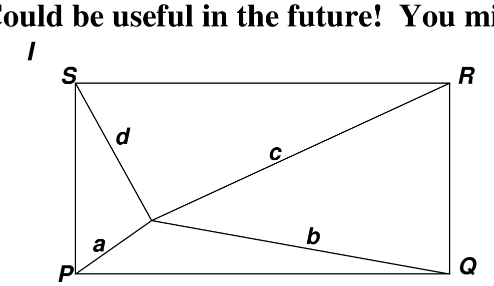
C) The length of the hypotenuse can not be $x - 1$ ($x + 3 > x - 1$ for all values of x).
Case 1: $h = x + 3$
$(x - 1)^2 + (2x - 6)^2 = (x + 3)^2$ → $x^2 - 2x + 1 + 4x^2 - 24x + 36 = x^2 + 6x + 9$
→ $4x^2 - 32x + 28 = 4(x^2 - 8x + 7) = 4(x - 1)(x - 7) = 0$ → **7** only
Check: sides are $x - 1 = 6$, $x + 3 = 10$ and $2x - 6 = 8$

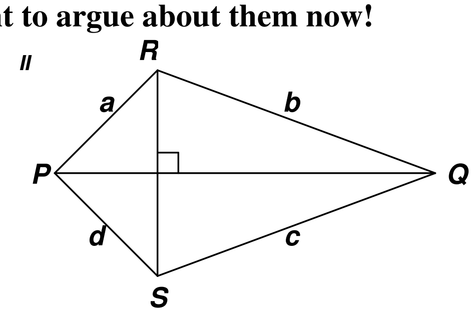
Case 2: $h = 2x - 6$
$(x - 1)^2 + (x + 3)^2 = (2x - 6)^2$ → $x^2 - 2x + 1 + x^2 + 6x + 9 = 4x^2 - 24x + 36$
→ $2x^2 - 28x + 26 = 2(x^2 - 14x + 13) = 2(x - 1)(x - 13) = 0$ → **13** only

Alternate solution:
Case 1: $2x - 6 < x + 3$, but $2x - 6 > x - 1$ → $x < 9$ and $x > 5$ → $5 < x < 9$
By trial and error and using the fact that one leg is 4 less than the hypotenuse, $x =$ **7** → sides of 6, 8 and 10

Case 2: $2x - 6 > x + 3$ → $x > 9$ ($2x - 6 > x - 1$ → $x > 5$ - a weaker condition)
By trial and error, 12 - 16 - 20 would be a right triangle in which the lengths of the legs differ by 4 and $x =$ **13** produces this triple.

Two nice relations derived from applying the Pythagorean Theorem.
Could be useful in the future! You might want to argue about them now!





I IF $PQRS$ is a rectangle, $a^2 + c^2 = b^2 + d^2$
II IF $\overline{PQ} \perp \overline{RS}$, then $a^2 + c^2 = b^2 + d^2$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008 SOLUTION KEY

Round 3

A) $0.1\overline{x} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Multiplying both sides by 10, $1.\overline{x} = \frac{10}{6} = \frac{5}{3}$

Subtracting 1 from both sides, $0.\overline{x} = \frac{2}{3} = 0.666666...$ → x = **<u>6</u>**

B) Clearly, fewer than half the members were absentees, so 60 is too high as a guesstimate.
40 is a good place to start, but it's not divisible by 3, so let's start with 39.
Two-thirds of 39 is 26 so there would be 65 members present, but 65 + 39 = 104 < 120.
(2/3)(42) → 28 → 70 present → 112 members – still too small
(2/3)(45) → 30 → 75 present → 120 members!
Since a quorum is 80 members, we need **<u>5</u>** more members.

An alternative solution (Let algebra do the heavy lifting.):
Let p denote the number of members present. Then # members absent = $a = (120 - p)$

$p = (120 - p) + \frac{2}{3}(120 - p) = \frac{5}{3}(120 - p)$ → $3p = 600 - 5p$ → $8p = 600$ → $p = 75$

$\frac{2}{3}(120) = 80$ members required for a quorum. Thus, **<u>5</u>** more members were needed.

<u>Note</u>: The problem does not say that the difference between the percentage of those present and the percentage of those absent is 66 2/3%, so $(a, p) = \left(\frac{1}{6}, \frac{5}{6}\right)$ or $\left(16\frac{2}{3}\%, 83\frac{1}{3}\%\right)$, where the difference between the fractions is 2/3 (or the percentages 66 2/3%) is a misinterpretation of the problem.

C) From $\begin{cases} x = 1 - 2t \\ y = \frac{t}{2} + 1 \end{cases}$, we have $y = \frac{t}{2} + 1$ → $t = 2(y - 1)$ → $x = 1 - 4(y - 1)$ → $x + 4y = 5$

→ (n, c) = **<u>(4, 5)</u>**

Alternate solution: Let $t = 0$ → $(x, y) = (1, 1)$. Let $t = 2$ → $(-3, 2)$ Substituting, $\begin{cases} 1 + n = c \\ -3 + 2n = c \end{cases}$

→ $-3 + 2n = 1 + n$ → (n, c) = **<u>(4, 5)</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008 SOLUTION KEY

Round 4

A) Replacing L and g by $100L$ and $g/4$ respectively, we have $\pi\sqrt{\dfrac{100L}{g/4}} = \pi\sqrt{\dfrac{400L}{g}} = 20\pi\sqrt{\dfrac{L}{g}}$.

Thus, the effect is an **<u>increase</u>** by a factor of **<u>20</u>**.

B) Converting the times to hours, we have times of $\dfrac{2}{5},\dfrac{1}{2},\dfrac{5}{3},\dfrac{25}{4}$.

We need only determine the LCM of the numerators. Clearly the LCM(1, 2, 5, 25) = 50
Each of these four fractions divides evenly into 50.
The quotients in order are: 125, 100, 30 and 8.
This means that the 125^{th} ringing of bell A coincides with the 100^{th} ringing of B which coincides with the 30^{th} ringing of C which coincides with the 8th ringing of D, 50 hours later.
Thus, 2 days plus two hours later the bells ring together (**<u>5:30 PM on Saturday</u>**)

C) Substituting, $1+\dfrac{rR}{f}=\dfrac{r}{F} \rightarrow 1+\dfrac{48R}{12F/5}=1+\dfrac{20R}{F}=\dfrac{48}{F} \rightarrow R=\left(\dfrac{48}{F}-1\right)\cdot\dfrac{F}{20} = \dfrac{48-F}{20}$

Clearly, (F, R) = (**<u>8</u>**, 2) and (**<u>28</u>**, 1) are the only solutions in positive integers.

Alternate solution: $1+\dfrac{48R}{f}=\dfrac{48}{F} \Leftrightarrow 1+\dfrac{5(48R)}{5f}=\dfrac{12(48)}{12F} \Leftrightarrow 1+\dfrac{5(48R)}{12F}=\dfrac{12(48)}{12F}$

$\Leftrightarrow 12F + 5(48R) = 12(48) \Leftrightarrow F + 20R = 48$ which is a linear equation with F intercept at (48, 0) and slope of $-1/20 \rightarrow$ (**<u>28</u>**, 1) and (**<u>8</u>**, 2)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008 SOLUTION KEY

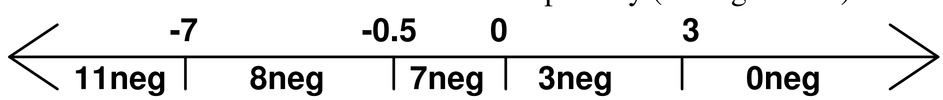
Round 5

A) $|2x - 5| \leq 13 \rightarrow -13 \leq 2x - 5 \leq 13 \rightarrow -4 \leq x \leq 9 \rightarrow \dfrac{-4+9}{2} = \underline{\mathbf{2.5}}$

B) $|\,101 - 8x\,| > 27$ is equivalent to $101 - 8x < -27$ OR $101 - 8x > +27$
$\Leftrightarrow 128 < 8x$ or $78 > 8x \Leftrightarrow 16 < x$ or $37/4 < x \Leftrightarrow x > 16$ or $x < 9.25$
Thus, the integer non-solutions are 10, 11, …, 16 and the sum is $7(10) + (6\cdot 7)/2 = \underline{\mathbf{91}}$.

C) In factored form, the left-hand side of the inequality is $\dfrac{(x+7)^3\left(x+\frac{1}{2}\right)x^4}{(x-3)^3}$.

In this quotient, we are dealing with 11 "factors". The critical values are -7, $-1/2$, 0 and 3. As we move along the number line from left to right, each factor takes on negative values, zero and then positive values. This chart summarizes the polarity (i.e. sign: + / −) of the 11 "factors":



Thus, the given expression is negative (or zero) where there are an odd number of negative factors, i.e. for $\underline{\boldsymbol{x \leq -7} \textbf{ or } \boldsymbol{-\frac{1}{2} \leq x < 3}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008 SOLUTION KEY

Round 6

A) The Thursdays in January fall on: 1, 8, 15, 22, 29, 36 (oops!) = Feb 5th
→ the Thursdays in February fall on: 5, 12, 19, 26 and 33 (oops!) = Mar 5th
→ the Thursdays in March fall on: 5, 12, 19, 26 and 33 (oops!) = April 2nd
→ the Thursdays in April fall on: 2, 9, 16, 23, 30, 37 (oops!) = May 7th

Alternate solution uses **John Conway's "Doom's Day" Formula.**
The following 11 dates fall on the **same** day of the week (DOW) as the <u>last day of February</u>.
3/7 (7 days later)
4/4 6/6 8/8 10/10 12/12 (even / day matches month)
7/11 and 11/7 5/9 and 9/5 (odd / month-day reversals)
A helpful mnemonic: "John is dyslexic and works from 9 to 5 at the '711'."

This takes care of every month except January
1/31 is 28 (or 29) days before the last day in February, depending on leap year status
Thus, **1/31 is the same DOW in a non-leap year and a day earlier in a leap year**.

Notice: The 11 listed dates all fall on Friday in 2008 (since 2/29 is a Friday) but
1/31 falls on a Thursday since 2008 is a leap year.

Proving this is not very difficult. Write up your arguments and give them to your coach!

The problem at hand: 1/1 = Thurs → 1/29 = Thurs → 1/31 = Sat.
The fact that 2025 is not a leap year means that "Doom's Day" in 2025 is on a Saturday.
3/1 = Sat → 1st Thurs = 5th, 4/4 = Sat → 4/2 = 1st Thurs, 5/9 = Sat → 5/7 = 1st Thurs
→ Month = **<u>MAY</u>**

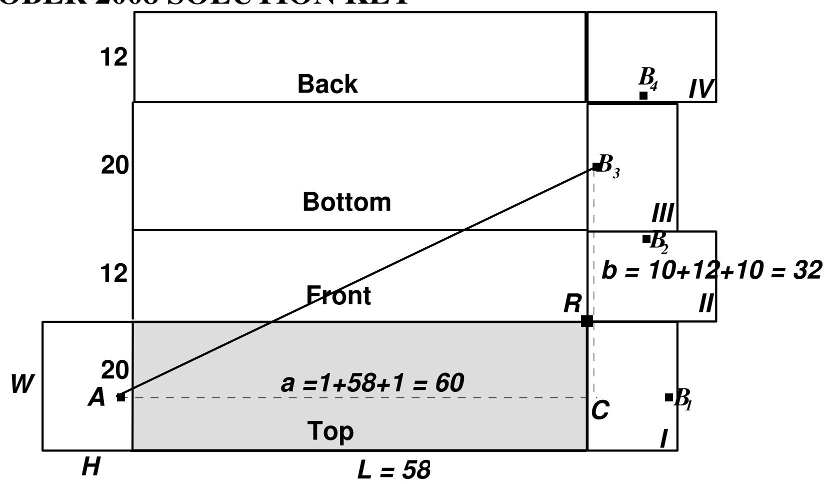
B) 29 (prime) → sum = 1 [29 is a <u>deficient</u> number (as are all primes)]
28 → factors of 1, 2, 4, 7 and 14 → sum = 28 [28 is called a <u>perfect</u> number.]
27 → 1, 3 and 9 → sum = 13
26 → 1, 2, 13 → sum = 16
25 → 1 and 5 → sum = 6 [25, 26 and 27 are all deficient numbers]
24 → 1, 2, 3, 4, 6, 8 and 12 → sum = 36 Bingo! - **<u>24</u>** is the abundant number we want!

C) We could directly substitute the 7 ordered pairs in the definition of the operation **♦.**
But a much better approach would be to determine what ordered pairs satisfied the equation
$a \ ♦\ b = b\ ♦\ a$ or $(a+1)(2-b)=(b+1)(2-a)$.
Multiplying out, $2a - ab + 2 - b = 2b - ab + 2 - a$ → $2a - b = 2b - a$ → $a = b$
Thus, we require ordered pairs (x, x^2) for which $x = x^2$.
$x = x^2$ → $x^2 - x = x(x - 1) = 0$ → $x = 0, 1$ → **<u>2</u>** solutions [(0,0), (1,1)]

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2008 SOLUTION KEY

Team Round

A) Consider the foldout at the right, where the left wall "flap" is fixed and the possible positions of the right wall "flap" are illustrated, along with the positions of point B, which must be 1 unit from the floor midway between the front and back walls. Note that Flap II (and the position of B) is obtained from flap I by rotating I CCW about the point R. Similarly, II maps to III and III to IV.

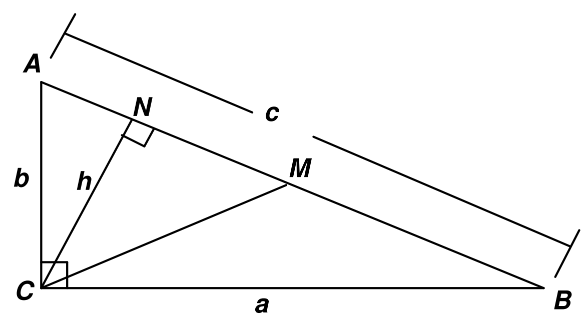


$AB_1 = 1 + 58 + 11 = 70$ $AB_2 = \sqrt{69^2+21^2} = 72.124\cdots$ $AB_4 = \sqrt{69^2+43^2} = 81.301\cdots$
However, ΔAB_3C has legs of 60 and 32. Factoring out a 4, we notice the $8 - 15 - 17$ triple and $AB = 4(17) =$ **<u>68</u>**.

B) $CM = \dfrac{c}{2}$. Let $(BC, AC) = (a, b)$. Area(ΔABC) =

$\dfrac{1}{2}ab = \dfrac{1}{2}hc \rightarrow c = \dfrac{ab}{h}$ In ΔCNM, $\left(\dfrac{c}{2}\right)^2 = NM^2 + h^2$

$\rightarrow NM = \dfrac{1}{2}\sqrt{c^2-4h^2}$. Thus, the ratio is



$$\frac{\frac{1}{2}ab}{\frac{1}{2}\cdot\frac{1}{2}\sqrt{c^2-4h^2}\cdot h} = \frac{2c}{\sqrt{c^2-4h^2}} \text{ and } c = 10 \rightarrow$$

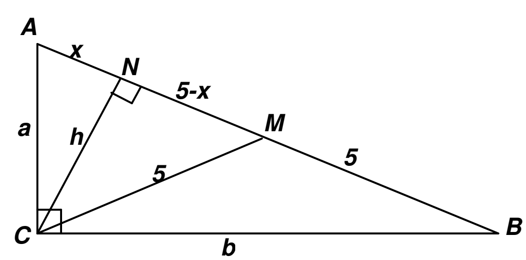
$$\frac{20}{\sqrt{100-4h^2}} = \frac{10}{\sqrt{25-h^2}} \text{ must be rational.}$$

For $h = 5$, $CM = CN$ and ΔCNM collapses and we must avoid division by zero. Thus, examining integer h over [1, 4], we obtain rational values of **<u>5/2</u>** for $h = 3$ and **<u>10/3</u>** for $h = 4$.

Alternate solution: [using altitude to the hypotenuse/geometric means]

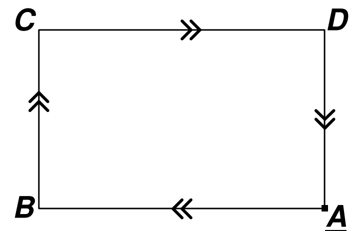
$$\frac{P}{Q} = \frac{(1/2)\cdot 10\cdot h}{(1/2)\cdot(5-x)\cdot h} = \frac{10}{5-x} \text{ must be rational.}$$

Clearly, $h = CN < 5$ and we need to examine $h = 1, 2, 3$ and 4.
$(AN)(BN) = CN^2 \rightarrow x(10-x) = h^2 = 1, 4, 9$ or 16
Only $x^2 - 10x + 9 = 0$ and $x^2 - 10x + 16 = 0$ have rational solutions, namely $(h, x) = (3, 1)$ and $(4, 2)$ and the required ratios are $10/4 =$ **<u>5/2</u>** and **<u>10/3</u>**.



Team Round - continued

C) Let $AB = 3x$ and $BC = 2x$. Then $10x = 1320 \rightarrow AB = 396$ and $BC = 264$
Let $T_{AB} = t$ denote the time needed to travel from A to B.
Given: $v_{AB} : v_{BC} : v_{CD} : v_{DA} = 1 : 2 : 4 : 40$, $2AB = 3BC$



$BC = \frac{2}{3}AB$ and $v_{BC} : v_{AB} = 2 : 1 \rightarrow 2 \cdot T_{BC} = 1 \cdot \frac{2}{3} \cdot T_{AB} \rightarrow T_{BC} = t/3$

$CD = AB$ and $v_{CD} : v_{AB} = 4 : 1 \rightarrow 4 \cdot T_{CD} = 1 \cdot t \rightarrow T_{CD} = t/4$

$DA = BC$ and $v_{DA} : v_{BC} = 40 : 2 \rightarrow 40 \cdot T_{DA} = 2 \cdot T_{BC} = 2(t/3) \rightarrow T_{DA} = t/60$

Summing the times, $96t/60 = 8t/5 = 12 \rightarrow t = 7.5$
Thus, $V_{AB} = 396/7.5 = 52.8 \rightarrow V_{DA} = 40(52.8) =$ **2112** feet/min
Alternate solution:
Let $AB = 3x$ and $BC = 2x$. Then $10x = 1320 \rightarrow AB = 396$ and $BC = 264$
Now let r denote the rate from A to B (solving for $40r$). $\frac{396}{r} + \frac{264}{2r} + \frac{396}{4r} + \frac{264}{40r} = 12$

$\rightarrow 396(40) + 264(20) + 396(10) + 264 = 12(40r)$
$\rightarrow 3960(5) + 264(21) = 12(40r)$
$\rightarrow 330(5) + 22(21) = 40r = 1650 + 462 =$ **2112** feet/min

D) Let w denote the total number of games won. Then $\frac{w}{82} > \frac{4}{5} \rightarrow w > \frac{328}{5} = 65.6 \rightarrow w \geq 66$

Since the team has already won 45 games, during the remainder of the season they must win at least 21 games out of the remaining 24 games [82 – (45 + 13)] Since $B > A$, winning 24 out of 24 is rejected, leaving 3 possibilities: 21… 23 out of 24 $\rightarrow (A, B) = (23, 24)$ (11, 12) or (7, 8)

Alternate solution:
There are $nB = 24$ games remaining $\rightarrow$
$B = 1, 2, 3, 4, 6, 8, 12$ or 24

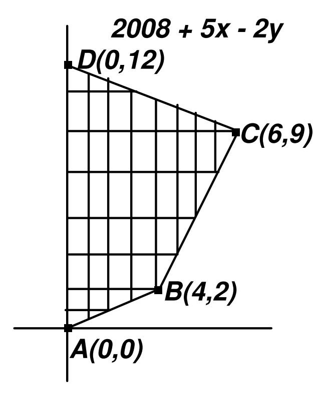
$\frac{45 + An}{82} > \frac{4}{5} \rightarrow 5An > 328 - 225 = 103$

$\rightarrow An > 20.6 \rightarrow A \geq \left\lceil \frac{20.6}{n} \right\rceil$

The chart at the right summarizes the possibilities:
5 out of every 6 $\rightarrow$ record 65/82 (0.793-)
7 out of every 8 $\rightarrow$ record 66/82 (0.804+)
11 out of every 12 $\rightarrow$ record 67/82 (0.817+)
23 out of every 24 $\rightarrow$ 68/82 (0.829+)

B	n	A	Verdict
1	24	**>0**	rejected
2	12	**>1**	rejected
3	8	**>2**	rejected
4	6	**>3**	rejected
6	4	**>5**	rejected
8	3	**>6**	7
12	2	**>10**	11
24	1	**>20**	23

E) The region defined by the system of inequalities is illustrated at the right. It can be shown that any maximum (or minimum) value occurs at a vertex on the boundary of the region. Evaluating the expression we have,
$A(0, 0)$: 2008, $B(4, 2)$: 2024, $C(6, 9)$: 2020 and $D(0, 12)$ 1984.
Thus, the maximum value is **2024**.



Team Round - continued

F) There are 31 fractions in this list of reduced fractions with denominators ≤ 10.

Denominator	2	3	4	5	6	7	8	9	10
Count	1	2	2	4	2	6	4	6	4

It seems reasonable that there would always be just as many fractions less than ½ as there would be greater than ½ . Thus, ½ is the 16th fraction in this increasing list, implying we want the next largest fraction. The possible suspects: 2/3, 3/4, 3/5, 4/7, 5/8, 5/9 and 7/10.

By comparing the decimal equivalents ($0.\overline{6}, 0.75, 0.6, 0.\overline{571428}, 0.625, 0.\overline{5}, 0.7$) or invoking the fact that $\frac{a}{b} > \frac{c}{d} \Leftrightarrow ad > bc$ for a, b, c, $d > 0$, we have the seventeenth fraction in the list, namely **5/9**.

Alternate solution (especially useful for longer lists)
For example, find the 495th fraction in list of reduced fractions w/ denominators ≤ 50.
There are 773 fractions in this list!

Start with the "seed" list: $\frac{0}{1}, \frac{1}{1}$ and apply this rule: Between successive elements $\frac{a}{b}, \frac{c}{d}$ insert $\frac{a+c}{b+d}$ as long as $b + d$ does not exceed the denominator of the previous list PLUS 1.
In each list, the fractions will automatically be in increasing order!
A programmable solution is now easily within reach.
These sequences are referred to as sequences of **Farey Fractions**.

List 2: $\frac{0}{1}, \mathbf{\frac{1}{2}}, \frac{1}{1}$

List 3: $\frac{0}{1}, \mathbf{\frac{1}{3}}, \frac{1}{2}, \mathbf{\frac{2}{3}}, \frac{1}{1}$

List 4: $\frac{0}{1}, \mathbf{\frac{1}{4}}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \mathbf{\frac{3}{4}}, \frac{1}{1}$ (2/5 and 3/5 were not added)

List 5: $\frac{0}{1}, \mathbf{\frac{1}{5}}, \frac{1}{4}, \frac{1}{3}, \mathbf{\frac{2}{5}}, \frac{1}{2}, \mathbf{\frac{3}{5}}, \frac{2}{3}, \frac{3}{4}, \mathbf{\frac{4}{5}}, \frac{1}{1}$ (2/7 and 5/7 were not added)

List 6: $\frac{0}{1}, \mathbf{\frac{1}{6}}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \mathbf{\frac{5}{6}}, \frac{1}{1}$

Lis 7: $\frac{0}{1}, \mathbf{\frac{1}{7}}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \mathbf{\frac{2}{7}}, \frac{1}{3}, \frac{2}{5}, \mathbf{\frac{3}{7}}, \frac{1}{2}, \mathbf{\frac{4}{7}}, \frac{3}{5}, \frac{2}{3}, \mathbf{\frac{5}{7}}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \mathbf{\frac{6}{7}}, \frac{1}{1}$

List 8: $\frac{0}{1}, \mathbf{\frac{1}{8}}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \mathbf{\frac{3}{8}}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \mathbf{\frac{5}{8}}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \mathbf{\frac{7}{8}}, \frac{1}{1}$

List 9: $\frac{0}{1}, \mathbf{\frac{1}{9}}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \mathbf{\frac{2}{9}}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \mathbf{\frac{4}{9}}, \frac{1}{2}, \mathbf{\frac{5}{9}}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \mathbf{\frac{7}{9}}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \mathbf{\frac{8}{9}}, \frac{1}{1}$

List 10:

$$\frac{0}{1}, \mathbf{\frac{1}{10}}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \mathbf{\frac{3}{10}}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \mathbf{\frac{7}{10}}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \mathbf{\frac{9}{10}}, \frac{1}{1}$$

Team Round – F) continued

The longer list!

Here's the distribution of those 773 positive reduced fractions with denominators ≤ 50.

Less than N and relatively prime to N

N	RP(N)	N	RP(N)
		26	**12**
2	**1**	27	**18**
3	**2**	28	**12**
4	**2**	29	**28**
5	**4**	30	**8**
6	**2**	31	**30**
7	**6**	32	**16**
8	**4**	33	**20**
9	**6**	34	**16**
10	**4**	35	**24**
11	**10**	36	**12**
12	**4**	37	**36**
13	**12**	38	**18**
14	**6**	39	**24**
15	**8**	40	**16**
16	**8**	41	**40**
17	**16**	42	**12**
18	**6**	43	**42**
19	**18**	44	**20**
20	**8**	45	**24**
21	**12**	46	**22**
22	**10**	47	**46**
23	**22**	48	**16**
24	**8**	49	**42**
25	**20**	50	**20**
	199		*574*
		Total:	**773**

Some usual facts for computing the number of positive integers less than N that are relatively prime to N, i.e. RP(N):

If N is prime, RP(N) = N – 1

If A and B are relatively prime, RP(AB) = RP(A) · RP(B)

The usual notation for RP(N) is $\phi(N)$ and it is referred to as the Euler ϕ function (pronounced "fee") after the master mathematician Leonard Euler (1707 – 1783).

Perhaps you can 'discover' a formula for computing the values in the table above!

When ordered from smallest to largest the 495th fraction is 23/36.

Code the algorithm suggested above and check it out!