MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2008
ROUND 1 COMPLEX NUMBERS (No Trig)

ANSWERS

A) (________ , ________)

B) (________ , ________)

C) ________________________________

A) In simplified form, $\dfrac{1-\dfrac{1}{\sqrt{-6}}}{\sqrt{-2}}-\left(\dfrac{1-\sqrt{-2}}{2}\right)^2$ may be expressed as $\dfrac{A\sqrt{3}+B}{12}$.

Determine the ordered pair (A, B).

B) $\left(\sqrt{2}+i\sqrt{3}\right)^{600}\cdot\left(\sqrt{2}-i\sqrt{3}\right)^{600}$ may be expressed as A^B in many ways, where A and B are positive integers. If A is the largest possible three-digit integer, determine the ordered pair (A, B).

C) If $\dfrac{2x+yi}{2+i}=\dfrac{5}{5+i}$, compute $x + y$, where x and y are real numbers.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2008
ROUND 2 ALGEBRA 1: ANYTHING

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Find <u>all</u> values of x for which $2009+\sqrt{x-2009}=x$

B) Find <u>all</u> values of x that satisfy $x-\frac{6}{x}=3-\frac{1}{2}$

C) On "Are You Smarter Than a 5th Grader", a contestant, had to answer the $1,000,000 question, "At what temperature does a Celsius thermometer and a Fahrenheit thermometer record the same temperature?". Math was not his strong suit and, under the pressure of the show, he incorrectly remembered the conversion formula as $F=\frac{5}{9}C-32$ and gave an answer $k°$ lower than the correct answer. He would have been able to "derive" the correct formula if he recalled the equivalent temperatures for the boiling and freezing of water, namely (212° F, 100° C) and (32° F, 0° C).

Compute $|k|$, the margin by which he missed becoming a millionaire.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2008
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

ANSWERS

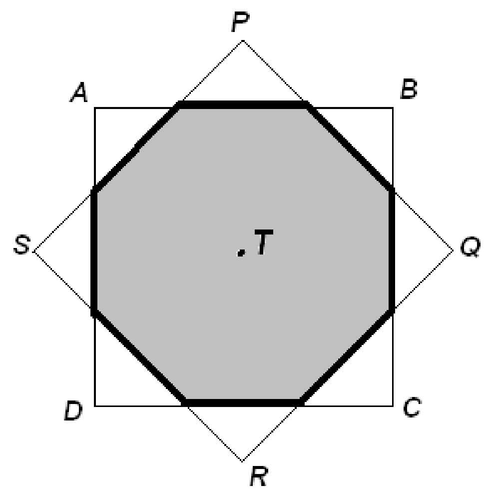
A) (____________ , ____________)

B) ______________________________

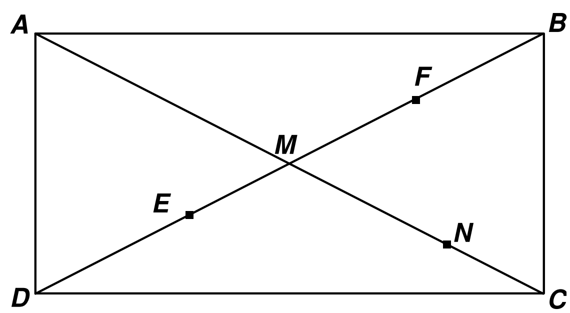
C) __________ : ________ : _________

A) The diagonal of a square has the same length as the altitude of an equilateral triangle. The simplified ratio of the area of the square to the area of the equilateral triangle may be expressed as $A : B$, where B is an <u>integer</u>. Determine the ordered pair (A, B).

B) If square $ABCD$ with $AB = 2$ is rotated 45° about its center T, a new square $PQRS$ is generated. Compute the area of the overlap, i.e. the area of the shaded region.



C) In rectangle $ABCD$, the diagonals intersect at point M. Points E and F lie on $\overline{BD}$ and point N lies on $\overline{AC}$ such that $DE : EB = 1 : 3$, $DF : FB = 11 : 5$ and $MN : NC = 3 : 1$. Compute the ratio of the areas of ΔAEF : ΔCBF : ΔDMN.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2008
ROUND 4 ALG 1: FACTORING AND ITS APPLICATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Factor completely over the integers. $4x^4 + 1 - 5x^2$

B) Factor completely: $9x^2 - 18 - 9A + 7x^2 - 7A^2 + 3A + 9 + 6A^2$

C) Factor completely over the integers: $8x^5 + 38x^3y^2 + 50xy^4$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2008
ROUND 5 TRIG: FUNCTIONS OF SPECIAL ANGLES

****** NO CALCULATORS ON THIS ROUND ******

ANSWERS

A) ______________________

B) ______________________

C) ______________________

A) Solve for θ over $0 \leq \theta < 2\pi$: $4(\sin\theta + 1) = 3\csc\theta$

B) Find <u>all</u> values of x over $-90 < x < 90°$ for which $\cos x = \dfrac{\cot(1035°)\cdot\tan(135°)}{\sec^2(-45°)}$.

C) Given: $\left(\tan\dfrac{4\pi}{3} - \sin\dfrac{5\pi}{2}\right)^{70} \cdot \left(\tan\dfrac{10\pi}{3} - \sin\dfrac{3\pi}{2}\right)^{70} = b^{140}$ Compute $|b|$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2008
ROUND 6 PLANE GEOMETRY: ANGLES, TRIANGLES AND PARALLELS

ANSWERS

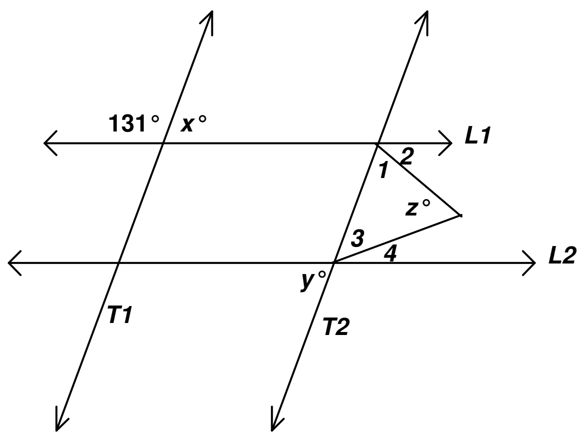
A) ______________________________°

B) ______________________________

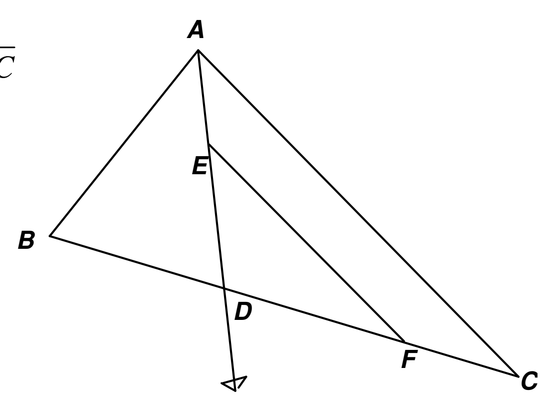
C) ______________________________

A) One exterior angle of a regular polygon measures 4.5°.
What is the sum of the interior angles of this polygon?

B) Given: $L_1 \parallel L_2$ and $T_1 \parallel T_2$, $m\angle 1 = m\angle 2$ and $m\angle 3 = m\angle 4$, and an obtuse angle of 131° as indicated in the diagram below. Compute $y + z$.



C) Given: $\overrightarrow{AD}$ bisects $\angle BAC$, $BD = DF$ and $\overline{EF} \parallel \overline{AC}$
If $AB = 7$, $AC = 13$ and $BC = 11$, compute FE.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2008
ROUND 7 TEAM QUESTIONS
ANSWERS

A) (________) + (________)i D) ____________________________

B) ___________________________________ E) ____________________________

C) ___________________________________ F) (_______ , _______)

A) For some integer k, $\left\| i + 2i^2 + 3i^3 + 4i^4 + 5i^5 + \ldots + (4k+1)i^{4k+1} \right\| = 29$,

where $\| \|$ denotes absolute value.

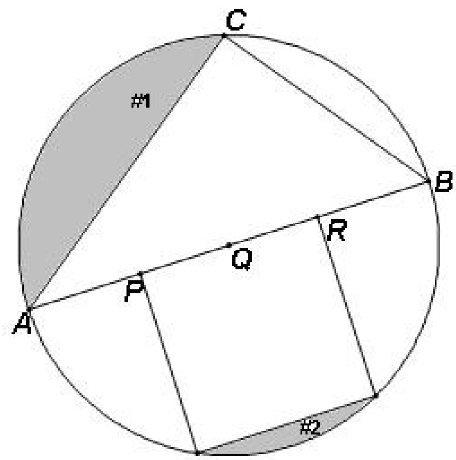
Recall: $\| a + bi \| = \sqrt{a^2 + b^2}$

$\sum$ denotes the sum of the indicated terms

$\prod$ denotes the product of the indicated terms.

Now for the value of k determined above, evaluate $\dfrac{\sum_{n=0}^{n=k} i^{2^n}}{\prod_{n=0}^{n=k} i^{2^n}}$ in simplified $a + bi$ form

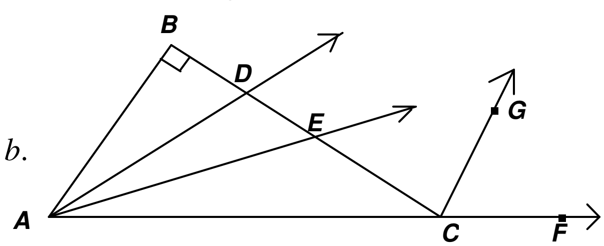
B) Solve: $$\frac{2}{1 - \dfrac{1}{1 - \dfrac{2}{t}}} \ge t^2 - 4$$



C) Find the dimensions of a rectangle with maximum area inscribed in an isosceles triangle with sides of length 13, 13 and 10. One side of this rectangle lies on the base of the isosceles triangle.

D) Solve for <u>all</u> real values of x for which
$12x(x-1) = 5 + \sqrt{(4x-7)(3x-1) + 13x}$.

E) ΔABC is inscribed in circle Q. $AC = 8$ and $BC = 6$.
Let the circle of maximum area with center S be inscribed in region #1 and the circle of maximum area with center T be inscribed in region #2.
Compute the $\sin(\angle SQT)$.



F) $\overline{AB} \perp \overline{BC}$, $\angle BAC$ is trisected by $\overrightarrow{AD}$ and $\overrightarrow{AE}$,
$\angle BCF$ is bisected by $\overrightarrow{CG}$ and m$\angle ADE$: m$\angle AEC$ = a : b.
If the GCF(a, b) = 1 and $m\angle GCF = 75$, compute (a, b).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2008 ANSWERS

Round 1 Algebra 2: Complex Numbers (No Trig)

A) (2, 3) B) (625, 150) C) $\frac{85}{52}$

Round 2 Algebra 1: Anything

A) 2009, 2010 B) $-3/2$ or 4 C) 32

Round 3 Plane Geometry: Area of Rectilinear Figures

A) $\sqrt{3}:2$ (or $\frac{\sqrt{3}}{2}:1$) B) $8\left(\sqrt{2}-1\right)$ C) 7 : 5 : 6

Round 4 Algebra 1: Factoring and its Applications

A) $(2x+1)(2x-1)(x+1)(x-1)$ B) $(4x-A-3)(4x+A+3)$ C) $2x\left(2x^2+xy+5y^2\right)\left(2x^2-xy+5y^2\right)$

-or equivalents -

Round 5 Trig: Functions of Special Angles

A) $\frac{\pi}{6},\frac{5\pi}{6}$ B) ± 60 C) $\sqrt{2}$

Round 6 Plane Geometry: Angles, Triangles and Parallels

A) 14040 (or 14,040) B) 139 C) 7

Team Round

A) -1 + 8i

B) $-3 \le t < 2$ ($t \neq 0$) (or equivalent)

C) 5 x 6

D) $-\frac{1}{2},\frac{3}{2}$

E) 3/5

F) (11, 13)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2008 SOLUTION KEY

Round 1

A) $\dfrac{1-\dfrac{1}{\sqrt{-6}}}{\sqrt{-2}}-\left(\dfrac{1-\sqrt{-2}}{2}\right)^2=\dfrac{\sqrt{-6}-1}{\sqrt{-2}\cdot\sqrt{-6}}-\dfrac{1-2\sqrt{-2}-2}{4}=\dfrac{i\sqrt{6}-1}{-2\sqrt{3}}-\dfrac{-1-2i\sqrt{2}}{4}$

$=\dfrac{i\sqrt{18}-\sqrt{3}}{-6}+\dfrac{1+2i\sqrt{2}}{4}=\dfrac{-3i\sqrt{2}+\sqrt{3}}{6}+\dfrac{1+2i\sqrt{2}}{4}=\dfrac{-6i\sqrt{2}+2\sqrt{3}}{12}+\dfrac{3+6i\sqrt{2}}{12}=\dfrac{2\sqrt{3}+3}{12}$

→ (A, B) = **<u>(2, 3)</u>**

B) $\left(\sqrt{2}+i\sqrt{3}\right)^{600}\cdot\left(\sqrt{2}-i\sqrt{3}\right)^{600}=\left(\left(\sqrt{2}+i\sqrt{3}\right)\cdot\left(\sqrt{2}-i\sqrt{3}\right)\right)^{600}=\left(2-i^2\cdot 3\right)^{600}=5^{600}$

$=\left(5^2\right)^{300}=\left(5^3\right)^{200}=\left(5^4\right)^{150}=\left(5^5\right)^{120}$ etc → (A, B) = **<u>(625, 150)</u>**

C) Cross Multiplying, $10x+2xi+5yi+yi^2=10+5i$ → $(10x – y) + (2x + 5y)i = 10 + 5i$

→ $\begin{cases}10x-y=10\\2x+5y=5\end{cases}$

The usual approach would be to:

multiply the 1st equation by 5 and then add the equations → $\left(x=\dfrac{55}{52}\right)$.

multiply the 2nd equation by −5 and then add the equations → $\left(y=\dfrac{15}{26}\right)$.

Adding these results we have $\mathbf{\dfrac{85}{52}}$ Nothing new here!

However, if the system had been $\begin{cases}13x-7y=4\\11x+9y=2.5\end{cases}$, then solving for x and y separately would have been very tedious and unnecessary. The following seems like magic, but it can be a real time saver.

We must find a <u>linear combination</u> of the left hand sides of each equation for which the coefficients of x and y are equal.
Suppose this happens when we multiply $(10x – y)$ by some A and $(2x + y)$ by some B.

Regrouping $A(10x – y) + B(2x + 5y)$, we have $(10A + 2B)x + (5B – A)y$
Equating the x and y coefficients, $10A + 2B = 5B – A$ → $11A = 3B$. Let's pick $A = 3$ and $B = 11$.
Multiplying the 1st equation by $A = 3$ and the 2nd equation by $B = 11$ produces

$\begin{cases}30x-3y=30\\22x+55y=55\end{cases}$ Adding, $52x + 52y = 85$ → $x + y =$ $\mathbf{\dfrac{85}{52}}$

Try the suggested system above the usual way and using the technique outlined above.

$x+y=\dfrac{21}{97}$. How much time did YOU save?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2008 SOLUTION KEY

Round 2

A) Subtracting 2009 from both sides and squaring, $x - 2009 = (x - 2009)^2$
Since this equation is of the form $k = k^2$, which has solutions $k = 0, 1$, we have $x =$ **2009, 2010**.

B) $x - \frac{6}{x} = \frac{5}{2}$ → $2x^2 - 12 = 5x$ → $2x^2 - 5x - 12 = 0$ → $(2x + 3)(x - 4) = 0$ → $x =$ **–3/2, 4**

C) $\begin{cases} F = \frac{5}{9}C - 32 \\ F = C \end{cases}$ → $5C - 288 = 9C$ → $4C = -288$ → $C = F = -72$

Either from recall or solving the system $\begin{cases} F = \frac{9}{5}C + 32 \\ F = C \end{cases}$, $C = F = -40$

Thus, $-40 - (-72) =$ **32**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2008 SOLUTION KEY

Round 3

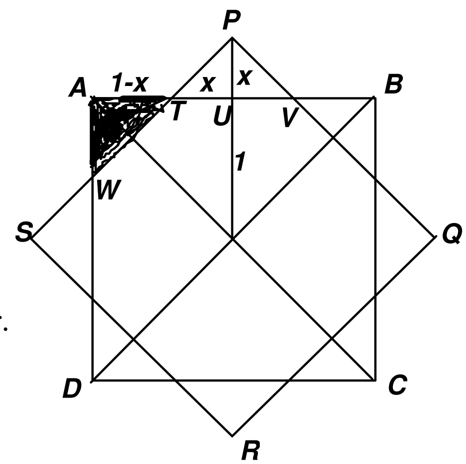
A) If the side of the square has length 1, then the diagonal and the altitude of the equilateral triangle will have lengths $\sqrt{2}$. Dividing by $\sqrt{3}$ and multiplying by 2, the side of the equilateral triangle is $\frac{\sqrt{2}}{\sqrt{3}} \cdot 2 = \frac{2\sqrt{6}}{3}$. Thus, the areas are $1^2 = 1$ and $\frac{1}{2} \cdot \frac{2\sqrt{6}}{3} \cdot \sqrt{2} = \frac{2\sqrt{3}}{3}$

$1 : \frac{2\sqrt{3}}{3} = 3 : 2\sqrt{3} = 3\sqrt{3} : 6 = \underline{\mathbf{\sqrt{3} : 2}}$ (or $\frac{\sqrt{3}}{2} : 1$)

B) $m\angle TPU = 45°$ and $PU = \sqrt{2} - 1 \rightarrow PT = 2 - \sqrt{2}$

$\rightarrow$ area(ΔTPV) = $\frac{1}{2}\left(2 - \sqrt{2}\right)^2 = 3 - 2\sqrt{2}$

$\rightarrow$ area of overlap = $4 - 4(3 - 2\sqrt{2}) = \underline{\mathbf{8\left(\sqrt{2} - 1\right)}}$



Alternate solution

A(regular octagon) = $\frac{1}{2}ap$, where a = apothem (perpendicular from center to side) and p = its perimeter.

$a = 1$, $x = PU = \sqrt{2} - 1$, $TV = 2x = 2\sqrt{2} - 2$, $TW = (1 - x)\sqrt{2} = \left(2 - \sqrt{2}\right)\sqrt{2} = 2\sqrt{2} - 2$

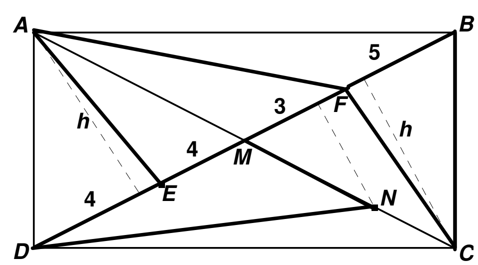
So the resulting octagon is in fact regular.

$p = 8\left(2\sqrt{2} - 2\right)$ and the area of the overlap is $\underline{\mathbf{8\left(\sqrt{2} - 1\right)}}$**.**

C) $DE : EB = 1 : 3 = 4 : 12$, $DF : FB = 11 : 5$

Each of these ratios divides segment $\overline{BD}$ into 16 parts.
Thus, without loss of generality, let $BD = 16$.
Then: $DE = 4$, $BF = 5$, $EF = 7$ and $DM = MC = 8$

The altitude from A to $\overline{BD}$ has the same length as the altitude from C to $\overline{BD}$ - call it h.



$MN : NC = 3 : 1 \rightarrow MN = \frac{3}{4}MC$ and the altitude from N to $\overline{BD}$ has length $\frac{3}{4}h$

$|\Delta AEF| : |\Delta CBF| : |\Delta DMN| = \frac{1}{2} \cdot 7 \cdot h : \frac{1}{2} \cdot 5 \cdot h : \frac{1}{2} \cdot 8 \cdot \frac{3}{4}h = \underline{\mathbf{7 : 5 : 6}}$

Note: $|\Delta AEF|$ denotes the <u>area</u> of ΔAEF.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2008 SOLUTION KEY

Round 4

Note: *Equivalents answers are allowed where terms within any individual factor have been rearranged or where pairs of factors have each been multiplied by -1.*

A) $4x^4 + 1 - 5x^2 = 4x^4 - 5x^2 + 1 = (4x^2 - 1)(x^2 - 1) =$ <u>$\mathbf{(2x + 1)(2x - 1)(x + 1)(x - 1)}$</u>

B) Combine like terms and regroup.
$9x^2 - 18 - 9A + 7x^2 - 7A^2 + 3A + 9 + 6A^2 = 16x^2 - A^2 - 6A - 9$
→ $(4x)^2 - (A^2 + 6A + 9) = (4x)^2 - (A + 3)^2 =$ <u>$\mathbf{(4x - A - 3)(4x + A + 3)}$</u>

C) $8x^5 + 38x^3y^2 + 50xy^4 = 2x\left(4x^4 + 19x^2y^2 + 25y^4\right) = 2x\left(4x^4 + 20x^2y^2 + 25y^4 - x^2y^2\right)$

$2x\left(\left(4x^4 + 20x^2y^2 + 25y^4\right) - x^2y^2\right) = 2x\left(\left(2x^2 + 5y^2\right)^2 - (xy)^2\right) =$
<u>$\mathbf{2x\left(2x^2 + xy + 5y^2\right)\left(2x^2 - xy + 5y^2\right)}$</u>

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2008 SOLUTION KEY

Round 5

A) $4(\sin\theta + 1) = 3\left(\frac{1}{\sin\theta}\right)$ → $4\sin^2\theta + 4\sin\theta - 3 = (2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\sin\theta = -3/2$ is impossible

$\sin\theta = 1/2$ → $\underline{\boldsymbol{\frac{\pi}{6}, \frac{5\pi}{6}}}$

B) $\cos x = \frac{\cot(1035°)\cdot\tan(135°)}{\sec^2(-45°)} = \frac{\cot(315°)\cdot\tan(135°)}{\sec^2(45°)} = \frac{-1\cdot -1}{2} = \frac{1}{2}$ → 60° family → $\underline{\mathbf{\pm 60}}$

C) $= \left(\left[\sqrt{3}-1\right]\left[\sqrt{3}+1\right]\right)^{70} = 2^{70} = \left(\sqrt{2}\right)^{140}$ → $|b| = \underline{\boldsymbol{\sqrt{2}}}$

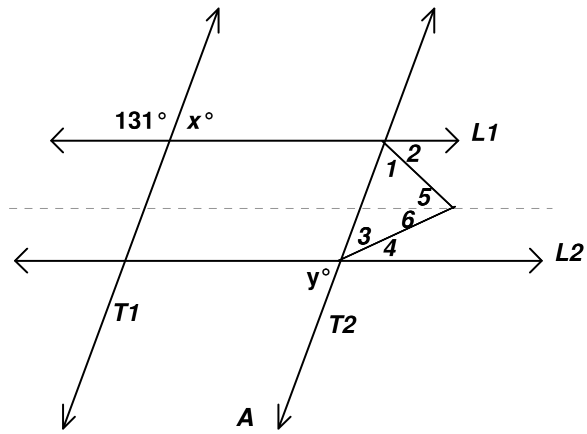
Note: The equation $b^{140} = \left(\sqrt{2}\right)^{140}$ has 140 roots, only two of which are real, but the absolute value of all of them is $\sqrt{2}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2008 SOLUTION KEY

Round 6

A) $360/n = 4.5 \rightarrow n = 80$ and the sum of the interior angles is determined by $180(n - 2)$.
$180\ (78) =$ **<u>14040</u>**

B) $x = 49 \rightarrow y = 49$
As vertical angles, $m\angle 3 + m\angle 4 = y$.
Since $m\angle 3 = m\angle 4$, $m\angle 4 = y/2$.
$m\angle 1 + m\angle 2 = 180 - y \rightarrow m\angle 2 = 90 - y/2$
Draw a line through the vertex of the angle whose measure is $z°$ parallel to L_1.
As alternate interior angles of ||s,
$m\angle 2 = m\angle 5$ and $m\angle 4 = m\angle 6$.
Thus, $z = y/2 + (90 - y/2) = 90$ and $y + z =$ **<u>139</u>**



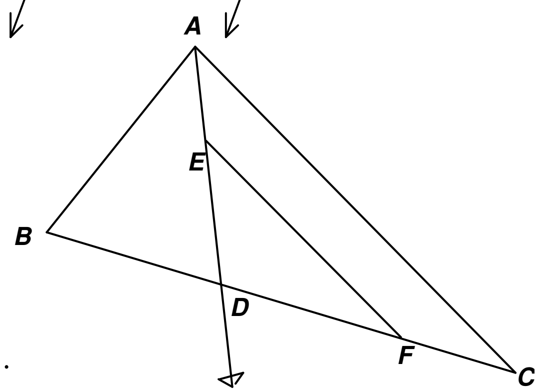
C) By the triangle angle bisector theorem, $\dfrac{BD}{DC} = \dfrac{AB}{AC}$.

Since $\Delta DEF \sim \Delta DAC$, $\dfrac{FE}{AC} = \dfrac{DF}{DC}$.

Using $BD = DF$, we have $\dfrac{FE}{AC} = \dfrac{BD}{DC}$

Using transitivity, we have a third proportion $\dfrac{FE}{AC} = \dfrac{AB}{AC}$.

Since the denominators are equal, we have $FE = AB =$ **<u>7</u>**.
No computations were necessary!



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2008 SOLUTION KEY

Team Round

A) Given: $\left|i+2i^2+3i^3+4i^4+5i^5+...+(4k+1)i^{4k+1}\right|=29$.

Grouping in blocks of 4, the first four terms sum to $i-2-3i+4=2-2i$
In fact, each successive block of four terms also sums to $2-2i$.
The absolute value expression consists of k blocks of 4 terms plus one additional.
The equation simplifies to
$|k(2-2i)+(4k+1)i|=|2k+(2k+1)i|=29 \rightarrow (2k)^2+(2k+1)^2=29^2$
Noting that $k = 10 \rightarrow 20 - 21 - 29$ which is a Pythagorean Triple avoids the necessity of solving this quadratic equation.

Therefore, the expression to be evaluated is the <u>sum</u> of 11 terms divided by the product of the same 11 terms.

The terms are: $i, i^2, i^4, i^8, i^{16}, \ldots i^{1024}$ The last 9 terms are all 1s.

Thus, the quotient is $\dfrac{i-1+9}{i(-1)(1)^9} = \dfrac{i+8}{-i} = \dfrac{i^2+8i}{-i^2} =$ **<u>$-1+8i$</u>**

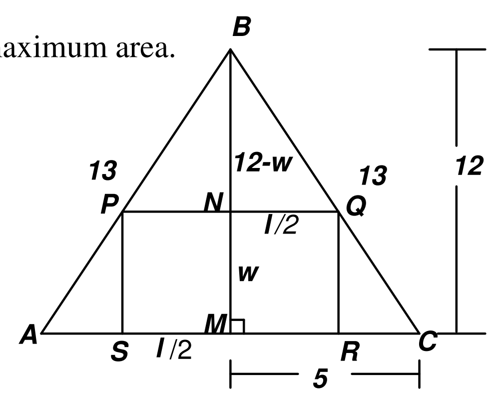
B) Given: $\dfrac{2}{1-\dfrac{1}{1-\dfrac{2}{t}}} \geq t^2-4$

Note: $t = 0, 2$ cause division by zero in the expression on the left side.

$$\frac{2}{1-\dfrac{1}{1-\dfrac{2}{t}}}=\frac{2}{1-\dfrac{1}{\dfrac{t-2}{t}}}=\frac{2}{1-\dfrac{t}{t-2}}=\frac{2}{\dfrac{t-2-t}{t-2}}=2\cdot\frac{t-2}{-2}=2-t$$

$2-t \geq t^2-4 \rightarrow t^2+t-6=(t+3)(t-2)\leq 0 \rightarrow -3\leq t\leq 2$
However, with the restriction that $t \neq 0, 2$, we have **<u>$-3 \leq t < 2$ ($t \neq 0$)</u>** (or equivalent)

C) Let l and w denote the length and width of the rectangle with maximum area.
Since $\Delta BNQ \sim \Delta BMC$,



$\dfrac{12-w}{12}=\dfrac{l/2}{5} \rightarrow l=\dfrac{60-6w}{5}=12-\dfrac{6}{5}w$

Area $= lw = \left(12-\dfrac{6}{5}w\right)w = -\dfrac{6}{5}w^2+12w =$

$-\dfrac{6}{5}\left(w^2-10w+25\right)+\dfrac{6}{5}\cdot 25 = -\dfrac{6}{5}(w-5)^2+30$

Clearly, an expression of this form always has a value less than or equal to 30 and attains its maximum value of 30 when $w = 5$ (and correspondingly $l = 6$) $\rightarrow$ dimensions: **<u>5 x 6</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2008 SOLUTION KEY

Team Round - continued

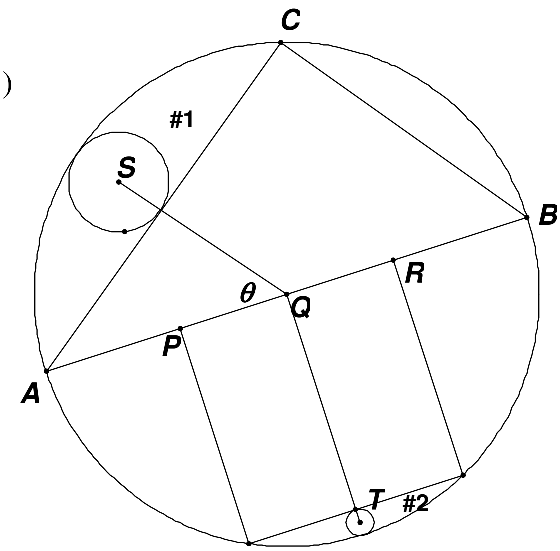
D) The given equation is equivalent to: $12x^2 - 12x = 5 + \sqrt{12x^2 - 12x + 7}$

Substituting $y = \sqrt{12x^2 - 12x + 7}$, we have $y^2 = 12 + y$

→ $y^2 - y - 12 = (y-4)(y+3) = 0$ → $y = 4$ only.

Thus, $12x^2 - 12x + 7 = 16$ → $12x^2 - 12x - 9 = 3(4x^2 - 4x - 3)$

$= 3(2x+1)(2x-3) = 0$ → $x = \underline{\mathbf{-\frac{1}{2}, \frac{3}{2}}}$



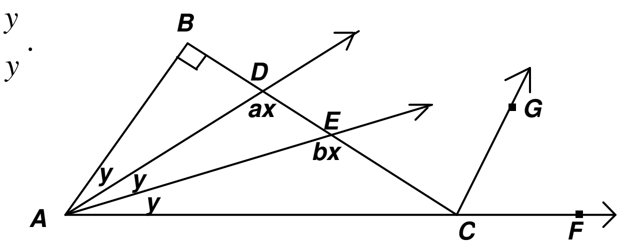
E) Point S lies on the perpendicular bisector of $\overline{AC}$ which will pass through point Q. Point T lies on the perpendicular bisector of the chord parallel to diameter $\overline{AB}$ which will pass through point Q. Let θ denote $\angle SQA$.

Then $\sin(\angle SQT) = \sin(\theta + 90) = \cos(\theta)$.

But $\angle\theta \cong \angle B$ and, therefore, $\cos\theta = \cos B = 6/10 =$ **<u>3/5</u>**.

F) Let $m\angle BAC = 3y, m\angle ADE = 3x$ and $m\angle AEC = 7x$.

Using exterior angles of ΔADE and ΔABD, $\begin{cases} bx = ax + y \\ ax = 90 + y \end{cases}$.



Subtracting, $x = \dfrac{90}{2a-b}$.

Substituting, $y = a\left(\dfrac{90}{2a-b}\right) - 90 = \dfrac{90(b-a)}{2a-b}$

$m\angle BCA = 90 - 3y$ and $m\angle BCF = 90 + 3y$

→ $m\angle GCF = 45 + \dfrac{3y}{2} = 45 + \dfrac{3}{2}\left(\dfrac{90(b-a)}{2a-b}\right) = 45 + \left(\dfrac{135(b-a)}{2a-b}\right) = 75$

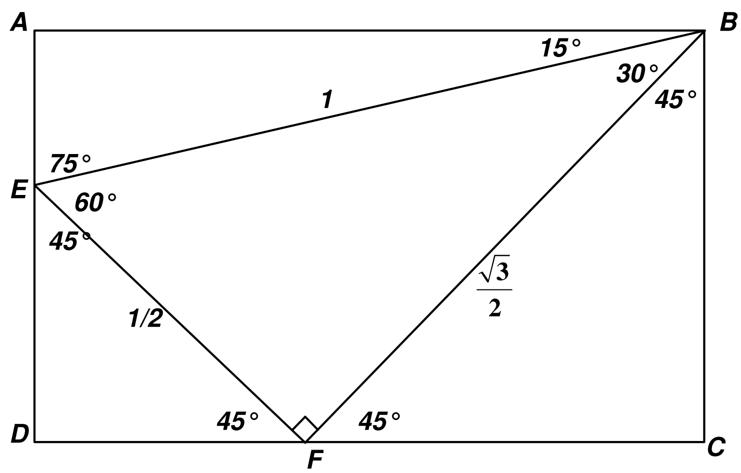
→ $\mathbf{135b - 135a = 60a - 30b}$ → $\mathbf{165b = 195a}$ → $\mathbf{11b = 13a}$ → $(\boldsymbol{a}, \boldsymbol{b})$ = <u>(11, 13)</u>

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2008 Notes

Ever need to explain how the sin(15°) is computed to a student with limited experience with trig formulas, i.e. no experience with formulas like $\sin(A \pm B)$?

Thanks to Mary Beth McGinn for the following gem.

Consider rectangle *ABCD* with an embedded 30 – 60 – 90 right triangle having sides as indicated.



$$FC = BC = AD = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{6}}{4}$$

$$DE = DF = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$AB = DF + FC = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$AE = AD - DE = \frac{\sqrt{6}-\sqrt{2}}{4}$$

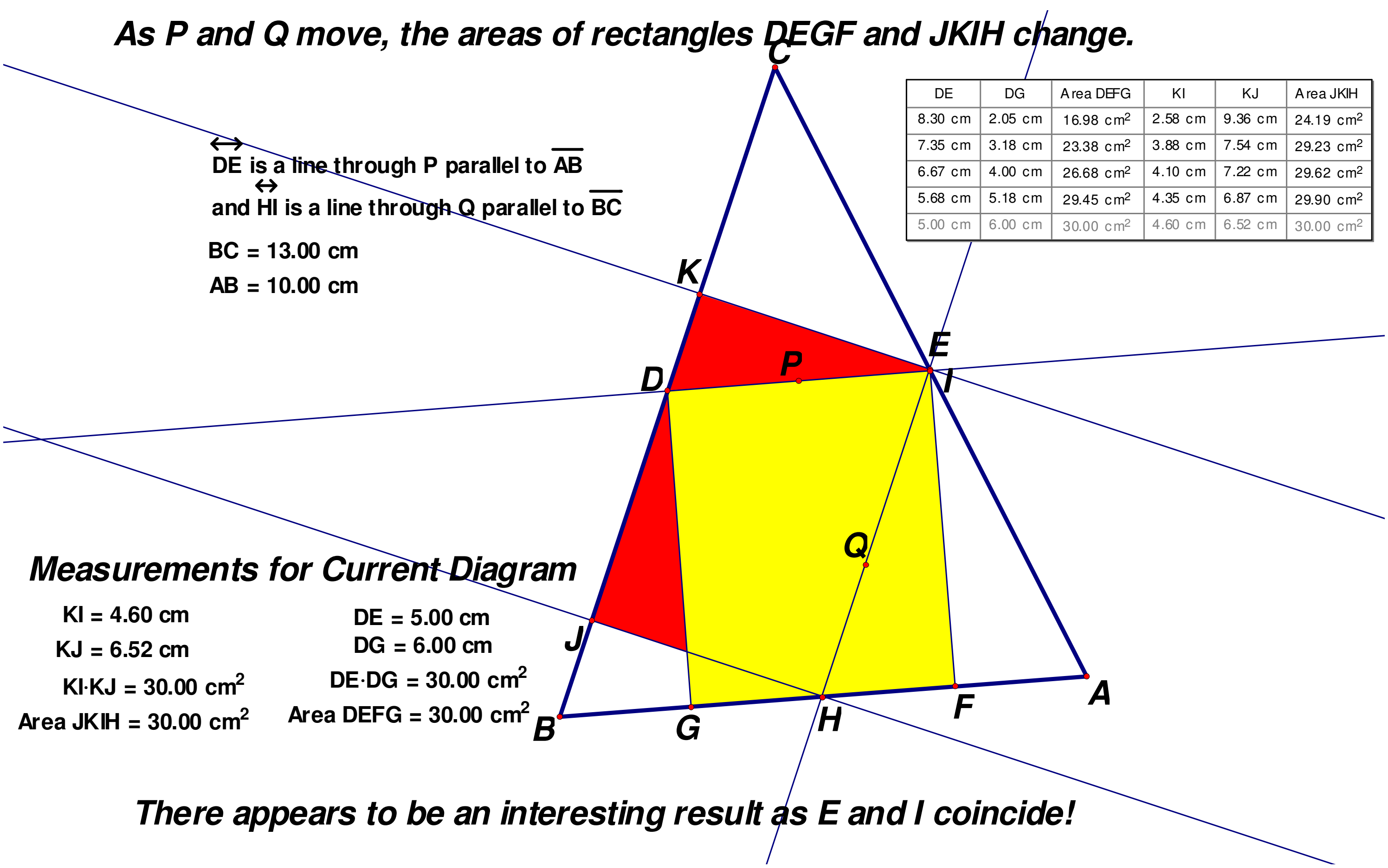
Now we have the <u>exact</u> lengths of the 3 sides in ΔABE, the 15 – 75 – 90 right triangle.

Using only the basic definitions of sine and cosine (SOH·CAH·TOA)

$$\mathbf{\sin 15° = \cos 75° = \frac{\sqrt{6}-\sqrt{2}}{4}} \quad \text{and} \quad \mathbf{\sin 75° = \cos 15° = \frac{\sqrt{6}+\sqrt{2}}{4}}$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2008 Addendum

For Team question C, the following Sketchpad diagram suggests that the rectangle of maximum area has an area of 30 square units regardless of whether a side lies along the base or the leg of the isosceles triangle. Is this a "coincidence"? Try proving (or disproving) your contention and sharing your ideas with your teammates and coaches.



DE	DG	Area DEFG	KI	KJ	Area JKIH
8.30 cm	2.05 cm	16.98 cm^2	2.58 cm	9.36 cm	24.19 cm^2
7.35 cm	3.18 cm	23.38 cm^2	3.88 cm	7.54 cm	29.23 cm^2
6.67 cm	4.00 cm	26.68 cm^2	4.10 cm	7.22 cm	29.62 cm^2
5.68 cm	5.18 cm	29.45 cm^2	4.35 cm	6.87 cm	29.90 cm^2
5.00 cm	6.00 cm	30.00 cm^2	4.60 cm	6.52 cm	30.00 cm^2