MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) c = ___________________________

A) Let $f(x)=\begin{cases} -x^3 & \text{for } x<0 \\ x+1 & \text{for } 0\le x\le 1 \\ -7 & \text{for } x>1 \end{cases}$. Compute: $f(f(3))+f(-1)+f(f(1/4))$

B) Let $f(x) = x - \dfrac{1}{x}$, $x > 0$. Compute $f^{-1}\left(\dfrac{5}{6}\right)$.

C) Given: $f(x)=ax^2+bx+3$, $f(2) = 3$ and $f(-1) = 9$
Find $c < 0$ such that $f(c) = f(4)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2009
ROUND 2 ARITHMETIC / NUMBER THEORY

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

***** NO CALCULATORS ON THIS ROUND *****

A) Definition: A semi-prime is a positive integer which is the product of exactly two primes. For example, 15, 25 and 145 are semi-primes.

How many semi-primes are there less than 50?

B) Find the sum of the 10 numbers in the 10^{th} row of the following triangular array.

			1			
		2		3		
	4		5		6	
7		8		9		10

C) Find all possible ordered pairs of positive integers (A, B), for which $A + B = 316$, A is a multiple of 13 and B is a multiple of 11.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2009
ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

ANSWERS

A) (________ , ________)

B) ________________________________

C) (________ , ________)

A) The numerical value of $\csc(2 Arc\cot 4)$ may be represented as a simplified ratio of integers $\frac{A}{B}$, where $B > 0$. Determine the ordered pair (A, B).

B) Find <u>all</u> values of x, where $0° \le x < 360°$, that satisfy

$$\frac{\sin 36° \sin 78° + \cos 36° \sin 12°}{\cos 72° \sin 66° + \sin 72° \sin 24°} = \tan x°$$

C) For some constant $\underline{\boldsymbol{B < 0}}$, $x = 135°$ is a solution of

$$\tan\left(Arc\sin\left(-\frac{2}{\sqrt{5}}\right) - Arc\cos B \right) = \cot(180° + x)\ .$$

The exact simplified value of B may be expressed in the rationalized form $\frac{P\sqrt{Q}}{Q}$.

Determine the ordered pair of integers (P, Q).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2009
ROUND 4 ALG 1: WORD PROBLEMS

ANSWERS

A) ________________________minutes

B) ______________________________

C) ______________________________

A) A machine can complete a job in 2 hours. A second, slower machine, can complete the same job in 3 hours. The faster machine begins the job 15 minute before the slower machine is available and then the two machines working together complete the job in *T* minutes. Compute *T*.

B) If a certain number is decreased by 2008 and the difference is multiplied by 2008, the result is the same as when the number is decreased by 2009 and the difference is multiplied by 2009. Find the number.

C) The ten's digit of a certain 3-digit number *N* is half of the units digit. The hundreds digit is equal to the sum of the units digit and the tens digit.
N is <u>never</u> divisible by *d*. Choose all possible values of *d* from: 3, 4, 6, 7, 13

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2009
ROUND 5 PLANE GEOMETRY: CIRCLES

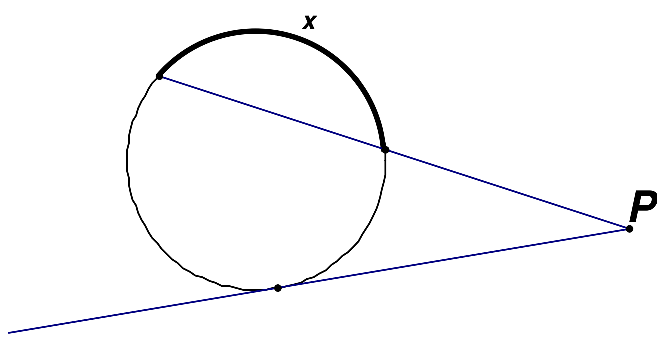
ANSWERS

A) ______________________________ °

B) ________________________________

C) ________________________________

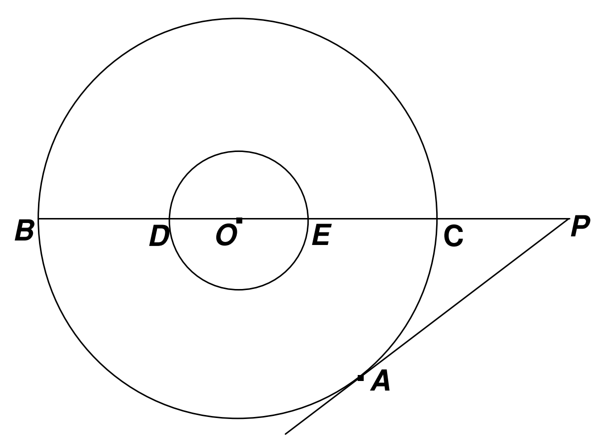
A) A secant and a tangent to circle O intersect at point P, forming a 42° angle. The two arcs intercepted by angle P have measures in a 7 : 3 ratio. Find x, the degree measure of the third arc.



B) In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect at point E, $AB = 20$ and $AE = 2$.
Chord $\overline{CD}$ has integer length $k < 20$. If, additionally, $\overline{CE}$ and $\overline{DE}$ have integer lengths, determine all possible values of k.

C) Given: two concentric circles
$\overrightarrow{PA}$ is tangent to the larger circle at point A
$PA = 6\sqrt{3}$, $BD = DE = EC = CP$

Find the area of the annulus, i.e. the "ring" between the two circles.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2009
ROUND 6 ALG 2: SEQUENCES AND SERIES

ANSWERS

A) ________________________

B) ________________________

C) (__________ , __________)

A) Find the nonnegative difference between the 8th number in the 3rd row and the 3rd number in the 8th row.
Assume the gap between consecutive terms across each row and down each column. increases by the same constant amount.
Note: <u>6</u> is the 2nd number in row 1.

2	6	12	20	30	42	56	72
4	10	18	28				
8	16	26					
14	24						
22							

B) Find the 4th term in <u>all</u> possible geometric sequences whose first three terms are:

$$x + 2, \quad 5x, \quad 13x + 6$$

C) Given: $a_1 = 4$, $a_2 = 5$ and $a_{n+2} = 2a_{n+1} - 3a_n$
For some <u>minimum</u> value of $k > 2$, $a_k > 0$. Determine the ordered pair (k, a_k).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2009
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) ____________________________

C) ____________________________ F) ____________________________

A) Consider the following function defined parametrically, i.e. in terms of a third variable (in this case *t*): $\begin{cases} x = 1 - a^{t} \\ y = 1 + a^{-t} \\ a > 1 \end{cases}$. If $x = \dfrac{2}{3}$, compute *y*.

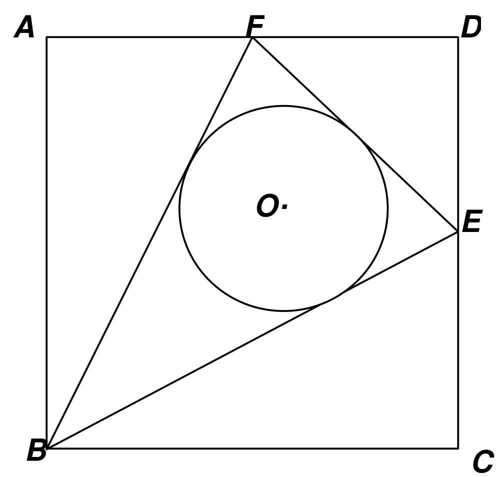
B) For <u>how many</u> positive integers $A < 360$ can the fraction $\dfrac{A}{360}$ be simplified?

C) If $\mathrm{Tan}^{-1}(x) + \mathrm{Sin}^{-1}(x) = \pi/2$ and $x > 0$, compute x^2.

D) The sum of the reciprocals of two integers *A* and *B*, not necessarily distinct, equals *k* times the reciprocal of the sum of *A* and *B*.

For certain integer values of *k*, the ratio $R = \dfrac{A}{B}$ is rational.

Determine <u>all</u> possible ordered pairs (k, R).



E) Given: Square *ABCD*, $AB = 12$, *E* is the midpoint of $\overline{CD}$, *F* is the midpoint of $\overline{AD}$ and circle *O* is inscribed in ΔBEF Compute the length of the radius of circle *O*.

F) For a finite sequence $A = (a_1, a_2, a_3, \ldots, a_n)$ of numbers, the Cesaro sum of *A* is defined to be $\dfrac{S_1 + S_2 + S_3 + \ldots + S_n}{n}$, where $S_k = a_1 + a_2 + a_3 + \ldots + a_k$.

Determine for <u>how many</u> integers $t > 1$, the Cesaro sum of $(a_1, a_2, a_3, \ldots, a_t)$ is 180 and the Cesaro sum of $(1, a_1, a_2, a_3, \ldots, a_t)$ is an integer.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009 ANSWERS

Round 1 Alg 2: Algebraic Functions

A) 337 B) $\frac{3}{2}$ C) -2

Round 2 Arithmetic/ Number Theory [Non-calculator Round]

A) 17 B) 505 C) (52, 264) and (195, 121)

Round 3 Trig Identities and/or Inverse Functions

A) (17, 8) B) 48, 228 or (48°, 228°) C) $(-3, 10)$

Round 4 Alg 1: Word Problems

A) 63 minutes B) 4017 C) 7 only

Round 5 Geometry: Circles

A) 150° B) 15, 13 and 12 (in any order) C) 54π

Round 6 Alg 2: Sequences and Series

A) 10 B) $-\frac{5}{3}, 135$ C) (7, 82)

Team Round

A) 4

B) 263

C) $\frac{\sqrt{5}-1}{2}$

D) $(4, 1)$ and $(0, -1)$

E) $2\sqrt{5}-\sqrt{2}$

F) 16

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009 SOLUTION KEY

Round 1

A) $f(f(3)) = f(-7) = -(-7)^3 = 343$

$f(-1) = -(-1)^3 = 1$

$f(f(1/4)) = f(5/4) = -7$

$343 + 1 - 7 =$ **<u>337</u>**

B) Inputting 5/6 to f^{-1} is equivalent to outputting 5/6 from f. Therefore, there is no need to determine the f^{-1} function rule. Simply set the given function equal to 5/6, i.e. solve $x - \frac{1}{x} = \frac{5}{6}$.

Thus, $6x^2 - 5x - 6 = (3x + 2)(2x - 3) = 0 \rightarrow x = \underline{\boldsymbol{\frac{3}{2}}}$

C) $x = 2 \rightarrow 4a + 2b + 3 = 3 \rightarrow 2a + b = 0$

$x = -1 \rightarrow a - b + 3 = 9 \rightarrow a - b = 6$ Thus, $(a, b) = (2, -4)$

$f(x) = 2x^2 - 4x + 3$ and $f(4) = 2(16) - 4(4) + 3 = 32 - 16 + 3 = 19$

$2c^2 - 4c - 16 = = \rightarrow c^2 - 2c - 8 = (c + 2)(c - 4) = 0 \rightarrow c =$ **<u>–2</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009 SOLUTION KEY

Round 2

A) $2 \cdot \left[2,3,5,7,11,13,17,19,23,\cancel{29}\right] \rightarrow 9$

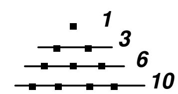
$3 \cdot \left[3,5,7,11,13,\cancel{17}\right] \rightarrow 5$

$5 \cdot \left[5,7,\cancel{11}\right] \rightarrow 2$

$7 \cdot \left[7,\cancel{11}\right] \rightarrow 1$

$11 \cdot \left[\cancel{11}\right] \rightarrow 0$

Thus, the number of semi-primes < 50 is **<u>17</u>**.



B) Note the numbers along the right edge, 1, 3, 6, 10, … . These are <u>triangular</u> numbers.
They numbers are generated by the formula $\dfrac{n(n+1)}{2}$. $n = 1 \rightarrow 1$, $n = 2 \rightarrow 3$, $n = 3 \rightarrow 6$ etc.

Thus, the last number in the 10^{th} row is $\dfrac{10(11)}{2} = 55$ and we must sum

$46, 47, 48, \ldots, 53, 54, 55 = 5(101) =$ **<u>505</u>**

101
101
101

B) <u>Alternate solution #1</u>
Note the numbers along the left edge. 1, 2, 4, 7, … = 1, 1 + 1, 1 + (1 + 2), 1 + (1 + 2 + 3), …
These numbers are generated by $1+\dfrac{(n-1)n}{2} = \dfrac{n^2-n+2}{2}$ Thus, the first number in the 10^{th} row is $\dfrac{10^2-10+2}{2} = 46$ and the solution follows as above.

<u>Alternate solution #2</u>
Note the number(s) in the "center" of each row. If there is a single number only, double it.
2(1), 2 + 3, 2(5), 8 + 9, … = 2, 5, 10, 17, … = 1 + 1, 1 + 4, 1 + 9, 1 + 16, …
These numbers are generated by $1 + n^2$. Thus, the "center" number is $\dfrac{1+n^2}{2}$ and, hence, the sum of the 10 numbers in the 10^{th} row is $10\left(\dfrac{10^2+1}{2}\right) = 5(101) =$ **<u>505</u>**.

Round 2 - continued

C) $13x + 11y = 316 \rightarrow y = \dfrac{316-13x}{11} = 28 - x + \dfrac{8-2x}{11} = 28 - x + 2\left(\dfrac{4-x}{11}\right)$

Thus, the smallest positive integer x to try is 4.
$x = 4 \rightarrow y = 24 + 2(0) = 24 \rightarrow$ partition $316 = 52 + 264 \rightarrow$ **(52, 264)**
$x = 15 \rightarrow y = 13 + 2(-1) = 11 \rightarrow$ partition $316 = 195 + 121 \rightarrow$ **(195, 121)**
Since the slope of the given linear equation is $-13/11$, we note that when x is increased by 11, y is decreased by 13. Therefore, x and y will no longer both be positive integers and the search stops.

Alternate solution (using congruence notation):
If you familiar with numerical congruence, read on!
If not, ask a teammate or teacher before continuing.

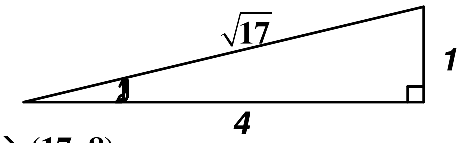
$11x + 13y = 316 \rightarrow 11x \equiv 316 \pmod{13}$ [$\equiv$ is the congruence operator]
Removing the largest multiple of 13, we have $11x \equiv 4 \pmod{13}$
The solution to this congruence is $x \equiv 11 \pmod{13}$, since $11(11) = 121 = 13(9) + 4$ and we see that 121 leaves a remainder of 4 when divided by 13. You may wish to verify that over the integer interval [0, 12], $x = 11$ is the only solution to $11x \equiv 4 \pmod{13}$.

$x = 11$ generates the partition $121 + 195 = 316 \rightarrow$ **(195, 121)**
$x = 11 + 13 = 24$ generates the partition $264 + 52 = 316 \rightarrow$ **(52, 264)**

In each ordered pair, the multiple of 13 must be listed first.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009 SOLUTION KEY

Round 3



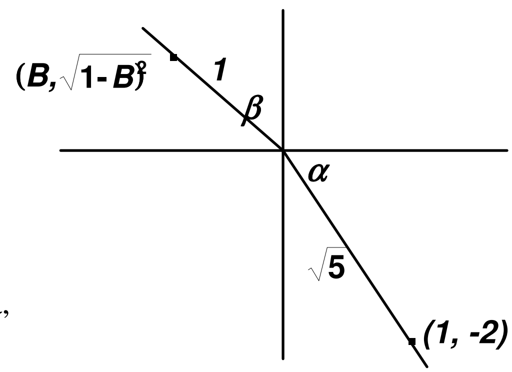
A) $\csc(2Arc\cot 4) = \csc\left(2Arc\tan\frac{1}{4}\right) =$

$$\frac{1}{2\sin\left(Arc\tan\frac{1}{4}\right)\cdot\cos\left(Arc\tan\frac{1}{4}\right)} = \frac{1}{2\left(\frac{1}{\sqrt{17}}\right)\left(\frac{4}{\sqrt{17}}\right)} = \frac{17}{8} \rightarrow \underline{\textbf{(17, 8)}}$$

(Even though there are no $\boxed{\csc}$ or $\boxed{Arc\cot}$ buttons on a scientific calculator, with a little thought, the calculator gives 2.125 = 17/8 and the same result follows)

B) $\dfrac{\sin 36°\sin 78° + \cos 36°\sin 12°}{\cos 72°\sin 66° + \sin 72°\sin 24°} = \dfrac{\sin 36°\cos 12° + \cos 36°\sin 12°}{\cos 72°\cos 24° + \sin 72°\sin 24°}$

Since the numerator and denominator are expansions of $\sin(A + B)$ and $\cos(A - B)$ respectively , we have $\dfrac{\sin(36° + 12°)}{\cos(72° - 24°)} = \dfrac{\sin 48°}{\cos 48°} = \tan 48°$ Over the specified range, we have two answers, namely x = **<u>48 , 228</u>** (or **<u> 48° , 228°</u>**).



C) Let $\alpha = Arc\sin\left(-\frac{2}{\sqrt{5}}\right)$ and $\beta = Arc\cos(B)$, where $B < 0$.

Because of principle values, α must be in quadrant 4 and β must be in quadrant 2, as diagrammed to the right.

If $x = 135$, $\tan\left(Arc\sin\left(-\frac{2}{\sqrt{5}}\right) - Arc\cos B\right) = \cot(180° + x)$

simplifies to $\tan(\alpha - \beta) = -1$. Applying the expansion formula,

$\dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = -1$. Substituting, $\dfrac{-2 - \left(\frac{\sqrt{1-B^2}}{B}\right)}{1 + (-2)\left(\frac{\sqrt{1-B^2}}{B}\right)} = \dfrac{1}{-1}$

Cross multiplying, $1 - 2\left(\frac{\sqrt{1-B^2}}{B}\right) = 2 + \left(\frac{\sqrt{1-B^2}}{B}\right) \rightarrow B - 2\sqrt{1-B^2} = 2B + \sqrt{1-B^2}$

$\rightarrow 3\sqrt{1-B^2} = -B$ (Note: B is negative, so we don't have a contradiction, $-B$ is <u>positive</u>!)

$\rightarrow 9\left(1-B^2\right) = B^2 \rightarrow 10B^2 = 9 \rightarrow B = \pm\dfrac{3}{\sqrt{10}}$

Since $B < 0$, we have, in rationalized form, $B = \dfrac{-3\sqrt{10}}{10} \rightarrow (P, Q) = \underline{\textbf{(–3, 10)}}$

Note: (3, –10) is unacceptable since it would correspond to $\dfrac{3\sqrt{-10}}{-10}$.

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 4

A) If T (hours) denotes the time both machines are working together, then $\frac{T+\frac{1}{4}}{2}+\frac{T}{3}=1$.

Multiplying by 24, $12T + 3 + 8T = 24 \rightarrow T = 21/20$ hours = **<u>63</u>** minutes

B) $2008(x - 2008) = 2009(x - 2009)$

$2008x - 2008^2 = 2009x - 2009^2$

$x = 2009^2 - 2008^2 = (2009 + 2008)(2009 - 2008) =$ **<u>4017</u>**

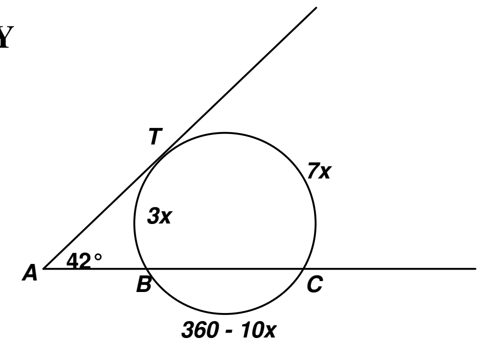
C) $\begin{cases} N = 100h + 10t + u \\ t = u/2 \rightarrow u = 2t \\ h = u + t \end{cases}$

Thus, $h = 3t$ and $t \le 4 \rightarrow N = 300t + 10t + 2t = 312t = 2^3 \cdot 3 \cdot 13t$

So, clearly, N is divisible by 3, 4, 6 and 13 $\rightarrow d =$ **<u>7</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009 SOLUTION KEY

Round 5



A) $42 = \frac{1}{2}(7x - 3x) \rightarrow x = 21 \rightarrow \overset{\frown}{BC} = 360 - 10(21) = \underline{\mathbf{150°}}$

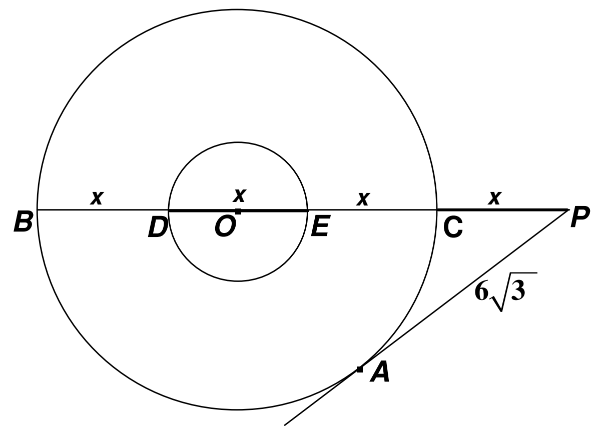
B) Let $(x, y) = (CE, DE)$. Thus, $CD = x + y$.

Applying the product chord theorem. $2(10 - 2) = 36 = xy$
As integer factors of 36, the possible ordered pairs are:
$(x, y) = (1, 36), (2, 18), (3, 12), (4, 9)$ and $(6, 6)$ - or vice versa.
Since $CD < 20$, the last three ordered pairs give us
$CD =$ **<u>15, 13 or 12</u>**.

C) Let $BD = DE = EC = CP = x$. Using the tangent-secant relationship $AP^2 = CP \cdot BP$. Substituting,
$108 = x(4x) \rightarrow x^2 = 27 \rightarrow x = 3\sqrt{3}$

Thus, the required area is $\pi\left(\frac{9\sqrt{3}}{2}\right)^2 - \pi\left(\frac{3\sqrt{3}}{2}\right)^2$

$= \pi\left(\frac{81 \cdot 3 - 9 \cdot 3}{4}\right) = \pi\left(\frac{3 \cdot 72}{4}\right) = \underline{\mathbf{54\pi}}$



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009 SOLUTION KEY

Round 6

A) Clearly, the gaps between succesive terms in every row and in every column are increasing by 2.
In the 2^{nd} row, the numbers are: 4, 10, 18, 28, 40, 54, 70, 88
In the 3^{rd} row, the numbers are: 8, 16, 26, 38, 52, 68, 86, <u>106</u>
In the 1^{st} column, the numbers are: 2, 4, 8, 14, 22, 32, 44, 58
In the 2^{nd} column, the numbers are: 6, 10, 16, 24, 34, 46, 60, 76
In the 8^{th} row, the numbers are: 58, 76, <u>96</u> Thus, the difference is **<u>10</u>**.

B) $\frac{13x+6}{5x}=\frac{5x}{x+2}$ → $3x^2 - 8x + 3 = (3x + 1)(x - 3) = 0$ → $x = 3, -1/3$

$x = -1/3$, $r = -1$ → terms alternate positive, then negative; hence the 4th term is $\underline{-\mathbf{\frac{5}{3}}}$

$x = 3$ → $r = 3$ → 5, 15, 45, **<u>135</u>**

C) The recursive rule states that the next term is twice the last known term minus three times the term before that. Thus, two terms must be known before the rule may be applied.
4, 5 → $a_3 = 2(5) - 3(4) = 10 - 12 = -2$
→ $a_4 = 2(-2) - 3(5) = -4 - 15 = -19$
→ $a_5 = 2(-19) - 3(-2) = -38 + 6 = -32$
→ $a_6 = 2(-32) - 3(-19) = -64 + 57 = -7$
→ $a_7 = 2(-7) - 3(-32) = -14 + 96 = +82$ → (k, a_k) = **<u>(7, 82)</u>**

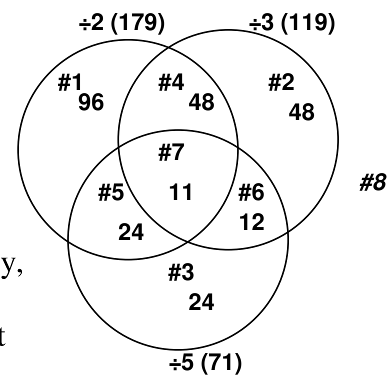
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009 SOLUTION KEY

Team Round

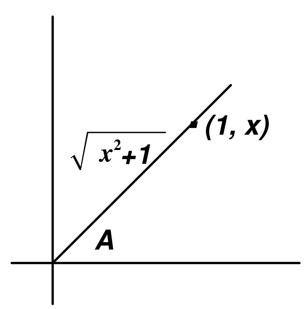
A) $y = 1 + \dfrac{1}{a^t}$, but $x = 1 - a^t \rightarrow a^t = 1 - x$

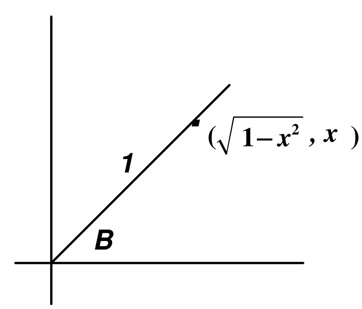
Thus, $y = 1 + \dfrac{1}{1-x} = \dfrac{1-x+1}{1-x} = \dfrac{2-x}{1-x} \rightarrow \dfrac{2-\frac{2}{3}}{1-\frac{2}{3}} = \dfrac{6-2}{3-2} = \underline{\mathbf{4}}$

B) Since $360 = 2^3 \cdot 3^2 \cdot 5^1$, the fraction can be reduced whenever A is a multiple of 2, 3 or 5.
Since multiples of 2, 3 and 5 overlap, counting each integer from 1 through 359 inclusive exactly once is made easier with the use of a Venn Diagram.
The upper left circle contains multiples of 2, the upper right multiples of 3 and the lower circle multiples of 5.
Section #7 contains integers divisible by 2, 3 <u>and</u> 5 (i.e. 30)
Sections (#4, #7), (#5, #7) and (#6, #7) contains integers divisible by 6, 10 and 15 respectively.
Section #4 contains integers divisible by 6, but <u>not</u> 5.
Section #5 contains integers divisible by 10, but <u>not</u> 3.
Section #6 contains integers divisible by 15, but <u>not</u> 2.
Sections #1, #2 and #3 respectively contain integers divisible by 2 only, 3 only and 5 only.
Totaling the number of integers in sections 1 – 7, we have <u>**263**</u> distinct integer values of A for which $\dfrac{A}{360}$ can be reduced, since numbers in each of these regions has at least one factor of 2, 3 or 5. (In region #8, outside all three circles, there are $359 - 263 = 96$ numbers, so $\dfrac{A}{360}$ is already simplified for 96 values of A.)



C) Let $A = \text{Tan}^{-1}(x)$ and $B = \text{Sin}^{-1}(x)$
$x > 0 \rightarrow$ both A and B are in quadrant 1.
Taking the cosine of both sides,
$\cos(A + B) = \cos(\pi/2)$
$\cos A \cos B - \sin A \sin B = 0$





$$\left(\frac{1}{\sqrt{x^2+1}}\right)\left(\sqrt{1-x^2}\right) - \left(\frac{x}{\sqrt{x^2+1}}\right)x = 0$$

$$\rightarrow \frac{\sqrt{1-x^2} - x^2}{\sqrt{x^2+1}} = 0$$

This is only possible if the numerator is equal to zero.

$\sqrt{1-x^2} - x^2 = 0 \rightarrow 1 - x^2 = x^4 \rightarrow x^4 + x^2 - 1 = 0$

Applying the quadratic formula and rejecting the negative result, we have $x^2 = \underline{\mathbf{\dfrac{\sqrt{5}-1}{2}}}$

MASSACHUSETTS MATHEMATICS LEAGUE
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Team Round

D) $\frac{1}{A}+\frac{1}{B}=k\left(\frac{1}{A+B}\right) \rightarrow \frac{A+B}{AB}=\frac{k}{A+B} \rightarrow (A+B)^2=kAB \rightarrow A^2+(2-k)AB+B^2=0$

Dividing through by B^2 ($\neq 0$), we have the quadratic $\left(\frac{A}{B}\right)^2+(2-k)\left(\frac{A}{B}\right)+1=0$

Appling the quadratic formula, $\frac{A}{B}=\frac{(k-2)\pm\sqrt{(2-k)^2-4}}{2}=\frac{(k-2)\pm\sqrt{k(k-4)}}{2}$

Clearly, for $k = 0$ and $k = 4$ the radical expression drops out, leaving $R = \frac{A}{B} = -1$ and $+1$ respectively and (k, R) = **<u>(4, 1)</u>** and **<u>(0, –1)</u>**. These are the only possibilities.

To prove this, the reasoning might go something like this:
For all other integer values of k, $k(k-4)$ is <u>not</u> a perfect square and $\sqrt{k(k-4)}$ is <u>irrational</u>.
Suppose $k(k-4) = m^2$, for some integer m. Then $k^2 - m^2 = 4k \rightarrow (k + m)(k - m) = 4k$ (****)
The factors on the left side are the sum and difference of the same two numbers.
Regardless of the values of k and m, the sum and difference will both be even or both be odd.
Since the right hand side of the equation must be even, the sum and difference must both be even. Let $k + m = 2p$ and $k - m = 2q$, for some integers p and q, i.e. the sum and difference are both even.

Solving for k and m in terms of p and q, $\begin{cases} k = p+q \\ m = p-q \end{cases}$

Substituting in (****), $(2p)(2q) = 4(p + q) \rightarrow pq = p + q$
This is satisfied by $(p, q) = (0, 0)$ and $(2, 2)$. Are there any others?

$pq = p + q \rightarrow q = 1+\frac{q}{p}$

If $p = 1$, then $q = 1 + q$ (a contradiction.)

If $p = 3$, then $\frac{2}{3}q = 1 \rightarrow q = \frac{3}{2}$ (a non-integer)

If $p = 4, 5, 6, \ldots, r$ then $q = \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots, \frac{r}{r-1}$ (all non-integers)

If $p = -1$, $2q = 1 \rightarrow q = \frac{1}{2}$ (a non-integer)

If $p = -2, -3, -4, \ldots, -r$, then $q = \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{r}{r+1}$ (all non-integers)

Thus, (0, 0) and (2, 2) are the only two possible ordered pairs and $k = p + q = 0, 4$, producing the ordered pairs (k, R) listed above.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2009 SOLUTION KEY

Team Round

E) $CE = ED = DF = 6$, $BE = 6\sqrt{5}$, $EF = 6\sqrt{2}$

Altitude $BG = \sqrt{BE^2 - EG^2} = \sqrt{180-18} = \sqrt{162} = 9\sqrt{2}$

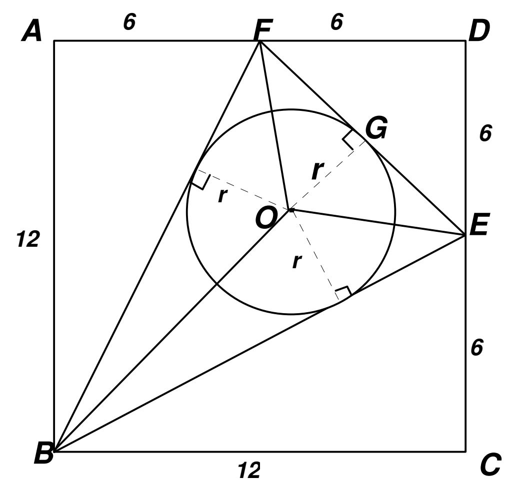
Area(ΔBEF) = $\frac{1}{2}\cdot 9\sqrt{2}\cdot 6\sqrt{2} = 54$

But Area(ΔBEF) = $\frac{1}{2}\cdot 6\sqrt{5}\cdot r + \frac{1}{2}\cdot 6\sqrt{5}\cdot r + \frac{1}{2}\cdot 6\sqrt{2}\cdot r$

$= 3r(2\sqrt{5}+\sqrt{2})$

Equating and solving for r,

$$r = \frac{18}{(2\sqrt{5}+\sqrt{2})} = \frac{18(2\sqrt{5}-\sqrt{2})}{20-2} = \underline{\mathbf{2\sqrt{5}-\sqrt{2}}}$$



F) Cesaro sum of $(a_1, a_2, a_3, \ldots, a_t)$ is 180 $\rightarrow S_1 + S_2 + S_3 + \ldots + S_t = 180t$

The Cesaro sum of $(1, a_1, a_2, a_3, \ldots, a_t)$ is $\frac{1+(1+S_1)+(1+S_2)+\ldots+(1+S_t)}{1+t}$

$= \frac{(1+t)+S_1+S_2+\ldots+S_t}{1+t} = \frac{(1+t)+180t}{1+t} = 1+\frac{180t}{1+t}$

For integers $t > 1$, $1 + t$ is never a factor of t. Thus, if $\frac{180t}{1+t}$ is to be an integer, $(1 + t)$ must be a factor of 180. Since $180 = 2^2\cdot 3^2\cdot 5^1$, 180 has $(2 + 1)(2 + 1)(1 + 1) = 18$ factors.
The only factors that must be excluded are 1 and 2 and, therefore, there are 16 values of t for which the required Cesaro sum will be an integer.