MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 – DECEMBER 2008
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

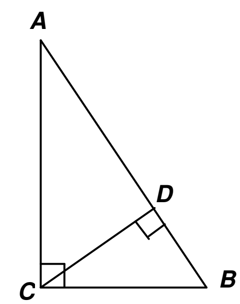
ANSWERS

A) _______________________________

B) _______________________________

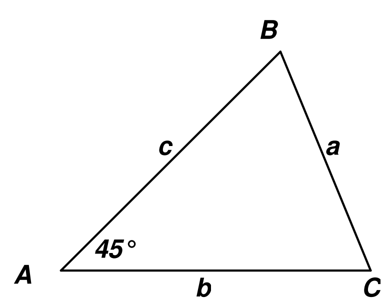
C) _______________________________

A) Given: In right triangle ABC, $AB = 17$, $BC = 8$ and $\overline{CD} \perp \overline{AB}$
Compute $\sin(\angle ACD)$.



B) In ΔABC, $a = 10$, $b = 9$ and $c = 11$
P is an angle of ΔABC, but it is neither the largest nor the smallest angle.
As a simplified fraction, $\cos P = \dfrac{m}{n}$. Compute $m + n$.

C) Let $m\angle A = 45°$, $\cos B = \dfrac{\sqrt{3}}{2}$ and $b = 2\sqrt{2}$
Compute $a^2 + b^2 + c^2$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008
ROUND 2 ARITHMETIC/NUMBER THEORY

ANSWERS

A) ______________________

B) (____ , ____ , ____ , ____)

C) ______________________ $_3$

A) Let $N = 158400$.
A denotes the largest odd factor of N.
B, whose only odd factor is 1, is the largest possible even factor of N.
Compute the product AB.

B) Given: $24^3 \cdot 120^2 \cdot 441 = 28^A \cdot 15^B \cdot 12^C \cdot 2^D$
Find the ordered quadruple (A, B, C, D).

C) Consider numbers written in "base 3", but rather than using the digits 0, 1 and 2, use the digits are A, B and C, where $A = 1$, $B = -1$ and $C = 0$.
For example, $21_{10} = ABAC_3$, since $ABAC_3 = 27(1) + 9(-1) + 3(1) + 1(0) = 27 - 9 + 3 + 0 = 21$.

How would you represent the base ten integer 211 in base 3 in terms of A, B and C?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES

ANSWERS

A) (________ , ________ , ________)

B) ________________________________

C) (______________ , ______________)

A) Given: $2x - 3y - 5 = 0$ and $4x - 6y + 1 = 0$
Let $Ax + By + C = 0$ denote the equation of the line equidistant from the given lines, where A, B and C are integers, $A > 0$ and the GCF(A, B, C) = 1.
Determine the ordered triple (A, B, C).

B) Compute the shortest distance between the circles defined by $\begin{cases} x^2 + y^2 - 4x + 10y + 4 = 0 \\ x^2 + y^2 + 12x - 20y + 55 = 0 \end{cases}$.

Note: Since these two circles do not intersect, the shortest distance between them lies along the segment connecting their centers.

C) There are an infinite number of ordered pairs (x, y) that satisfy the linear equation $6x + 9y = 12$. Let S be set of all ordered pairs (x, y) that satisfy this equation, where both x and y are single digit integers. Compute (A, B), where A is the sum of all the x-coordinates in S and B is the sum of all y-coordinates in S.

Note: The values of x and y may be positive, negative or zero.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

******* NO CALCULATORS ON THIS ROUND ******

ANSWERS

A) ______________________

B) ______________________

C) ______________________

A) Compute the numerical value of $49^{\log_7 3 - 4\log_7 2}$.

B) Compute <u>all</u> real values of x for which $2^x = 2^{-x} - \frac{3}{2}$.

C) Compute <u>all</u> possible real values of x for which $3^{1+5\log_3 x} - 4^{0.5+4\log_4 x} + 2^{3+2\log_2 x} = 12x^3$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) If $\begin{cases} y = 10 - 3t \\ x = 4t + 1 \end{cases}$, then compute $\dfrac{y}{t}$ when $x = -3$

B) A varies directly as the square root of B and inversely as the cube of C.
When $B = 9$ and $C = 2$, the value of A is 36. Find the value of B when $A = 8$ and $C = 4$.

C) Given $\dfrac{z}{x+y} = 2$ and $\dfrac{y}{x+z} = 3$, compute the numerical value of $\dfrac{z}{y+z}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

ANSWERS

A) ________________________________

B) _______________ : _______________

C) ________________________________

A) The vertices of rhombus *PQRS* are midpoints of consecutive sides of square *ABCD*.
J, *K*, *L* and *M* are midpoints of consecutive sides of rhombus *PQRS*.
If $AB = 4$, compute the perimeter of quadrilateral *JKLM*.

B) All diagonals of regular polygon *A* with *a* sides are equal in length.
All diagonals of regular polygon *B* with *b* sides are also equal in length.
Regular polygon *C* has $a + b$ sides and $b > a$.
Let *m* and *n* denote the degree measures of an interior and exterior angles of polygon *C* respectively. Compute the ratio $m : n$.

C) The diagonals of a rhombus have lengths in a 4 : 3 ratio. If the ratio of the numerical value of the area to the numerical value of the perimeter is 9 : 4, compute the length of the longer diagonal.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ______ : ______ AM PM

B) ____________________________ E) ____________________________

C) (___________ , ___________) F) ____________________________

A) A triangle has sides of length $x + 1$, $x + 3$ and $11 - 2x$. Find <u>two</u> values of x so that one angle of the triangle has a measure of 120°.

B) A positive 5-digit integer N has 13 as its rightmost two digits, has a digit sum of 13 and is divisible by 13. Determine the <u>sum</u> of the smallest and largest such integers.

C) In the xy-plane, the graph of $(a + 2)x + ay + b = 0$ is 1 unit from the origin. Determine the ordered pair (a, b), if b must be the <u>minimum</u> positive value for which this is true.

D) Connie was last seen alive at 10pm last evening. According to the police report, she was found dead the next morning. It was a cool fall night and the air temperature was 65° F all night long. The CSI unit arrived on the scene and at 6:45am determined that the body temperature was 90° F and a half hour later had dropped to 89.5° F . Applying Newton's law of cooling, the time of the murder can be determined. *Help solve this crime*! Rounding to the nearest 15 minutes, at what time did the murder occur?

$$T(t) = T_a + (T_0 - T_a)e^{-kt}\text{, where}$$

$T(t)$ denotes temperature at time t.
T_a denotes the air temperature.
T_0 denotes the initial temperature of the deceased's body.
Assume at the time of her death Connie's body temperature was 98.6° F.

Note: In the formula above, e denotes the base of natural logarithms.
On the scientific calculator, these functions are accessed using the $\boxed{\ln}$ and $\boxed{e^x}$ keys.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008
ROUND 7 TEAM QUESTIONS – continued

E) In the game of baseball, a <u>*hit*</u> (H) falls into one of the following four categories:
$1B$ (singles), $2B$ (doubles), $3B$ (triples) and HR (homeruns)

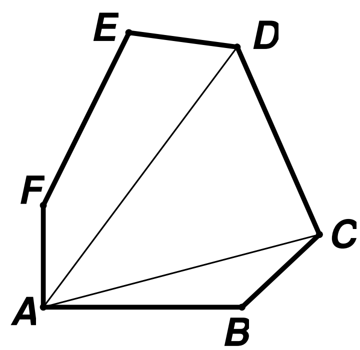
The following abbreviations are used in the formula for a player's slugging percentage:
AB (at bats), SAC (sacrifices), BB (base-on-balls), HBP (hit-by-pitch).

(Additional knowledge of the game is **<u>not</u>** essential.)

Given: $AB = 120$, $H = 35$, $SAC = BB + HBP = 5$ and $2B : 3B : HR = 3 : 1 : 2$

How many singles ($1B$) has a player hit, if this player's slugging percentage,
$\left(\dfrac{\text{1B} + 2(\text{2B}) + 3(\text{3B}) + 4(\text{HR})}{\text{AB} - (\text{SAC} + \text{BB} + \text{HBP})}\right)$ is 0.618 (rounded to 3 decimal places)?

F) In hexagon $ABCDEF$, $AB = CD = EF = 4$, $BC = DE = FA = 3$,
$m\angle A = m\angle C = m\angle E = 90°$, and $m\angle B = m\angle D = m\angle F = 150°$
Compute $AC^2 - AD^2$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008 ANSWERS

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

A) $\frac{15}{17}$ B) 50 C) $40+8\sqrt{3}$ or $8\left(5+\sqrt{3}\right)$

Round 2 Arithmetic/Elementary Number Theory

A) 158400 B) (2, 2, 5, 1) C) $ACBBAA_3$

Round 3 Coordinate Geometry of Lines and Circles

A) (8, −12, −9) B) 3 C) (3, 6)

Round 4 Alg 2: Log and Exponential Functions

A) $\frac{9}{256}$ B) −1 C) 2/3, 2

Round 5 Alg 1: Ratio, Proportion or Variation

A) −13 B) $\frac{256}{9}$ C) $\frac{8}{17}$

Round 6 Plane Geometry: Polygons (no areas)

A) 8 B) 7 : 2 C) 15

Team Round

A) 2, 4

B) 81926
[smallest: 11713 largest: 70213]

C) $\left(-1,\sqrt{2}\right)$

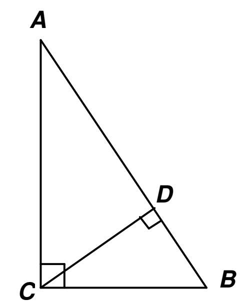
D) 11:30 pm

E) 17

F) 0

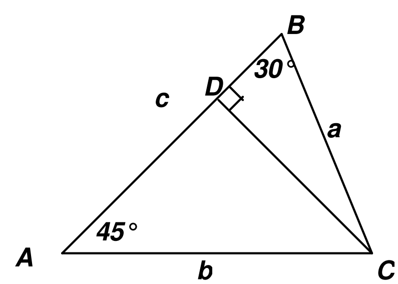
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008 SOLUTION KEY

Round 1



A) The sides of right triangle ABC are 8 – 15 – 17.
If $m\angle A = \theta$, $m\angle ACD = m\angle CBD = 90 - \theta$.
Therefore, $\sin(\angle ACD) = \sin(\angle CBD) = \underline{\mathbf{\dfrac{15}{17}}}$

B) Since the largest angle is opposite the longest side and the smallest angle is opposite the shortest side, the medium sized angle is A.
According to the law of cosines, $10^2 = 9^2 + 11^2 - 2(9)(11)\cos A$
$\rightarrow \cos A = \dfrac{10^2-(9^2+11^2)}{-2\cdot 9\cdot 11} = \dfrac{102}{2\cdot 9\cdot 11} = \dfrac{17}{33} \rightarrow m + n = \underline{\mathbf{50}}$



C) As an angle in ΔABC, $\cos B = \dfrac{\sqrt{3}}{2} \rightarrow B = 30°$

and $C = (180 - 45 - 30) = 105°$

By the law of sines, $\dfrac{\sqrt{2}/2}{a} = \dfrac{1/2}{2\sqrt{2}} \rightarrow a = 4$

$a^2 + b^2 = \left(2\sqrt{2}\right)^2 + 4^2 = 24$

By the law of cosines, $c^2 = a^2 + b^2 - 2ab\cos(105°) \rightarrow c^2 = 24 - 16\sqrt{2}\cos(60° + 45°)$
$= 24 - 16\sqrt{2}\cdot\dfrac{\sqrt{2}-\sqrt{6}}{4} = 24 - 8 + 4\sqrt{12} = 16 + 8\sqrt{3}$. Thus, $a^2 + b^2 + c^2 = 24 + 16 + 8\sqrt{3} =$
$\underline{\mathbf{40+8\sqrt{3}}}$

Alternately, drop an altitude from C to $\overline{AB}$, creating 45-45-90 and 30-60-90 triangles.
This quickly gives $AB = 2+\sqrt{3}$ and the same result follows.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008 SOLUTION KEY

Round 2

A) The prime factorization of N is of the form $2^x \cdot 3^a \cdot 5^b \cdot \ldots$ or $2^x \cdot P$, where P denotes a product of all powers of odd primes. Clearly, $A = P$, since an odd product requires all odd factors. Similarly, since 1 is a factor of every integer, 1 must be the <u>only</u> odd factor of B, implying B is a power of 2. Thus, $B = 2^x$ and $AB = 2^x \cdot P = N$
Thus, without bothering to factor N, AB = **<u>158400</u>**

B) $24^3 \cdot 120^2 \cdot 441 = 2^{9+6} \cdot 3^{3+2+2} \cdot 5^2 \cdot 7^2 = 2^{15} \cdot 3^7 \cdot 5^2 \cdot 7^2$
$28^A \cdot 15^B \cdot 12^C \cdot 2^D = 2^{2A+2C+D} \cdot 3^{B+C} \cdot 5^B \cdot 7^A$
Thus, $A = B = 2$, $2 + C = 7$ and $2A + 2C + D = 15 \rightarrow (A, B, C, D)$ = **<u>(2, 2, 5, 1)</u>**

C) The base 3 place values are: 1, 3, 9, 27, 81, 243, ….
Since 243 is the smallest power of 3 greater than 211, the representation will have 6 digits.
$243 - 27 = 216 \rightarrow$ (ACB _ _ _)
$216 - 9 = 207 \rightarrow$ ($ACBB$ _ _)
$207 + (3 + 1) = 211$
Thus, 211_{10} = ***<u>ACBBAA</u>***$_3$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008 SOLUTION KEY

Round 3

A) The required line is a line halfway between the given two parallel lines.

The given equations are equivalent to $\begin{cases} 4x-6y-10=0 \\ 4x-6y+1=0 \end{cases}$

Clearly, $4x-6y+\dfrac{-10+1}{2}=0$ is the required equation.

Multiplying by 2 to clear fractions: $8x - 12y - 9 = 0$ → **<u>(8, –12, –9)</u>**

B) Completing the square we have: $\begin{cases} (x-2)^2+(y+5)^2=25 \\ (x+6)^2+(y-10)^2=81 \end{cases}$

The shortest distance between the circles lies on the segment connecting the centers of the circles. The distance between the centers (2, –5) and (–6, 10) is 17.
Thus, the required distance is 17 – (5 + 9) = **<u>3</u>**.

C) $2x + 3y = 4$ → linear with slope = –2/3 and passes through (2, 0) Applying the slopes, we have additional solutions of: (5, –2), (8, –4) (–1, 2), (–4, 4) and (–7, 6)
For any other solutions, either the x- or y - coordinate (or both) has (have) more than one digit.
Adding, we have $A = (2 + 5 + 8 - 1 - 4 - 7) = 3$ and $B = 0 - 2 - 4 + 2 + 4 + 6 = 6$ → **<u>(3, 6)</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008 SOLUTION KEY

Round 4

A) $49^{\log_7 3-4\log_7 2} = 49^{\log_7 3-\log_7 16} = 49^{\log_7 \frac{3}{16}} = 7^{2\log_7 \frac{3}{16}} = 7^{\log_7 \frac{9}{256}} = \underline{\mathbf{\frac{9}{256}}}$

B) $2^x = \frac{1}{2^x} - \frac{3}{2}$ Let $N = 2^x$. Then: $N = \frac{1}{N} - \frac{3}{2}$ → $2N^2 + 3N - 2 = (2N-1)(N+2) = 0$

→ $N = \frac{1}{2}, -2$ Since a power function never produces a negative value, the latter value is extraneous. Thus, $2^x = \frac{1}{2}$ → $x = \underline{\mathbf{-1}}$

C) Since $A^{\log_A B} = B$, the equation simplifies to $3x^5 - 2x^4 + 8x^2 = 12x^3$
→ $x^2(3x^3 - 2x^2 - 12x + 8) = 0$
→ $x^2[x^2(3x - 2) - 4(3x - 2)] = x^2(3x - 2)(x^2 - 4) = x^2(3x - 2)(x + 2)(x - 2) = 0$
→ $x = 0, 2/3, \pm 2$
Since x is the argument of the log function, $x > 0$. Thus, 0 and -2 are rejected → **<u>2/3, 2</u>** only

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008 SOLUTION KEY

Round 5

A) Given: $\begin{cases} y = 10 - 3t \\ x = 4t + 1 \end{cases}$ $\quad x = -3 \rightarrow t = -1$

Substituting in the 1st equation, $y = 13$. Thus, $\frac{y}{t} = \underline{\mathbf{-13}}$

B) $A = \frac{k\sqrt{B}}{C^3} \rightarrow 36 = \frac{k\sqrt{9}}{2^3} \rightarrow k = 36(8)/3 = 96$

$(A, C) = (8, 4) \rightarrow 8 = \frac{96\sqrt{B}}{4^3} \rightarrow \sqrt{B} = \frac{8(64)}{96} = \frac{2^9}{2^5 \cdot 3} = \frac{2^4}{3} \rightarrow B = \underline{\mathbf{\frac{256}{9}}}$

C) We need to find $\frac{y}{z}$ since $\frac{z}{y+z} = \frac{1}{\frac{y}{z}+1}$ (***).

From the given equations, we have $z = 2x + 2y$ and $y = 3x + 3z$.
Thus, $z = 2x + 2(3x + 3z) = 8x + 6z$ or $-5z = 8x$
Also $y = 3x + 3(2x + 2y) = 9x + 6y$ or $-5y = 9x$

So $\frac{y}{z} + 1 = \frac{\frac{y}{x}}{\frac{z}{x}} + 1 = \frac{-9/5}{-8/5} + 1 = \frac{9}{8} + 1 = \frac{17}{8}$

Substituting in (***), $\frac{z}{y+z} = \underline{\mathbf{\frac{8}{17}}}$

Notice that if the values of the two given expressions were reversed, the answer would be $\frac{9}{17}$ and $\frac{8}{17} + \frac{9}{17} = 1$.

Round 5 - continued

C) Alternate solution:

$$\frac{z}{z+y}=2 \rightarrow \frac{\frac{z}{x}}{1+\frac{y}{x}}=2 \quad \text{and} \quad \frac{y}{x+z}=3 \rightarrow \frac{\frac{y}{x}}{1+\frac{z}{x}}=3$$

Rearranging, we get: $\begin{cases} \frac{y}{x}-3\frac{z}{x}=3 \\ 2\frac{y}{x}-\frac{z}{x}=-2 \end{cases}$ Solving we get $\frac{z}{x}=-\frac{8}{5}$ and $\frac{y}{x}=-\frac{9}{5} \rightarrow \frac{y}{z}+1=\frac{17}{8}$

Thus, $\frac{z}{y+z}=\frac{1}{\frac{y}{z}+1}=\underline{\boldsymbol{\frac{8}{17}}}$

Generalization

(contributed by Shing S. So Dept. Math and Computer Science – University of Central Missouri)

Given: $\frac{z}{x+y}=a$ and $\frac{y}{x+z}=b$, find $\frac{z}{y+z}$ in terms of a and b.

$\frac{z}{y+z}=\frac{1}{\frac{y}{z}+1}$ so we solve for $\frac{y}{z}$.

From $\frac{z}{x+y}=a$ we have $z = ax + ay$

From $\frac{y}{x+z}=b$ we have $y = bx + bz$

Substituting we have $z = ax + a(bx + bz) = ax + abx + abz$, so $\frac{z}{x}=\frac{a+ab}{1-ab}$.

Substituting again we have $y = bx + b(ax + ay) = bx + abx + aby$, so $\frac{y}{x}=\frac{b+ab}{1-ab}$.

Now $\frac{y}{z}=\frac{\frac{y}{x}}{\frac{z}{x}}=\frac{b+ab}{a+ab}$ and $\frac{y}{z}+1=\frac{b+ab+a+ab}{a+ab}=\frac{2ab+a+b}{a+ab}$

Inverting we have the required ratio, namely $\frac{z}{y+z}=\underline{\boldsymbol{\frac{a+ab}{2ab+a+b}}}$

Note if the values of the two given ratios are reversed we have $\frac{z}{y+z}=\underline{\boldsymbol{\frac{b+ab}{2ab+a+b}}}$,

and these two values sum to 1.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008 SOLUTION KEY

Round 6

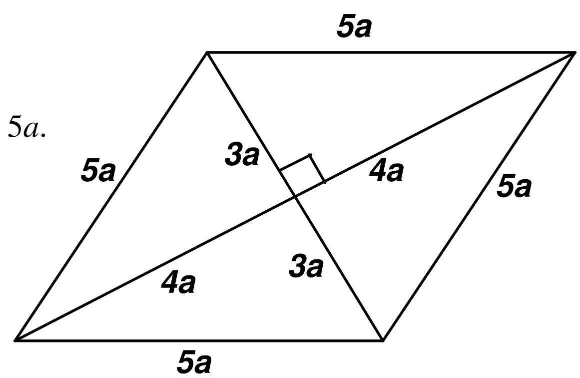
A) Rhombus *PQRS* must also be a square and $PS = 2\sqrt{2}$
JKLM must also be a square.
If J is the midpoint of $\overline{PS}$, then $PJ = \sqrt{2} \rightarrow JK = \sqrt{2}\cdot\sqrt{2} = 2$
Thus, the perimeter of *JKLM* = **<u>8</u>**

B) A must be a square and B must be a pentagon, since these are the only regular polygons in which all diagonals could have the same length. Thus, C has 9 sides.
$n = 360/9 = 40$ and $m = 180 - 40 = 140 \rightarrow m : n = 140 : 40 =$ **<u>7 : 2</u>**

C) Since the diagonals of a rhombus are perpendicular, the fact that the diagonals have lengths in a 4 : 3 ratio implies that the sides of the rhombus must have lengths $5a$.
Thus,

$$\frac{Area(\text{Rhom})}{Per(\text{Rhom})} = \frac{\frac{1}{2}8a\cdot 6a}{20a} = \frac{9}{4} \rightarrow \frac{6a}{5} = \frac{9}{4} \rightarrow a = \frac{15}{8}$$

and the long diagonal has length **<u>15</u>**.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008 SOLUTION KEY

Team Round

A) $x + 1$ can not be the longest side and therefore, can not be opposite the largest angle of 120°.

Case 1: $(x + 3)^2 = (x + 1)^2 + (11 - 2x)^2 - 2(x + 1)(11 - 2x)(-1/2)$
$\rightarrow x^2 + 6x + 9 = 5x^2 - 42x^2 + 122 \;+\; 11 + 9x - 2x^2$
$\rightarrow 2x^2 - 39x + 124 = (2x - 31)(x - 4) = 0 \rightarrow 4$ only

Case 2: $(11 - 2x)^2 = (x + 1)^2 + (x + 3)^2 - 2(x + 1)(x + 3)(-1/2)$
$\rightarrow 121 - 44x + 4x^2 = 2x^2 + 8x + 10 \;+\; x^2 + 4x + 3$
$\rightarrow x^2 - 56x + 108 = (x - 2)(x - 54) = 0 \rightarrow 2$ only

Note: In both cases, the triangle has sides of lengths 3, 5 and 7.

Alternate solution:
Case 1: $x + 3 > 11 - 2x \rightarrow x > 2\ 2/3$, but $11 - 2x > 0 \rightarrow x < 5\ ½$
Thus, $2.4 < x < 5.5$
By trial and error, 3 → 4, 5, 6 rejected **<u>4</u>** → 3, 5, 7 OK 5 → 1, 6, 8 rejected

Case 2: $11 - 2x > x + 3$ and $x + 1 > 0 \rightarrow -1 < x < 2\ 2/3$
By trial and error, 0 → 1, 3, 11 rejected 1 → 2, 4, 9 rejected **<u>2</u>** → 3, 5, 7 OK

B) The leftmost three digits must sum to 9. They must also be a multiple of 13; otherwise, there would be a positive remainder r and the three-digit integer r13 is never a multiple of 13.

Notice the pattern of remainders?

Remainders upon division by 13

r	r13	remainder	r	r13	remainder
1	113	9	7	713	11
2	213	5	8	813	7
3	313	1	9	913	3
4	413	10	10	1013	12
5	513	6	11	1113	8
6	613	2	12	1213	4

Thus, only 013 is a multiple of 13.
Examining the three-digits multiples of 13, the smallest is 104 (but it is not a multiple of 9)
117 is the smallest with a digit sum of 9 → $N_{min} = 11713$

Finding N_{max} :
Three-digit multiples of 13 must be of the form $104 + 13k$.
9xy: Only 900 could work and it is not a multiple of 13
8xy: The smallest are 806, 819, 832, …. and clearly none of these will have a digit-sum of 9.
7xy: The smallest are 702, 715, 728, …
702 is a multiple of 13 and clearly none of the other 7xy integers will have a digit-sum of 9
→ $N_{max} = 70213$ and the required sum is **<u>81926</u>**

Team Round - continued

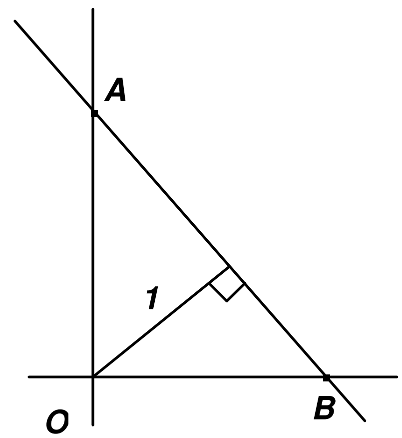
C) The distance form the origin to this line is measured along the perpendicular and this distance is given by:

$$\frac{|0(a+2)+0(a)+b|}{\sqrt{(a+2)^2+a^2}} = 1 \rightarrow b^2 = 2a^2 + 4a + 4 \rightarrow 2a^2 + 4a + (4 - b^2) = 0$$

Solving for a in terms of b using the quadratic formula, the discriminant must be nonnegative to guarantee a real solution. Thus, $16 - 8(4 - b^2) \geq 0 \rightarrow 8b^2 \geq 16 \rightarrow b^2 \geq 2$

Since $b > 0$, $b_{\min} = \sqrt{2}$. Substituting, $2a^2 + 4a + 2 = 2(a^2 + 2a + 1) = 2(a+1)^2 = 0 \rightarrow a = -1$

$\rightarrow (a, b) = \underline{\left(\mathbf{-1}, \sqrt{\mathbf{2}}\right)}$



Alternate solution: Note that $A\left(0, -\frac{b}{a}\right)$ and $B\left(\frac{-b}{a+2}, 0\right)$

The area of ΔAOB is $\frac{1}{2}\cdot\left|\frac{b}{a}\right|\cdot\left|\frac{b}{a+2}\right|$ using $\overline{OB}$ as the base or

$\frac{1}{2}\cdot 1\cdot\sqrt{\left(\frac{b}{a}\right)^2+\left(\frac{b}{a+2}\right)^2}$ using $\overline{AB}$ as the base. Equating, we have

$$\frac{b^2}{|a(a+2)|} = |b|\cdot\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{a+2}\right)^2} = |b|\cdot\sqrt{\frac{(a+2)^2+a^2}{a^2(a+2)^2}} = \frac{|b|}{|a(a+2)|}\cdot\sqrt{2a^2+4a+4} =$$

Cancelling, $b = \sqrt{2(a+1)^2+2} = \sqrt{2(a+1)^2+2}$

To minimize b is to minimize the radical expression, which is equivalent to minimizing the quadratic expression in the radicand. Clearly, $a = \underline{\mathbf{-1}}$ does this and the minimum value of b is $\underline{\sqrt{\mathbf{2}}}$.

D) $89.5 = 65 + (90-65)e^{-0.5k} \rightarrow 24.5 = 25e^{-0.5k} \rightarrow k = -2\ln\left(\frac{24.5}{25}\right) \approx 0.040405$

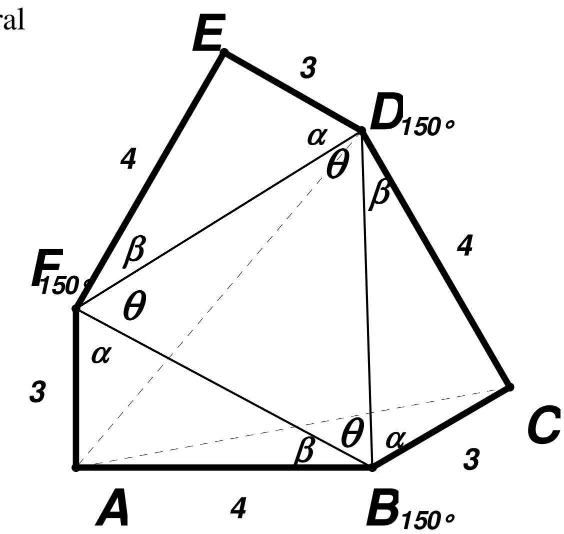
$98.6 = 65 + 25e^{-0.040405t} \rightarrow t = \dfrac{\ln\left(\frac{33.6}{25}\right)}{-0.040405} \approx -7.317 \rightarrow$ 7 hours 19^{+} minutes $\rightarrow$ **<u>11:30 pm</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2008 SOLUTION KEY

Team Round - continued

E) Let doubles ($2B$) = $3n$, triples ($3B$) = n and homeruns $HR = 2n$. Then:
Singles ($1B$) = $H - (2B + 3B + HR)$ → $1B = 35 - (3n + n + 2n) = 35 - 6n$
Numerator = $(35 - 6n) + 2(3n) + 3(n) + 4(2n) = 35 + 11n$
Denominator = $120 - SAC - (BB + HBP) = 120 - 5 - 5 = 110$
$\frac{35+11n}{110} = 0.618$ → $n = 2.998 \approx 3$ → $1B = 35 - 6(3) = \underline{\mathbf{17}}$

F) Clearly, $BD = DF = FB = 5$ and ΔBDF is both equilateral and equiangular ($\theta = 60°$).



Using the law of cosines in ΔABC,
$AC^2 = 3^2 + 4^2 - 2\cdot 3\cdot 4\cdot \cos 150°$
→ $AC^2 = 25 + 12\sqrt{3}$
Using the law of cosines on ΔFAD,
$AD^2 = 3^2 + 5^2 - 2\cdot 3\cdot 5\cdot \cos(\alpha + 60°)$
$= 34 - 30\cos(\alpha + 60°)$
$= 34 - 30(\cos\alpha\cos 60° - \sin\alpha\sin 60°)$
$= 34 - 30\left(\frac{3}{5}\cdot\frac{1}{2} - \frac{4}{5}\cdot\frac{\sqrt{3}}{2}\right)$
$= 34 - 30\left(\frac{3-4\sqrt{3}}{10}\right) = 34 - 9 + 12\sqrt{3} = 25 + 12\sqrt{3} = AC^2$
Thus, $AC^2 - AD^2 = \underline{\mathbf{0}}$

[In this version of the problem, 3-4-5 triangles were 'attached' to the sides of an equilateral triangle. Can this problem be generalized by starting with an equilateral triangle and 'attaching' other types of congruent triangles to each side?]