MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2009
ROUND 1 ANALYTIC GEOMETRY: ANYTHING

ANSWERS

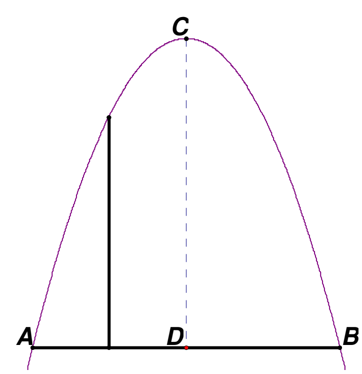
A) ______________________________

B) ______________________________

C) ______________________________

A) The points $A(7, a)$ and $B(b, 1)$ lie on the hyperbola $x^2 - y^2 = 24$.
Compute the largest possible value for the distance AB.

B) A parabolic arch has a span (AB) of 12 units and a maximum height (CD) of 8 units.
Find the height of the arch $\frac{1}{4}$ of the way across the span.



C) A circle of radius 5 is tangent to $x = 3$ and $y = -2$. Let S be the sum of all possible x-intercepts.
Compute S.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009
ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) _______________________

B) _______________________

C) _______________________

A) Clearly $x = 3$ is a solution of the equation $(2x-1)(x+2) = 25$. Find the non-integer solution.

B) Determine <u>all</u> values of x that satisfy $12x^5 - 36x^3 = 46x^4$

C) Factor the following expression completely over the integers.

$$72x^3 - 4x^2 + 9 - 32x^5$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009
ROUND 3 TRIG: EQUATIONS WITH A RESAONABLE NUMBER OF SOLUTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Solve for x, where $0 \le x < 2\pi$. Give <u>exact</u> answers in terms of π.

$$3\cos 2x = 2\cos^2 x$$

B) Solve for θ, where $0° \le \theta < 360°$: $\left(\sqrt{2}\cos\theta - \sqrt{2}\sin\theta\right)^2 = 3$

C) Solve for θ, where $0° \le \theta < 360°$: $\sqrt{3}\tan^2\theta + \tan\theta = \sqrt{3}\tan\theta + \sec^2\theta - \tan^2\theta$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009
ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) $y =$ ___________________________

A) It is known that x is positive and that A and B have opposite signs.
Solve for x in terms of A and B:

$$x^2 - 3A^2B^2 = 2ABx$$

B) The quadratic expression $Ax^2 - 4x + B$ has a value of 9 when $x = -3$ and a value of -7 when $x = 5$.
Determine the minimum value of this expression.

C) Given: $3x^2 - 6x + xy + 4y - 2y^2 = 0$
Determine all possible values of y in terms of x.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009
ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

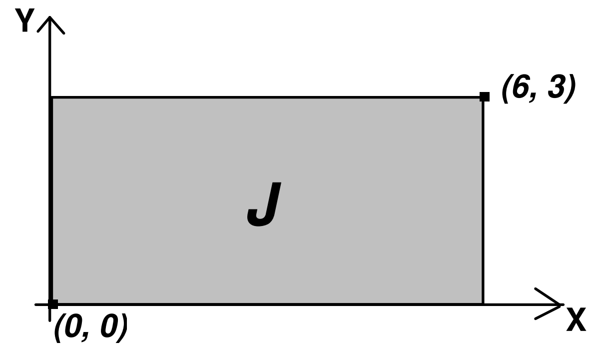
ANSWERS

A) _______________________________

B) _______________________________

C) ______________ : ______________

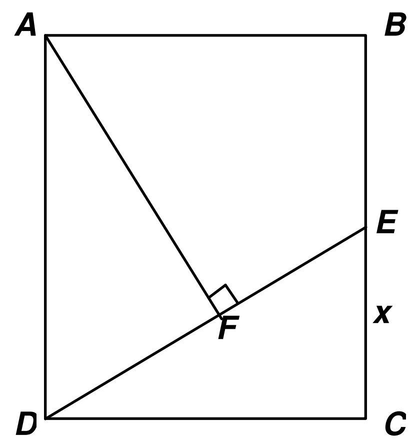
A) Rectangle J contains all points (x, y) in the shaded region. What is the area of the rectangle that contains all the points $(2x + 3, 6 - 2y)$?



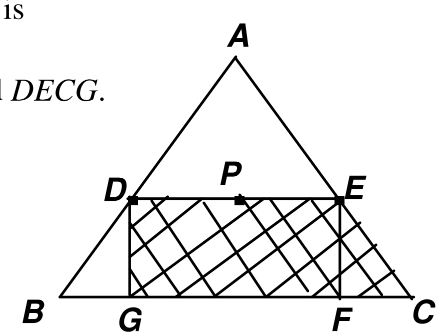
B) In rectangle $ABCD$, $EC = x$, $AB = 13$, $AD = 15$ and

$$\frac{Area(\Delta DCE)}{Area(\Delta DFA)} = \frac{10}{9}$$

(E is on $\overline{BC}$ and F is on $\overline{DE}$)
Compute x.



C) In isosceles triangle ABC, $\overline{DE}$ contains the centroid P of ΔABC and is parallel to base $\overline{BC}$. D and E lie on $\overline{AB}$ and $\overline{AC}$, respectively.
Find the simplified ratio of the area of ΔADE to the area of trapezoid $DECG$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009
ROUND 6 ALG 1: ANYTHING

ANSWERS

A) ______________________________

B) ______________________________

C) $y =$ ___________________________

A) Find the value(s) of x for which

$$\sqrt{8^4+4^6+16^3+64^2}\cdot\sqrt{x^2}=\sqrt{64^3}$$

B) At 2:00 the hour and minute hands of an analog clock form a 60° angle.
Between 2 and 3 o'clock, this happens again at exactly x minutes past 2 o'clock.
Determine the value of x. Express your answer as the ratio of two relatively prime integers.

C) Consider the following operations on real numbers: $\begin{array}{l} x\circ y=2x-y \\ \overline{x}=x^2 \end{array}$

If $\overline{(x\circ y)}=\overline{x}\circ\overline{y}$, then find a simplified expression for y in terms of x.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) ____________________________

C) ____________________________ F) ____________________________

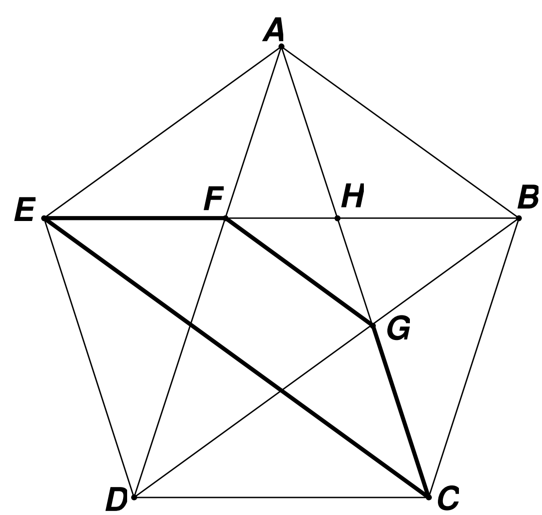
A) Find the length of the minor axis of the ellipse $2x^2 + 3xy + 2y^2 = 63$.
Note: This ellipse has been rotated 45° and its axes lie along the lines $y = x$ and $y = -x$.

B) Factor completely over the integers: $G^4 + T^4 - 8\left(G^2 + T^2 - 2\right) - 2GT\left(2G^2 + 2T^2 - 3GT - 8\right)$

C) One of the solutions of the equation $\sqrt{1 + a\sin x} = -\cos x$ over $0° \le x < 360°$ is 150°.
Determine the other solution(s).

D) The parabola $y = x^2$ intersects the line $y = 7x + 13$ at points A and B.
Find the coordinates of the midpoint of $\overline{AB}$.

E) Given: $ABCDE$ is a regular pentagon, $AD = 1$
Compute the perimeter of isosceles trapezoid $CEFG$.



F) If A is a positive integer, determine the largest possible value of x, given: $3 - \dfrac{1}{\frac{1}{A} + \frac{1}{x}} = \dfrac{1}{3}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009 ANSWERS

Round 1 Analytic Geometry: Anything

A) $6\sqrt{5}$ B) 6 C) 12

Round 2 Alg1: Factoring

A) $-\frac{9}{2}$ B) $0,-\frac{2}{3},\frac{9}{2}$ C) $(2x+1)(4x^2-2x+1)(3+2x)(3-2x)$

- or equivalent - (exactly 4 factors required)

Round 3 Trig: Equations

A) $\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$ B) 105°, 165°, 285°, 345° C) 45°,150°,225°,330°

Round 4 Alg 2: Quadratic Equations

A) $-AB$ B) -16 C) $y=\frac{3x}{2},2-x$

-or equivalent simplified fractions-

Round 5 Geometry: Similarity

A) 72 B) 9 C) 8 : 9

Round 6 Alg 1: Anything

A) ± 4 B) $\frac{240}{11}$ C) x

Team Round

A) $6\sqrt{2}$ D) $\left(\frac{7}{2},\frac{75}{2}\right)$

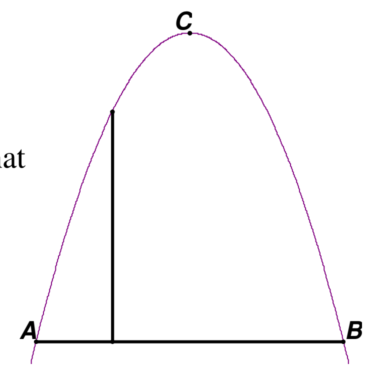
B) $(G-T-2)^2(G-T+2)^2$ E) $\frac{11-3\sqrt{5}}{2}$

C) Other solutions: 180° only F) 24

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009 SOLUTION KEY

Round 1

A) Substituting, $A(7, a)$: $49 - a^2 = 24 \rightarrow a = \pm 5$ $B(b, 1)$: $b^2 - 1 = 25 \rightarrow b = \pm 5$
The longest distance is between $A(7, -5)$ and $B(-5, 1)$ $AB = \sqrt{(7 - {}^-5)^2 + ({}^-5 - 1)^2} = \sqrt{180} = \underline{\mathbf{6\sqrt{5}}}$



B) Let $A(0, 0)$, $B(12, 0)$ and $C(6, 8)$. Then the equation of the arch is $y = x(12 - x) \cdot c$, for some fudge-factor c to adjust the height.
Substituting, $8 = 6(12 - 6) \cdot c$ and $c = \dfrac{2}{9}$. The perpendicular segment that is a quarter of the way across the span connects $(3, 0)$ and $(3, h)$.
Thus, substituting, $h = \dfrac{2}{9} \cdot 3 \cdot (12 - 3) = \underline{\mathbf{6}}$

C) Unless the center of the circle is <u>above</u> $y = -2$, there would be no x-intercepts. Additionally, the center must be 5 units left or right of $x = 3$, i.e. at $(-2, 3)$ or $(8, 3)$.
When $y = 0$,
$(x+2)^2 + (y-3)^2 = 25 \rightarrow (x + 2)^2 = 16 \rightarrow x = -6, 2$
$(x-8)^2 + (y-3)^2 = 25 \rightarrow (x - 8)^2 = 16 \rightarrow x = 4, 12$
Thus, the required sum is $-6 + 2 + 4 + 12 = \underline{\mathbf{12}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009 SOLUTION KEY

Round 2

A) $(2x-1)(x+2)=25 \rightarrow 2x^2+3x-27=0 \rightarrow (2x+9)(x-3)=0 \rightarrow x=$ $\underline{\mathbf{-\frac{9}{2}}}$

B) $12x^5-36x^3=46x^4 \rightarrow 2x^3(6x^2-23x-18)=0 \rightarrow 2x^3(3x+2)(2x-9) \rightarrow x=$ $\underline{\mathbf{0,-\frac{2}{3},\frac{9}{2}}}$

C) $72x^3-4x^2+9-32x^5 = 8x^3(9-4x^2)+(9-4x^2) = \left(8x^3+1\right)\left(9-4x^2\right)$

As the sum of perfect cubes and the difference of perfect squares, this product factors to
$\underline{\mathbf{(2x+1)(4x^2-2x+1)(3+2x)(3-2x)}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009 SOLUTION KEY

Round 3

A) $3(2\cos^2 x - 1) = 2\cos^2 x \rightarrow 4\cos^2 x = 3 \rightarrow \cos x = \pm\frac{\sqrt{3}}{2} \rightarrow x = \underline{\boldsymbol{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}}$

B) $\left(\sqrt{2}\cos\theta - \sqrt{2}\sin\theta\right)^2 = 3 \rightarrow \left(\cos\theta - \sin\theta\right)^2 = \frac{3}{2} \rightarrow 1 - 2\sin\theta\cos\theta = \frac{3}{2}$

$\rightarrow \sin(2\theta) = -0.5 \rightarrow 2\theta = \begin{cases} 210 + 360n \\ 330 + 360n \end{cases} \rightarrow \theta = \begin{cases} 105 + 180n \\ 165 + 180n \end{cases}$

$n = 0 \rightarrow \theta =$ **<u>105°, 165°</u>**
$n = 1 \rightarrow \theta =$ **<u>285°, 345°</u>**

C) Since $\tan^2\theta + 1 = \sec^2\theta$, the original equation simplifies to $\sqrt{3}\tan^2\theta + \tan\theta = \sqrt{3}\tan\theta + 1$
$\rightarrow \sqrt{3}\tan^2\theta - \sqrt{3}\tan\theta + \tan\theta - 1 = 0 \rightarrow \sqrt{3}(\tan\theta - 1) + \left(\tan\theta + 1\right) = 0$
$\rightarrow \left(\tan\theta - 1\right)\left(\sqrt{3}\tan\theta + 1\right) = 0 \rightarrow \theta =$ **<u>45°, 225°, 150°, 330°</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009 SOLUTION KEY

Round 4

A) $x^2 - 3A^2B^2 = 2ABx$ → $x^2 - 2ABx - 3A^2B^2 = (x - 3AB)(x + AB) = 0$
→ $x = 3AB$ or $-AB$ Thus, $x =$ **$\underline{-AB}$** only.

B) $\begin{cases} 9 = 9A + 12 + B \\ -7 = 25A - 20 + B \end{cases}$ → $-16 = 16A - 32$ → $A = 1$ and $B = -12$
Thus, the expression $x^2 - 4x - 12$ is equivalent to $(x - 2)^2 - 16$.
The minimum value of **$\underline{-16}$** occurs when $x = 2$.

C) Re-arranging the terms of $3x^2 - 6x + xy + 4y - 2y^2 = 0$, we have $3x^2 + xy - 2y^2 - 2(3x - 2y) = 0$
→ $(3x - 2y)(x + y) - 2(3x - 2y) = 0$
→ $(3x - 2y)(x + y - 2) = 0$ → $3x - 2y = 0$ or $x + y - 2 = 0$
→ $y =$ $\underline{\boldsymbol{\frac{3x}{2}, 2 - x}}$

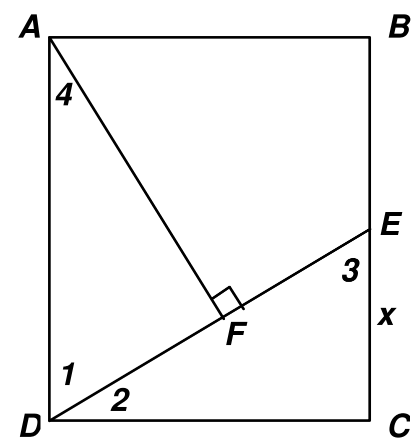
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009 SOLUTION KEY

Round 5

A) The area of the original rectangle J is 18. Since $(2x - 3, 6 - 2y) = \left(2\left(x+1.5\right),-2\left(y-3\right)\right)$
The original rectangle has been translated 1.5 units to the right and 3 units down. This has no effect on the area. However, the original rectangle has been dilated (stretched) by a factor of 2 in both the x – direction and y – direction (as well as reflected across the x – axis). So the new rectangle is similar to the original rectangle with 4 times the area → **72**

B) $m\angle 1 = m\angle 3$ and $m\angle 2 = m\angle 4$ → $\Delta ECD \sim \Delta DFA$ →

$$\frac{x}{DF} = \frac{13}{AF} = \left(\frac{DE}{15} = \frac{\sqrt{10}}{3}\right)$$



Cross multiplying, $DE = 5\sqrt{10}$ Using the Pythagorean theorem on ΔECD, $\left(5\sqrt{10}\right)^2 = 250 = x^2 + 169$ → $x = \mathbf{9}$

The following is an alternate algebraic solution:
From the proportion, $AF = \dfrac{13\cdot 15}{DE}$ and $DF = \dfrac{15x}{DE}$ Then:

$$\frac{\frac{1}{2}\cdot 13x}{\frac{1}{2}DF\cdot AF} = \frac{10}{9} \rightarrow \frac{13x}{\frac{13\cdot 15^2\cdot x}{DE^2}} = \frac{DE^2}{15^2} = \frac{10}{9} \rightarrow DE^2 = 250$$

But, applying the Pythagorean Theorem to ΔDEC, $DE^2 = x^2 + 13^2$ → $x^2 = 81$ → $x = \mathbf{9}$

The following is a trigonometric solution: Let $\theta = m\angle 2 = m\angle 4$.
Then: $AF = 15\cos(\theta)$, $DF = 15\sin(\theta)$ and $x = EC = 13\tan(\theta)$

$$\frac{\frac{1}{2}\cdot 13\cdot(13\tan\theta)}{\frac{1}{2}\cdot 15\sin\theta\cdot 15\cos\theta} = \frac{10}{9} \rightarrow \frac{13^2}{10\cdot 5^2} = \cos^2\theta$$

Finally, $\tan^2\theta = \sec^2\theta - 1$ → $\tan^2\theta = \dfrac{10\cdot 5^2 - 13^2}{13^2} = \dfrac{81}{13^2}$ → $EC = \mathbf{9}$

C) The altitude and median to the base of an isosceles triangle are one and the same. The centroid divides the median into segments whose lengths are in a 2 : 1 ratio. $\overline{DE} \parallel \overline{BC}$ → $\Delta ADE \sim \Delta ABC$ → $DE : BC = 2 : 3$.
Then lengths are represented in the diagram at the right:
The bases of the trapezoid $DECG$ are $DE = 4x$ and $GC = 5x$

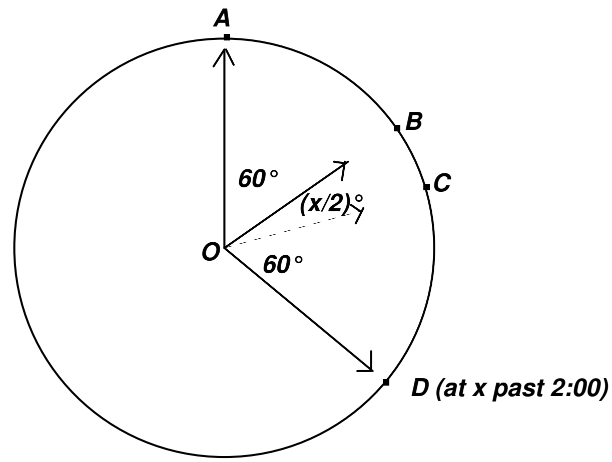
$$\frac{A(\Delta ADE)}{A(DECG)} = \frac{\frac{1}{2}\cdot 4x\cdot 2h}{\frac{1}{2}\cdot h\cdot(4x+5x)} = \frac{8xh}{9xh} = \mathbf{8 : 9}$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009 SOLUTION KEY

Round 6

A) Each of the expressions 8^2, 4^6, 16^3 and 64^2 is equivalent to 2^{12}. Thus, we have:
$\sqrt{4\cdot 2^{12}}\cdot\sqrt{x^2}=\sqrt{2^{14}}\cdot\sqrt{x^2}=2^7\cdot|x|$ = $128|x| = 8^3$ → $2^7|x| = 2^9$ → $|x| = 4$ → $x =$ **<u>± 4</u>**

B) In 1 minute the minute hand travels through 6°.
In x minutes the minute hand travels through $(6x)$ °.
The hour hand travels at 1/12 the rate of the minute hand and, therefore travels through $(x/2)$°
.



From the diagram at the right, we see that:
$$60+\frac{x}{2}+60=6x$$
→ $240 = 11x$ → $x =$ $\underline{\mathbf{\dfrac{240}{11}}}$

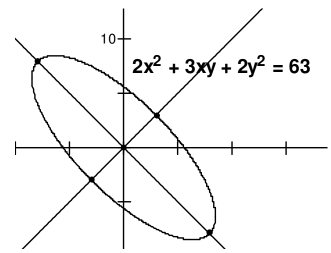
Note: A 60° is <u>not</u> formed again before 3:00. During the remainder of the hour, the angle between the hour and minute hand increases to 180° (when they point in diametrically opposite directions) and then decreases to 90° at 3:00.

C) $\overline{(x\circ y)}=\overline{2x-y}=(2x-y)^2$
$\overline{x}\circ\overline{y}=x^2\circ y^2=2x^2-y^2$
Expanding and equating we have: $4x^2-4xy+y^2=2x^2-y^2$
→ $2x^2+2y^2-4xy=0$ → $2\left(x^2-2xy+y^2\right)=0$ or $2(x-y)^2 = 0$
→ $y =$ **<u>x</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2009 SOLUTION KEY

Team Round

A) The center of the ellipse is at (0, 0).
The graph is symmetric with respect to:
$y = x$, since interchanging x and y does not affect the equation.
$y = -x$, since replacing x and $-y$ and y by $-x$ does not affect the equation.
So the minor axis lies along one of these lines.
Replacing y by x, we have: $2x^2 + 3x^2 + 2x^2 = 63 \rightarrow x^2 = 9 \rightarrow x = \pm 3$
Replacing y by $-x$, we have: $2x^2 - 3x^2 + 2x^2 = 63$
$\rightarrow x^2 = 63 \rightarrow x = \pm 3\sqrt{7}$



Since $3\sqrt{7} > 3$, the endpoints of the minor axis are (–3, 3) and (3, 3) and its length is $\underline{\mathbf{6\sqrt{2}}}$.

<u>Notes for future contests:</u>
The general 2nd degree equation is $Ax^2 + Bxy + Cx^2 + Dx + Ey + F = 0$.
This equation normally graphs as a circle, a parabola, an ellipse or a hyperbola.
Bxy is a rotation term: If $B = 0$, all axes of symmetry are parallel to either the x – or y – axis.
If $B \neq 0$, then $B^2 - 4AC$ is called a discriminant.

$$B^2 - 4AC = \begin{cases} < 0 & \textit{ellipse} \\ > 0 & \textit{hyperbola} \\ = 0 & \textit{parabola} \end{cases} \quad (\underline{\text{graph}})$$

Possible *degenerate* cases	$B^2 - 4AC$
no graph: $(x-3)^2 + (y+2)^2 = -1 \Leftrightarrow x^2 + y^2 - 6x + 4y + 14 = 0$	-4
a single point: $(x-3)^2 + (y+2)^2 = 0 \Leftrightarrow x^2 + y^2 - 6x + 4y + 13 = 0$	-4
a single line: $(x-y+1)^2 = 0 \Leftrightarrow x^2 - 2xy + y^2 + 2x - 2y + 1 = 0$	0
a pair of parallel lines: $(x-y+1)(x-y-1) = 0 \Leftrightarrow x^2 - 2xy + y^2 - 1 = 0$	0
a pair of intersecting lines: $(x-y+1)(x+y+1) = 0 \Leftrightarrow x^2 - y^2 + 2x + 1 = 0$	+4

Team Round – continued

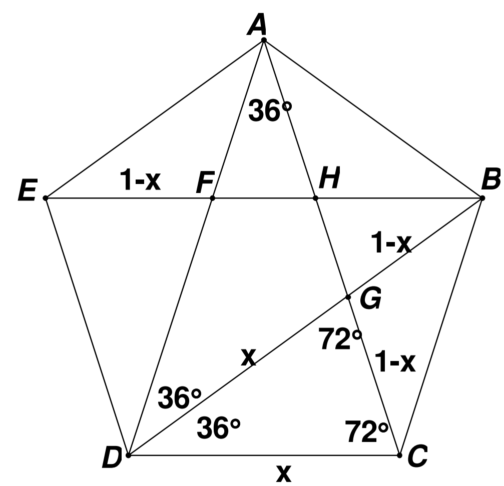
B) Multiply out and re-arrange terms: $G^4+T^4-8G^2-8T^2+16-4G^3T-4GT^3+6G^2T^2+16GT$
$\rightarrow \left(G^4-4G^3T+6G^2T^2-4GT^3+T^4\right)-8\left(G^2-2GT+T^2\right)+16$
$= \left(G-T\right)^4-8\left(G-T\right)^2+16$
$= \left((G-T)^2-4\right)^2 = \underline{\mathbf{(G-T-2)^2(G-T+2)^2}}$

C) Squaring both sides, $1+a\sin x=\cos^2 x \rightarrow a\sin x=\cos^2 x-1=-\sin^2 x$
$\rightarrow \sin x(\sin x+a)=0$
$\rightarrow$ sin x = 0 $\rightarrow$ x = 0°, 180° (180° checks, but 0° is extraneous)
or sin x = –a and x = 150° $\rightarrow$ a = –1/2
Check: sin x = 1/2 $\rightarrow$ x = 30° , 150°
30: $\sqrt{1+\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}=\frac{\sqrt{3}}{2}\neq-\frac{\sqrt{3}}{2}$
150: $\sqrt{1+\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}=\frac{\sqrt{3}}{2}=-\left(-\frac{\sqrt{3}}{2}\right)$
Thus, a must be $-\frac{1}{2}$ and the only additional solution is **<u>180°</u>**.

D) If you tried finding the actual coordinates of points A and B, the computation quickly became painful. How can this be avoided?
Suppose $x^2 = m\text{x} + b$ and that r and s are the roots of $x^2 - mx - b = 0$. Then:
The coordinates of A, B and the midpoint M would be (r, r^2), (s, s^2) and $\left(\frac{r+s}{2},\frac{r^2+s^2}{2}\right)$.
From the root coefficient relationship, r + s = m and $rs = -b$.
Squaring and substituting, $m^2=r^2+2rs+s^2 = r^2-2b+s^2 \rightarrow r^2+s^2=m^2+2b$
Thus, the midpoint M has coordinates $\left(\frac{m}{2},\frac{m^2+2b}{2}\right)$ and m = 7, b = 13 $\rightarrow \left(\mathbf{\frac{7}{2},\frac{75}{2}}\right)$

Team Round – continued

E) (*) $\frac{x}{1} = \frac{1-x}{x}$ → (**) $x^2 = 1 - x$ → $x^2 + x - 1 = 0$



Applying the quadratic formula, $x = \mathbf{\frac{-1+\sqrt{5}}{2}}$ and

$x^2 = 1 - x = CG = EF = 1 - \left(\frac{-1+\sqrt{5}}{2}\right) = \frac{3-\sqrt{5}}{2}$

$FH = EB - (EF + HB) = 1 - 2(1 - x) = 2x - 1$

$\Delta AFE \sim \Delta FHG$ → $\frac{AE}{AF} = \frac{FG}{FH}$ → $\frac{x}{1-x} = \frac{FG}{2x-1}$

(*) → $\frac{1}{x} = \frac{FG}{2x-1}$

or $FG = \frac{2x-1}{x} = 2 - \frac{1}{x} = 2 - \left(\frac{\sqrt{5}+1}{2}\right) = \frac{3-\sqrt{5}}{2} = x^2$

Thus, the perimeter of the trapezoid $CGFE$ is $1 + 2(1 - x) + x^2$

(**) → $1 + 2(x^2) + x^2 = 3x^2 + 1 = 3\left(\frac{3-\sqrt{5}}{2}\right) + 1 = \frac{9-3\sqrt{5}+2}{2} = \underline{\mathbf{\frac{11-3\sqrt{5}}{2}}}$

Oops, I missed a much simpler way of showing that $FG = x^2$.

Note that $\Delta DFG \sim \Delta DAB$ → $\frac{DF}{DA} = \frac{FG}{AB}$ → $\frac{x}{1} = \frac{FG}{x}$ → $FG = x^2$.

[There are many occurrences of the constant ϕ (referred to as the golden ratio) in the regular pentagon. The value of ϕ is $\frac{1+\sqrt{5}}{2} \approx 1.61803\cdots$. The value of x above is $\frac{1}{\phi}$.
You can read about this amazing constant in many outstanding books on mathematical topics, e.g. chapter 15 of The Loom of God – Mathematical Tapestries at the Edge of Time by Clifford A. Pickover (a writer for both Discover and OMNI magazines).]

Team Round – continued

F) $3 - \dfrac{1}{\dfrac{1}{A}+\dfrac{1}{x}} = \dfrac{1}{3} \rightarrow \dfrac{8}{3} = \dfrac{1}{\dfrac{1}{A}+\dfrac{1}{x}}$

$\rightarrow \dfrac{8}{3} = \dfrac{Ax}{x+A} \rightarrow 8x + 8A = 3Ax$

$\rightarrow 8A = x(3A-8) \rightarrow x = \dfrac{8A}{3A-8} = \dfrac{8}{3-\dfrac{8}{A}}$

A = 1, 2, 3, 4 5, 6, … → x = −8/5, −8, **24**, 8, 40/7, 24/5…

Alternate solution:

After getting $x = \dfrac{8A}{3A-8}$. If x is an integer then so is $3x$.

Since $3x = \dfrac{24A}{3A-8}$, by long division, $3x = 8 + \dfrac{64}{3A-8}$

Clearly, the values of $3x$ increase until the fractional component on the right hand side becomes negative, i.e. when $A < 3$, and thereafter they decrease. $A = 3 \rightarrow 3x = 8 + \dfrac{64}{1} = 72 \rightarrow x =$ **24**.