MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2009
ROUND 1 ALG 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Solve for x. $\begin{vmatrix} 1 & -3 & 5 \\ 0 & 2 & 1 \\ x & 2 & 8 \end{vmatrix} = 38 - 5x$

B) For what value(s) of a does the system $\begin{cases} (1) & x + 2y - 3z = 5 \\ (2) & 2x + y + 2z = 7 \\ (3) & 7x + 8y - 5z = a \end{cases}$ have an infinite number of solutions?

C) It has been stated that we count in base ten because we have ten fingers.
A race of humanoids (called Cylons) have only one hand with many fingers.
Suppose a Cylon solves the quadratic equation $x^2 + 21x + 99 = 0$ and claims that the difference between its roots is 12. How many fingers does a Cylon have?
It is understood that all numerical values have been expressed in the Cylon base b, which is equal to the total number of fingers that a Cylon has.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2009
ROUND 2 ALG1: EXPONENTS AND RADICALS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Solve for x. $8^{x-1} = 2 \cdot 4^{x+1}$

B) Find all possible values of x for which $3x^{5/3} + 4x^{2/3} = 15x^{-1/3}$

Note: $x^{-1/3} = \frac{1}{\sqrt[3]{x}}$

C) Solve for x: $\sqrt{48x^2} - \sqrt{\frac{16}{3}} - 12^{\frac{1}{2}} = 0$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2009
ROUND 3 ALG 2: ANYTHING

ANSWERS

A) ______________________________

B) (__________ , __________)

C) ______________________________

A) Solve for x: $x^2 + (1 + 2i)x = 1 - i$
Express the answers in $a + bi$ form.

Note: An answer of 2 would be expressed as $2 + 0i$
and an answer of $-3i$ would be expressed as $0 - 3i$.

B) If $f(x) = 3^{mx+b}$, $f(0) = \frac{1}{3}$ and $f(x+2) = 27f(x-2)$, determine the ordered pair (m, b).

C) The system $\begin{cases} y = 7 - |Ax - 2| \\ y = 0 \end{cases}$ bounds a region whose area is 2009.
If $A > 0$, compute A.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2009
ROUND 4 ALG 1: ANYTHING

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) If ¼ of 20 were 6, then what would 1/5 of 10 be, assuming the incorrect answers are proportional to the correct answers?

B) A 57 gallon acid-water solution contains 19 gallons of pure acid. Using this solution, a mixture of 20% acid was to have been produced by adding k gallons of water. However, k gallons of pure acid were added by mistake, producing a p% acid solution. Compute p.

C) Solve for x: $$1-\cfrac{1}{1-\cfrac{3}{2+\cfrac{1}{x}}}=2x$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2009
ROUND 5 PLANE GEOMETRY: ANYTHING

ANSWERS

A) ________________________________

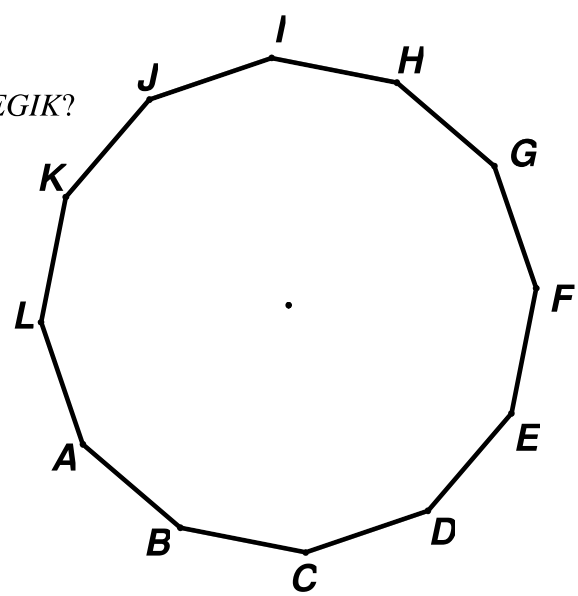
B) ________________________________

C) ________________________________

A) I am a regular polygon, have more than 100 diagonals and my interior angles have an integral measure. What is the smallest number of sides I can have?

B) In circle O, a chord of length 28 units is $3\sqrt{6}$ units from the center of the circle. $\overline{AB}$ and $\overline{CD}$ are parallel chords in circle O on the same side of a diameter. If $AB = 26$ and $CD = 30$, compute the distance between the chords.

C) A regular duo-decagon (12 sides) has area 972. Compute the area of the inscribed regular hexagon $ACEGIK$?



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2009
ROUND 6 ALG 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) ______________________________

B) (__________ , __________)

C) ______________________________

A) A committee of 5 is chosen at random from 4 juniors and 7 seniors. What is the probability that this committee will contain exactly 2 juniors and 3 seniors?
Leave your answer as a simplified fraction.

B) The first term in the expansion of $\left(\dfrac{x^3}{4}+8x^{-n}\right)^5$ is $\dfrac{x^{15}}{2^{10}}$. For a unique integer n, the second term in the expansion is a constant c. Determine the ordered pair (n, c).

C) Four members of the *Jones* family and six members of the *Smith* family participate in a Superbowl pool every year. Every person has the same probability of winning the pool. What is the exact probability that a *Jones* family member will win at least 4 of the pools during the next 6 years?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2009
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ______________________________ D) ______________________________

B) ______________________________ E) ______________________________

C) ______________________________ F) ______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Find <u>all</u> possible ordered pairs (x, y) of real numbers that satisfy

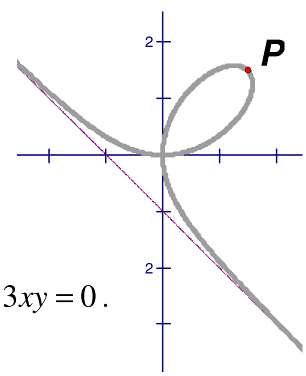
$$x^2 + xy + y^2 = 8 \quad \text{and} \quad x + y = xy + 2$$

given that x and y have the <u>same</u> sign.

.

B) Given: N, A and B are positive integers

Determine all values of N for which $\sqrt{N - 24\sqrt{2}} = A - B\sqrt{2}$.



C) The graph at the right is called the Folium of Descartes. Its equation is $x^3 + y^3 - 3xy = 0$.
It is almost indistinguishable from the line $\mathscr{L}$: $x + y + 1 = 0$
as x gets arbitrarily large or arbitrarily small. The line $\mathscr{L}$ is called an asymptote.
If P is the point on the folium farthest from $\mathscr{L}$, compute this distance from P to $\mathscr{L}$.

D) Players in consideration for the baseball Hall of Fame are (O) outfielder Jim Rice, (I) infielder Alan Trammell and (P) pitcher Bert Blyleven. Voters received these two instructions:

“You may vote for the outfielder provided you do not vote for the pitcher.”
“You may vote for the pitcher provided you do not vote for the outfielder.”

(This is equivalent to “You may vote for the outfielder if and only if you do not vote for the pitcher.”)
The results of fan voting were as follows:
500 fans voted (each for at least one these players)
78 voted the infielder and the outfielder (They may or may not have voted for the pitcher.)
80 voted for only the outfielder
320 voted for the infielder or the outfielder, but not the pitcher
90 voted for the infielder and the pitcher (They may or may not have voted for the outfielder.)
96 voted for the outfielder and the pitcher (They may or may not have voted for the infielder.)
30 voted for only the pitcher
How many fans followed the voting instructions?

E) Definition: In a <u>semi-golden rectangle</u>, the ratio of the length of a short side to the length of a long side is the same as the ratio of the length of a long side to the sum of the lengths of a short side and a diagonal.
Find the area of a semi-golden rectangle if the length of a diagonal is $10\sqrt{6}$.

F) Martha makes ¾ of her free throws. The success or failure of making a free throw is independent from one shot to the next. Martha shoots until she hits three free throws or misses two. Find the probability of making three free throws <u>before</u> missing two.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2009 ANSWERS

Round 1 Alg 2: Simultaneous Equations and Determinants

A) -3 B) 29 C) 13

Round 2 Alg 1: Exponents and Radicals

A) 6 B) $\frac{5}{3}, -3$ C) $\pm\frac{5}{6}$

Round 3 Alg2: Anything

A) $0 - i,\quad -1 - i$ B) $\left(\frac{3}{4}, -1\right)$ C) $\frac{1}{41}$

Round 4 Alg 1: Anything

A) $\frac{12}{5}$ (or 2.4) B) 60 C) $\frac{5}{2}$ (0 is extraneous)

Round 5 Plane Geometry: Anything

A) 18 B) 4 C) $486\sqrt{3}$

Round 6 Alg 2: Probability and the Binomial Theorem

A) $\frac{5}{11}$ B) $\left(12, \frac{5}{32}\right)$ or $(12, 0.15625)$ C) $\frac{112}{625}$ $(= 0.1792)$

Team Round [Non-Calculator Round]

A) $\left(\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right)$ and $\left(\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right)$ D) 206

B) 44 and 146 E) $150\sqrt{3}$

C) $2\sqrt{2}$ F) $\frac{189}{256}$ $(= 0.73828125)$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2009 SOLUTION KEY

Round 1

A) Computing the determinant on the left side of the equation:

$$\begin{array}{ccccc} 1 & -3 & 5 & 1 & -3 \\ 0 & 2 & 1 & 0 & 2 \\ x & 2 & 8 & x & 2 \end{array} = (1\cdot 2\cdot 8 - 3\cdot 1\cdot x + 5\cdot 0\cdot 2) - (x\cdot 2\cdot 5 + 2\cdot 1\cdot 1 + 8\cdot 0\cdot -3)$$

$= 16 - 3x - 10x - 2 = 14 - 13x$

Therefore, $14 - 13x = 38 - 5x \rightarrow -8x = 24 \rightarrow x =$ <u>**-3**</u>

B) If the 3rd equation is a linear combination of the 1st two equations (or any two equations equal a linear combination of the remaining equation) then there are an infinite number of solutions. (Each equation represents a plane and the planes intersect along a common line.)

Find constants p and q so that (3rd) = p(1st) + q(2nd).

Equating coefficients of x and y, $\begin{cases} 7 = p + 2q \\ 8 = 2p + q \end{cases} \rightarrow (p, q) = (3, 2)$

Note that the coefficients of z are also equal. $-3p + 2q = -5$

Equating the constant terms, $5p + 7q = a \rightarrow a =$ <u>**29**</u>

Alternate solution:

The system $\begin{cases} (1) \quad x + 2y - 3z = 5 \\ (2) \quad 2x + y + 2z = 7 \\ (3) \quad 7x + 8y - 5z = a \end{cases}$ can be easily be solved using matrix algebra.

Here's the background. This technique allows the development of algorithms for solving systems of any number of linear equations mechanically on a computer.

Let M denote the matrix of coefficients, i.e. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 2 \\ 7 & 8 & -5 \end{bmatrix}$ and $\|M\|$ the determinant of M.

(M_x, M_y and M_z) denote related matrices, where the x-, y- and z-coefficients have been replaced by the constants on the right hand side of the equations, i.e.

$\begin{bmatrix} 5 & 2 & -3 \\ 7 & 1 & 2 \\ a & 8 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 5 & -3 \\ 2 & 7 & 2 \\ 7 & a & -5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 7 \\ 7 & 8 & a \end{bmatrix}$ and $\left(\|M_x\|, \|M_y\|, \|M_z\|\right)$ the respective determinants.

The solution of the system is $\left(\dfrac{\|M_x\|}{\|M\|}, \dfrac{\|M_y\|}{\|M\|}, \dfrac{\|M_z\|}{\|M\|}\right)$, provided $M \neq 0$.

If $M = 0$ and $M_x \neq 0$ (M_y or $M_z \neq 0$), then the system has <u>no</u> solution.

If $M = 0$ and $M_x = 0$ (M_y or $M_z = 0$), then the system has an <u>infinite</u> number of solutions.

Round 1 - continued

B) continued

If you are unfamiliar with how to take the determinant of a 3 x 3 matrix, ask a teammate or your coach to explain the technique using the space below:

Computing the determinant of M: $\begin{array}{|ccccc|} 1 & 2 & -3 & \mathbf{1} & \mathbf{2} \\ 2 & 1 & 2 & \mathbf{2} & \mathbf{1} \\ 7 & 8 & -5 & \mathbf{7} & \mathbf{8} \end{array}$

$M = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 2 \\ 7 & 8 & -5 \end{bmatrix} \rightarrow \|M\| = (-5 + 28 - 48) - (-21 + 16 - 20) = -25 + 25 = 0$

Thus, if $\|M_z\| = 0$, the system has an infinite number of solutions.

$M_z = \begin{bmatrix} 1 & 2 & \mathbf{5} \\ 2 & 1 & \mathbf{7} \\ 7 & 8 & \boldsymbol{a} \end{bmatrix} \rightarrow \|M_z\| = (a + 98 + 80) - (35 + 56 + 4a) = 0 \rightarrow 3a = 87 \rightarrow a = \underline{\mathbf{29}}$

C) Let b (base 10) denote the Cylon base. Then

$$\begin{cases} 1)\ r_1 + r_2 = -2b - 1 \\ 2)\ r_1 r_2 = 9b + 9 \\ 3)\ r_1 - r_2 = b + 2 \end{cases}$$

Adding 1) and 3) and dividing by 2, $r_1 = \dfrac{1-b}{2}$. Substituting, $r_2 = -\dfrac{3b+3}{2}$

Substituting in 2), $\left(\dfrac{1-b}{2}\right)\left(-\dfrac{3b+3}{2}\right) = 9b + 9$

$\rightarrow (b-1)(3b+3) = 4(9b+9) \rightarrow (b-1)(b+1) = 12(b+1)$

Canceling ($b > 0 \rightarrow b + 1 \neq 0$), $b - 1 = 12 \rightarrow b = \underline{\mathbf{13}}$

Alternate solution

Since the digit 9 is used in the equation, b must be greater than 9.

Consider the equation factored as $(x + r)(x + s) = x^2 + 21x + 99$, for integer roots r and s.

$r + s = 21_b = 2b + 1$ (always odd) and $r - s = 12_b = b + 2$ (even or odd, depending on b)

Since the sum and the difference of the integer roots must have the same parity, b must be odd.

So try $b = 11$. Adding, $2r = 33_b = 3b + 3 = 36$ so $r = 18$. Substituting, $18 + s = 21_b = 23 \rightarrow s = 5$.

But $rs = 18(5) = 90$ and $rs = 99_b = 9(11) + 9 = 108$ Oops!

Next try $b = 13$ and, as indicated below, it works.

Adding, $2r = 33_b = 42 \rightarrow r = 21$ Substituting, $21 + s = 3(13)+1 = 27 \rightarrow s = 6$

$rs = 21(6) = 126$ and $rs = 99_b = 9(13) + 9 = 126$ Bingo!

How do you argue that there are no other solutions larger than 13?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2009 SOLUTION KEY

Round 2

A) $\left(2^3\right)^{x-1} = 2^{3x-3} = 2\cdot\left(2^2\right)^{x+1} = 2^{2x+2+1}$ → $= 2^{3x-3} = 2^{2x+3}$ → $3x - 3 = 2x + 3$ → $x =$ **<u>6</u>**

B) $3x^{5/3}+4x^{2/3}=15x^{-1/3}$ → $3x^{5/3}+4x^{2/3}-15x^{-1/3} = x^{-1/3}\left(3x^2+4x-15\right)=0$

→ $x^{-1/3}(3x-5)(x+3)=0$ Since $x^{-1/3}$ never equals 0, the only solutions are $\underline{\boldsymbol{\frac{5}{3},-3}}$

C) $\sqrt{48x^2}-\sqrt{\frac{16}{3}}-12^{\frac{1}{2}}=0$ → $4\sqrt{3}\,|x|-\frac{4\sqrt{3}}{3}-2\sqrt{3}=0$

Dividing by $2\sqrt{3}$, $2|x|-\frac{2}{3}-1=0$ → $|x|=\frac{5}{6}$ → $x =$ $\underline{\boldsymbol{\pm\frac{5}{6}}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2009 SOLUTION KEY

Round 3

A) Applying the quadratic formula to $x^2+(1+2i)x+(i-1)=0 \rightarrow x=\dfrac{-1-2i\pm\sqrt{(1+2i)^2-4(i-1)}}{2}$

$=\dfrac{-1-2i\pm\sqrt{(1+4i-4-4i+4)}}{2}=\dfrac{-1-2i\pm 1}{2} \rightarrow$ $\underline{\mathbf{0-}\boldsymbol{i}}$, $\underline{\mathbf{-1-}\boldsymbol{i}}$

An alternate solution notes that $x^2+(1+2i)x+(i-1)=0 \Leftrightarrow (x+i)\big(x+(1+i)\big)=0$ and the solutions follow immediately.

B) $f(x)=3^{mx+b}$ and $f(x+2)=27f(x-2) \rightarrow m(x+2)+b=m(x-2)+b+3$

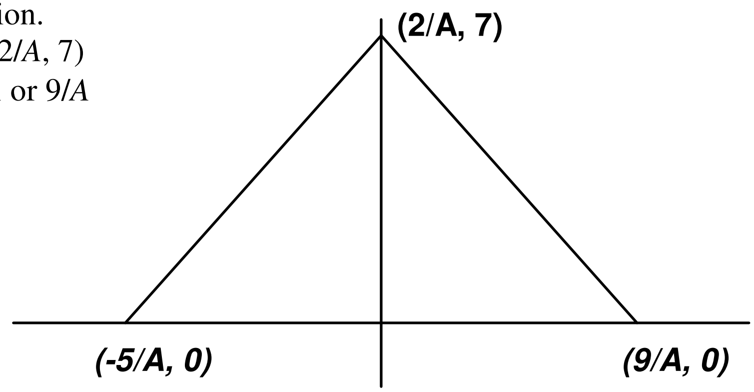
$\rightarrow 2m=-2m+3 \rightarrow m=3/4$

$f(0)=1/3 \rightarrow 3^{m\cdot 0+b}=\dfrac{1}{3}=3^{-1} \rightarrow b=-1$ Thus, $(m,b)=\underline{\left(\boldsymbol{\frac{3}{4},-1}\right)}$

C) The diagram at the right shows region in question.
Maximum occurs when $Ax-2=0 \rightarrow$ Max at $(2/A, 7)$
$y=0 \rightarrow |Ax-2|=7 \rightarrow Ax=2\pm 7 \rightarrow x=-5/A$ or $9/A$

The area is $\dfrac{1}{2}\left(\dfrac{14}{A}\right)(7)=2009$

$\rightarrow \dfrac{49}{A}=2009 \rightarrow A=\dfrac{49}{2009}=\dfrac{49}{7^2\cdot 41}=\underline{\boldsymbol{\frac{1}{41}}}$



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2009 SOLUTION KEY

Round 4

A) Let x denote the correction factor. Then: $5x = 6$ or $x = 6/5$

Since 1/5 of 10 should be 2, applying the correction factor, we have $2x = \underline{\mathbf{\frac{12}{5}}}$

B) $\frac{19}{57+k} = \frac{1}{5} \rightarrow 95 = 57 + k \rightarrow k = 38$ Thus, the new solution will contain (19 + 38) gallons of acid and the solution is $\frac{19+38}{57+38} = \frac{57}{95} = \frac{3}{5}$ acid, i.e. $\underline{\mathbf{60}}$%

C) $1 - \cfrac{1}{1 - \cfrac{3}{2 + \cfrac{1}{x}}} = 2x \rightarrow x \neq 0$ (to avoid division by zero)

$$1 - \cfrac{1}{1 - \cfrac{3}{\cfrac{2x+1}{x}}} = 1 - \cfrac{1}{1 - \cfrac{3x}{2x+1}} = 1 - \cfrac{1}{\cfrac{2x+1-3x}{2x+1}} = 1 - \frac{2x+1}{1-x} = 2x \rightarrow \frac{1-2x}{1} = \frac{2x+1}{1-x}$$

Cross multiplying, $2x + 1 = 1 - 3x + 2x^2 \rightarrow 2x^2 - 5x = x(2x-5) = 0 \rightarrow x = \underline{\mathbf{\frac{5}{2}}}$

Alternate solution:
Replace $1/x$ with x (and vice-versa).

Then $1 - \cfrac{1}{1 - \cfrac{3}{2+x}} = \frac{2}{x} \rightarrow 1 - \frac{x+2}{x-1} = \frac{2}{x} \rightarrow \frac{-3}{x-1} = \frac{2}{x} \rightarrow -3x = 2x - 2 \rightarrow x = 2/5$

But x was replaced by $1/x$ (and vise versa)! Letting the real x stand up, $x = 5/2$
The only possible extraneous roots would have resulted from attempts to divide by zero and this would have occurred only if x came out to be 0, –2 or +1.
Therefore, our answer will check and the actual substitution below is unnecessary!

$$1 - \cfrac{1}{1 - \cfrac{3}{2 + \cfrac{2}{5}}} = 1 - \cfrac{1}{1 - \cfrac{3}{\cfrac{12}{5}}} = 1 - \cfrac{1}{1 - \cfrac{5}{4}} = 1 - (-4) = \boxed{5}$$

$2(2.5) = \boxed{5}$

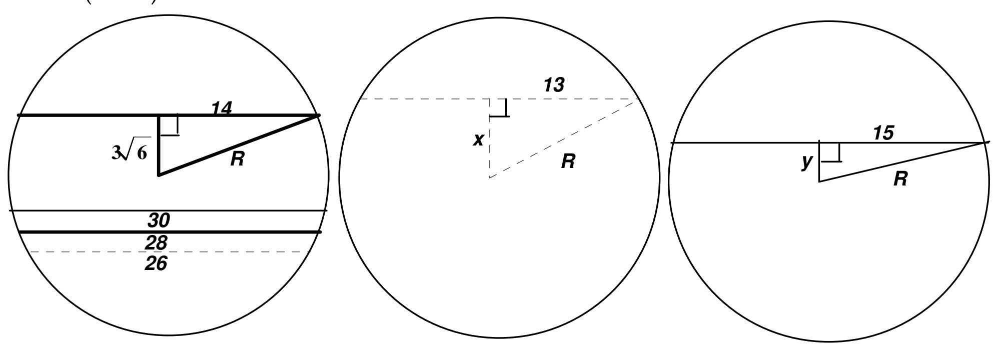
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2009 SOLUTION KEY

Round 5

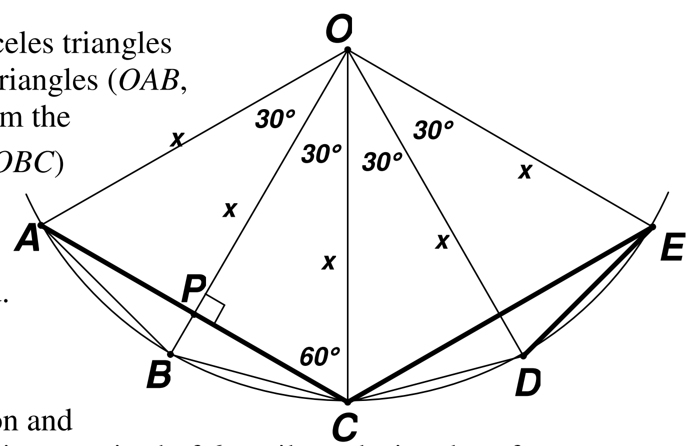
A) The number of diagonals is given by $d = \dfrac{n(n-3)}{2}$.
For $n = 15$, $d = 90$, but for $n = 16$, $d = 104$. Thus, the polygon must have at least 16 sides.
The measure of each angle is given by $A = \dfrac{180(n-2)}{n}$.
For $n = 16$ and 17, A is not an integer, but for $n = 18$, $A = 160°$.
Thus, the minimum number of sides is **18**.

B) Since the distance to a chord is measured along a radius drawn perpendicular to the chord and any perpendicular radius bisects the chord to which it is drawn, the radius of circle O may be determined from $14^2 + \left(3\sqrt{6}\right)^2 = 196 + 54 = 250 = R^2$. Then:

$x^2 + 13^2 = 250$
$\rightarrow x = 9$
$y^2 + 15^2 = 250$
$\rightarrow y = 5$
Therefore, the distance between the chords on the same side of a diameter is $9 - 5 =$ **4**



C) $360/12 = 30° \rightarrow$ vertex angle of each of the 12 isosceles triangles comprising the 12-gon. Dealing with 30°-75°-75° triangles (OAB, OBC, etc) appears difficult, but draw an altitude from the vertex of a base angle to the opposite leg ($\overline{CP}$ in ΔOBC) and a 30°-60°-90° triangle is created, ΔOCP
$\rightarrow CP = x/2$. Now, using $\overline{OB}$ as the base and $\overline{CP}$ as the height, the area may be easily determined.
Area $= 12(\frac{1}{2} \cdot x \cdot \frac{x}{2}) = 972 \rightarrow x^2 = 324 \rightarrow x = 18$
$\overline{AC}$ and $\overline{CE}$ represent sides of the inscribed hexagon and they clearly have lengths 18. A hexagon of side 18 is comprised of 6 equilateral triangles of side 18. Thus, the area of the inscribed hexagon is $6(\frac{18^2}{4}\sqrt{3}) =$ $\mathbf{486\sqrt{3}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2009 SOLUTION KEY

Round 6

A) P(3 sen <u>and</u> 2 jr) = $\dfrac{\left({}_4C_2\right)\left({}_7C_3\right)}{{}_{11}C_5} = \dfrac{\dfrac{4!}{2!2!}\cdot\dfrac{7!}{3!4!}}{\dfrac{11!}{6!5!}} = 6\cdot\cancel{7}\cdot 5\cdot\dfrac{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}{11\cdot\cancel{10}^{2}\cdot\cancel{9}^{3}\cdot\cancel{8}\cdot\cancel{7}} = \underline{\mathbf{\dfrac{5}{11}}}$

B) The second term in the expansion is $\binom{5}{1}\left(\dfrac{x^3}{4}\right)^4\left(8x^{-n}\right)^1$. Expanding, this is

$5(4)^{-4}8^1\left(x^3\right)^4\left(x^{-n}\right) = 5\cdot 2^{-8}\cdot 2^3\cdot x^{12-n} = \dfrac{5}{32}x^0 = \dfrac{5}{32}$ provided $n = 12$ Thus, $(n, c) = \underline{\left(\mathbf{12, \dfrac{5}{32}}\right)}$

C) P(*Jones* winner) = 4/10 = 2/5
Exactly 6 *Jones* winners can occur in only one way (*JJJJJJ*).
Exactly 5 *Jones* winners could occur in 6 possible ways, each represented by a sequence of 5 *J*'s and 1 *S* (e.g. *JJJJJS*), since the *S* could occur in anyone of 6 positions in the sequence .
Exactly 4 *Jones* winners requires permuting the sequence *JJJJSS*. This can be done in
$\binom{6}{2} = \binom{6}{4} = \dfrac{6!}{2!\,4!} = \dfrac{6\cdot 5}{2} = 15$ ways.
Thus, P($\geq$ 4 wins) = $\left(\dfrac{2}{5}\right)^6 + 6\left(\dfrac{2}{5}\right)^5\left(\dfrac{3}{5}\right) + 15\left(\dfrac{2}{5}\right)^4\left(\dfrac{3}{5}\right)^2 = \dfrac{64+576+2160}{5^6} = \dfrac{2800}{5^6} = \underline{\mathbf{\dfrac{112}{625}}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2009 SOLUTION KEY

Team Round

A) $\begin{cases} x^2+xy+y^2=8 \\ x+y=xy+2 \end{cases} \rightarrow \begin{cases} (1)\ \ x^2+2xy+y^2=(x+y)^2=8+xy \\ (2)\ x+y=xy+2 \end{cases}$

Subtracting (2) from (1), $(x+y)^2-(x+y)-6=0 \rightarrow \big((x+y)-3\big)\big((x+y)+2\big)=0$

$\rightarrow$ Case 1: $(x + y) = 3$ (and $xy = 1$) or Case 2: $(x + y) = -2$ (and $xy = -4$)

Since it is given that x and y have the same sign, case 2 is eliminated.

[An aside: Case 2 generates solutions of $(-1+\sqrt{5},-1-\sqrt{5})$ and $(-1-\sqrt{5},-1+\sqrt{5})$ which satisfy the system of equations, but are rejected because the x-coordinate and y-coordinates have opposite signs.]

Substituting $y=\frac{1}{x}$ in equation (2), $x+\frac{1}{x}=3 \rightarrow x^2-3x+1=0 \rightarrow x=\frac{3\pm\sqrt{5}}{2}$ and

$$y=\frac{2}{3\pm\sqrt{5}}\cdot\frac{\mathbf{3\mp\sqrt{5}}}{\mathbf{3\mp\sqrt{5}}}=\frac{2\left(3\mp\sqrt{5}\right)}{9-5}=\frac{3\mp\sqrt{5}}{2}$$

$\rightarrow$ two ordered pairs $\underline{\left(\mathbf{\frac{3+\sqrt{5}}{2},\frac{3-\sqrt{5}}{2}}\right),\left(\mathbf{\frac{3-\sqrt{5}}{2},\frac{3+\sqrt{5}}{2}}\right)}$

Alternate Solution (using the Method of Symmetric Equations):
Note: Replacing x by y and y by x does not change the system of equations.
This technique eliminates the xy- term.

Let $x = u + v$ and $y = u - v$. Then:

$$\begin{cases} (u+v)^2+(u+v)(u-v)+(u-v)^2=8 \\ u+v-(u+v)(u-v)+u-v=2 \end{cases} \rightarrow \begin{cases} 3u^2+v^2=8 \\ -u^2+v^2+2u=2 \end{cases}$$

Subtracting, $4u^2-2u=6 \rightarrow 2u^2-u-3=(2u-3)(u+1)=0 \rightarrow u=\frac{3}{2},-1$

Substituting, $3\left(\frac{3}{2}\right)^2+v^2=8 \rightarrow v^2=8-\frac{27}{4}=\frac{5}{4}$ and $3(-1)^2+v^2=8 \rightarrow v^2=5$

$(u, v) = \left(\frac{3}{2},+\frac{\sqrt{5}}{2}\right) \rightarrow (x, y) = \left(\frac{3+\sqrt{5}}{2},\frac{3-\sqrt{5}}{2}\right)$

$(u, v) = \left(\frac{3}{2},-\frac{\sqrt{5}}{2}\right) \rightarrow (x, y) = \left(\frac{3-\sqrt{5}}{2},\frac{3+\sqrt{5}}{2}\right)$ (both are ok since x and y are each positive)

$(u,v)=\left(-1,+\sqrt{5}\right) \rightarrow (x, y) = \left(-1+\sqrt{5},-1-\sqrt{5}\right)$

$(u,v)=\left(-1,-\sqrt{5}\right) \rightarrow (x, y) = \left(-1-\sqrt{5},-1+\sqrt{5}\right)$ (both rejected since x and y have opposite signs)

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B) $N-24\sqrt{2}$ and $A-B\sqrt{2}$ must be positive.
Squaring both sides, $N-24\sqrt{2}=(A^2+2B^2)-2AB\sqrt{2}$

$$\rightarrow \begin{cases} N = A^2+2B^2 \\ AB=12 \end{cases}$$

There are only 6 possible ordered pairs:

(A, B)	$N\ (A^2 + 2B^2)$
(1, 12)	1 + 288 = 289
(2, 6)	4 + 72 = 76
(3, 4)	9 + 32 = 41
(4, 3)	16 + 18 = 34
(6, 2)	36 + 8 = **44**
(12, 1)	144 + 2 = **146**

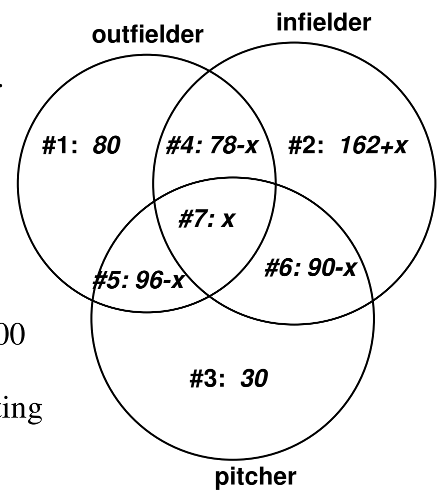
The first 4 ordered pairs are rejected since $A-B\sqrt{2}$ is negative.
For the last two ordered pairs, both radical expressions are positive. → N = **44, 146**

C) Since the equation defining the folium is unchanged when x is replaced by y (and vice versa), the graph is symmetric to the line $y = x$. Thus, point P lies on this line.

$$\begin{cases} x^3+y^3-3xy=0 \\ y=x \end{cases} \rightarrow 2x^3-3x^2=0 \rightarrow x^2(2x-3)=0 \rightarrow x = 0,\ 3/2$$

The distance from $\left(\frac{3}{2},\frac{3}{2}\right)$ to $x + y + 1 = 0$ is given by $\dfrac{\left|1\left(\frac{3}{2}\right)+1\left(\frac{3}{2}\right)+1\right|}{\sqrt{1^2+1^2}} = \dfrac{4}{\sqrt{2}} = \mathbf{2\sqrt{2}}$

D) Let x denote the number of voters who voted for all three players.
The voters who followed the voting instructions either voted for an outfielder and did not vote for a pitcher (#1 and #4) – the IF part - or voted for a pitcher and did not vote for an outfielder (#3 and #6) - the ONLY IF part. The given information can be summarized in the Venn diagram at the right.



Thus, $80 + (78 - x) + (162 + x) + (96 - x) + x + (90 - x) + 30 = 500$
→ $536 - x = 500$ → $x = 36$
Therefore, the number of voters who correctly deciphered the voting instructions: $(80 + 42) + (30 + 54) =$ **206**

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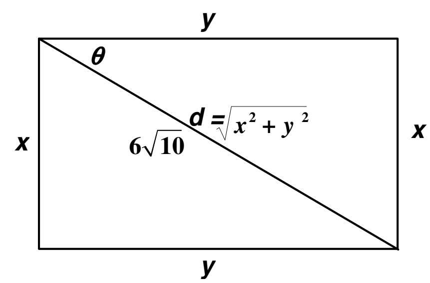
E) Let x and y denote the lengths of the short and long sides, respectively. Then:

$$\begin{cases} \dfrac{x}{y} = \dfrac{y}{x+10\sqrt{6}} \to y^2 - x^2 = 10x\sqrt{6} \\ x^2 + y^2 = \left(10\sqrt{6}\right)^2 = 600 \end{cases}$$



Subtracting, $2x^2 = 600 - 10x\sqrt{6}$

→ $2x^2 + 10x\sqrt{6} - 600 = 0$ → $x^2 + 5x\sqrt{6} - 300 = 0$

→ $x = \dfrac{-5\sqrt{6} \pm \sqrt{150+1200}}{2} = \dfrac{-5\sqrt{6} \pm \sqrt{25(54)}}{2} = \dfrac{-5\sqrt{6} \pm 15\sqrt{6}}{2} = 5\sqrt{6}$

→ $y^2 = 600 - 150 = 450 = 25(18)$ → $y = 15\sqrt{2}$ → Area = $75\sqrt{12}$ = $\underline{\mathbf{150\sqrt{3}}}$

More general solutions

#1 Let x and y denote the lengths of the short and long sides, respectively.

The definition gives the equation: $\dfrac{x}{y} = \dfrac{y}{x+\sqrt{x^2+y^2}} = \dfrac{1}{\dfrac{x}{y} + \sqrt{\left(\dfrac{x}{y}\right)^2 + 1}}$ Let $s = \dfrac{x}{y}$. Then:

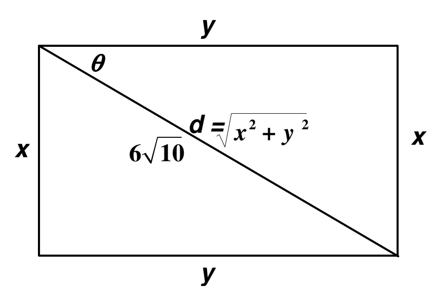
(***) $s = \dfrac{1}{s+\sqrt{s^2+1}}$ → $s^2 + s\sqrt{s^2+1} = 1$ → $s\sqrt{s^2+1} = 1 - s^2$

Squaring both sides, $s^2\left(s^2+1\right) = \left(1-s^2\right)^2$ → $s^4 + s^2 = 1 - 2s^2 + s^4$

→ $3s^2 = 1$ → s = $\dfrac{1}{\sqrt{3}} = \dfrac{x}{y}$ → $y^2 = 3x^2$.

Thus, the sides of the semi-golden rectangle form 30-60-90 right triangles. Since $d = 10\sqrt{6}$, $100(6) = x^2 + 3x^2$ → $x^2 = 150$ → $x = 5\sqrt{6}$ and $y = 15\sqrt{2}$ and, therefore, the area is $\underline{\mathbf{150\sqrt{3}}}$

#2 Let $s = \tan\theta = \dfrac{x}{y}$.



Since $1 + \tan^2\theta = \sec^2\theta$, this trig substitution greatly simplifies equation (***) above.

$\tan\theta = \dfrac{1}{\tan\theta + \sec\theta}$ → $\dfrac{\sin\theta}{\cos\theta}\left(\dfrac{\sin\theta + 1}{\cos\theta}\right) = 1$

→ $\sin^2\theta + \sin\theta = \cos^2\theta = 1 - \sin^2\theta$

→ $2\sin^2\theta - \sin\theta - 1 = (2\sin - 1)(\sin\theta + 1) = 0$ → $\sin\theta = \dfrac{1}{2}, -1$ (extraneous) → θ = 30° and the rest of the argument proceeds as above.

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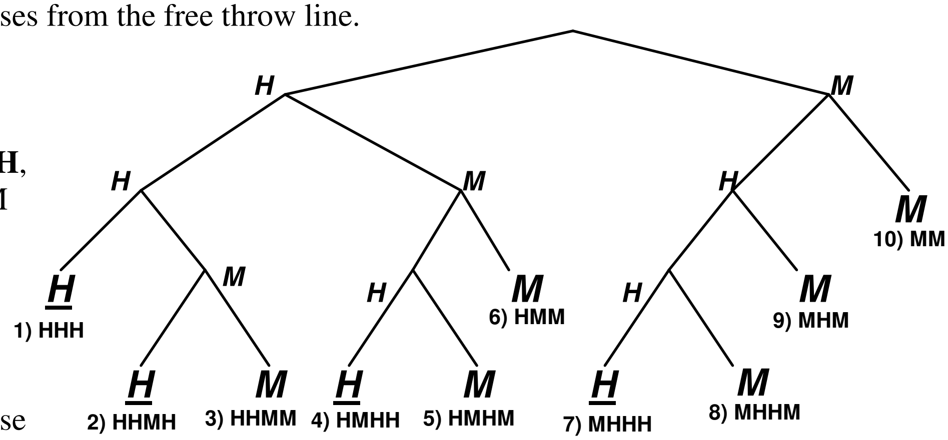
F) Let (H, M) denote Hits and Misses from the free throw line.

\# Free Throws
2 (1) MM
3 (3) **HHH**, MHM, HMM
4 (6) **HHMH**, **HMHH**, **MHHH**, HHMM, HMHM, MHHM



The 10 cases where Martha hits 3 or misses 2 are summarized in the diagram at the right.

The underlined cases denote those cases where she has hit 3 before she missed 2.
Going left, P(Hit) = ¾. Going right, P(Miss) = ¼.

Thus, success occurs in 4 out of the 10 cases and the required probability is

$$\frac{\left(\frac{3}{4}\right)^3+3\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)^3+3\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)+3\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2+2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^2}=\frac{\frac{27}{64}+\frac{81}{256}}{\frac{27}{64}+\frac{81}{256}+\frac{27}{256}+\frac{6}{64}+\frac{1}{16}}$$

$$=\frac{108+81}{108+81+27+24+16}=\underline{\mathbf{\frac{189}{256}}}$$

Note that the denominators in the first line above are each equal to 1, so simply evaluating the numerators gives us the same result.