MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 – OCTOBER 2009
ROUND 1 VOLUME & SURFACES

******* NO CALCULATORS IN THIS ROUND *******

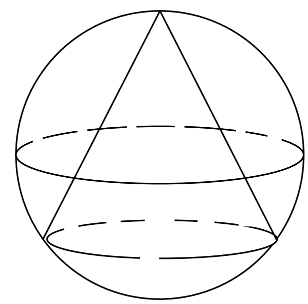
ANSWERS

A) _____________ : ______________

B) _____________________ units

C) ________________________inches^3

A) A cone is inscribed in a sphere. The radius of the base of the cone is 3 and the radius of the sphere is 5. Find the ratio of the volume of the sphere to the volume of the cone.



B) A rectangular sheet of cardboard has dimensions of $\frac{9x}{2}$ by $\frac{11x}{2}$ units. Squares x units on a side are cut from each corner of the sheet. The sheet is then folded upward to form an open box. The volume of this box is 560 units^3. What was the perimeter of the original rectangular sheet of cardboard?

C) A square pyramid has a volume of 108 cubic inches and the ratio of length of its altitude to the perimeter of its base is 3 : 8. A plane parallel to its base divides the pyramid into two solids one of which is a smaller pyramid whose slant height is $\sqrt{10}$.
Compute the volume of the smaller pyramid.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009
ROUND 2 PYTHAGOREAN RELATIONS IN RECTILINEAR FIGURES

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

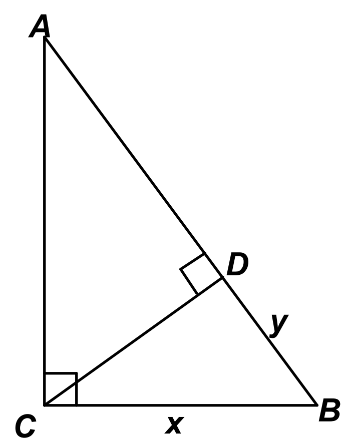
A) _____________ : ______________

B) (__________ , __________)

C) ________________________________

A) In square $ABCD$, $AB = 4$, E and F are midpoints of $\overline{BC}$ and $\overline{CD}$ respectively. Compute the ratio of the area of ΔAEF to the area of $ABCD$.

B) Given: $AD = 8$, $CD = 6$
Compute the ordered pair (x, y).



C) A right triangle has a hypotenuse of length 65. If the length of the long leg is increased by 4 and the length of the short leg is decreased by 8, the length of the hypotenuse is unchanged. What is the perimeter of the original right triangle?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009
ROUND 3 ALG 1: LINEAR EQUATIONS

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) ________________________________

B) ________________________________

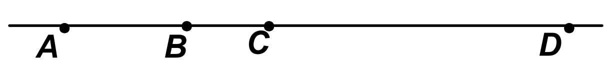
C) ________________________________

A) A number is 1 more than half of its opposite. Compute the reciprocal of this number.

B) If $0.06x + 0.15(10 - x) = \dfrac{3x}{4}$, compute $[-x]$.

Note: The greatest integer in x, denoted $[x]$ is the largest integer less than or equal to x.

C) A, B, C and D are 4 collinear points ordered on a line as indicated in the diagram below. If $AD = 203$, $\dfrac{AC}{BD} = \dfrac{4}{9}$ and $\dfrac{AB}{CD} = \dfrac{10}{27}$, compute BC.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) (______ , ______ , ______)

B) ________________________________

C) ________________________________

A) A triathlete swims 2 miles in three hours, jogs 10 miles in four hours and bicycles 40 miles in $1\frac{1}{2}$ hours. The average rate for these three events lies between two consecutive integers *A* and *B*, closer to one of these values than the other. Let *C* denote the closer value. Determine the ordered triple (*A*, *C*, *B*).

B) If x is 80% of y and y is $33\frac{1}{3}$% of w, then x^2 is k% of w^2.
Determine the value of k to the nearest tenth.

C) If $\frac{a+b}{c}=2$ and $\frac{a+c}{b}=3$, compute $\frac{b+c}{a}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) ________________________________

B) (__________ , __________)

C) ________________________________

A) The complete set of *x*-values satisfying the inequality $\left(x^2-4\right)\left(x^2-9\right)>0$

B) N is 2 more than a multiple of 3, that is, $N = 3k + 2$ for integer values of k.
It is also known that N is <u>at most</u> 96 and <u>at least</u> 16.
The values of k (and only those values) for which this is true satisfy the relation
$|k - a| \le b$, where a and b are <u>integers</u>.
Determine the ordered pair (a, b).

C) A and B are distinct two-digit positive integers with digits reversed.
A and B are both prime, with $A < B$.

Let p denote the number of ordered pairs (A, B).
Let C = the <u>minimum</u> value of $|A - B|$ and
D = the <u>maximum</u> value of $|A - B|$.

<u>How many</u> integers x are there in the range $p \cdot C < x < p \cdot D$?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009
ROUND 6 ALG 1: EVALUATIONS

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Compute $2^{-1} - \sqrt{\dfrac{25}{9} - \dfrac{64}{25}} + (3 \cdot 5)^{-1}$

B) Given: $\begin{cases} a : b = 2 : 3 \\ b : c = 4 : 5 \\ a + b + c = 70 \end{cases}$

Compute the value of c.

C) Let the binary operation $(*)$ be defined as follows:

$$a * b = \begin{cases} a + ab, & \text{when } b \text{ is a proper fraction} \\ b - ab, & \text{when } b \text{ is an improper faction} \end{cases}$$

Compute $\left(6 * \dfrac{2}{3}\right) * \left(\dfrac{3}{4} * \dfrac{3}{2}\right)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009
ROUND 7 TEAM QUESTIONS

******* CALCULATORS ARE PERMITTED IN THIS ROUND *******

ANSWERS

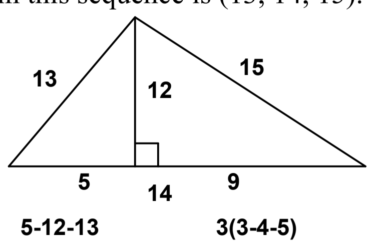
A) ___________ : ___________ D) ______________________ miles

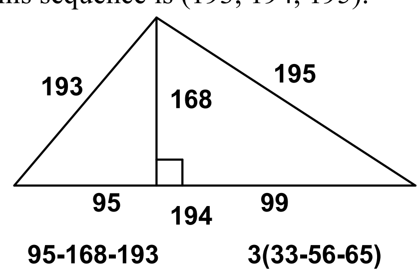
B) (______ , ______ , ______) E) ______________________ units^2

C) (___________ , ___________) F) (___________ , ___________)

A) A rectangular block of ice cream has dimensions 7″, 8″ and 9″. It is completely covered with chocolate (like a Klondike Bar). Suppose it is then cut up into one inch cubes by making cuts parallel to the faces. Let A be the number of cubes with 2 or 3 faces covered with chocolate. Let B be the number of cubes with 0 or 1 face covered with chocolate. Compute the ratio of $A : B$.

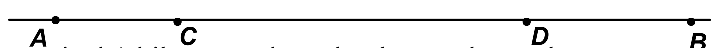
B) Several non-right triangles have all of these properties:
its sides have lengths that are consecutive integers
one of its altitudes has an integer length and
that altitude divides the triangle into two right triangles each with integer sides
Let S be a sequence of triplets (a, b, c) representing the lengths of sides of such triangles in increasing order of perimeter (where $a < b < c$).
The first term in this sequence is (13, 14, 15). The third term in this sequence is (193, 194, 195).





Determine the <u>second</u> term in this sequence.

C) Find the ordered pair (x, y) that satisfies $4x - 7y = 451$, x and y are positive integers and $x + y$ is the largest possible three-digit multiple of 3.



D) Alice and Barbara (starting at points A and B respectively) bike towards each other on the track above. Alice, biking 10 mph faster than Barbara, would meet Barbara at point D in 1 hour. If , however, Barbara increased her speed by 10 mph and Alice decreased her speed by k mph, they would meet at point C in 40 minutes. Alice's reduced speed is three-quarters of her original speed. Compute the distance (in miles) between C and D.

E) Compute the area of the region containing all points (x, y) that satisfy $\begin{cases} y \le 15 \\ y \ge (2x + 5)\left(\dfrac{|x|}{x}\right) \end{cases}$.

F) In each of the years from 1999 through 2008, 5 state quarters were issued at each of the mints in Philadelphia, Denver and San Francisco. Living on the east coast, I have found in circulation 100% of the quarters minted in Philadelphia and at least 75% of the quarters minted in Denver.
San Francisco only issues mint coins and I have found at most 12.5% of these quarters.
Let m and M denote the exact <u>m</u>inimum and <u>M</u>aximum percentages of all the quarters minted that I have found. Compute the ordered pair (m, M).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009 ANSWERS

Round 1 Geometry Volumes and Surfaces

A) 500 : 81   B) 80 units   C) 4 inches^3

Round 2 Pythagorean Relations

A) 3 : 8   B) $\left(\frac{15}{2}, \frac{9}{2}\right)$ C) 154

Round 3 Linear Equations

A) 3/2 (or 1.5)   B) −2   C) 18

Round 4 Fraction & Mixed numbers

A) (6, 6, 7)   B) 7.1   C) $\frac{7}{5}$ (or 1.4)

Round 5 Absolute value & Inequalities

A) $x < -3,\ -2 < x < 2,\ x > 3$ B) (18, 13)   C) 143

Round 6 Evaluations

A) $\frac{1}{10}$ B) 30   C) $\frac{55}{4}$ $\left(13\frac{3}{4} \text{ or } 13.75\right)$

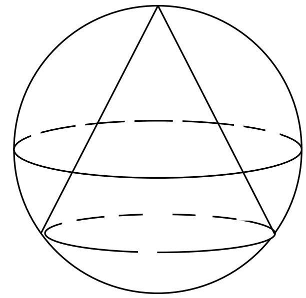
Team Round [Calculators allowed]

A) 10 : 53   D) 6 miles

B) (51, 52, 53)   E) 125

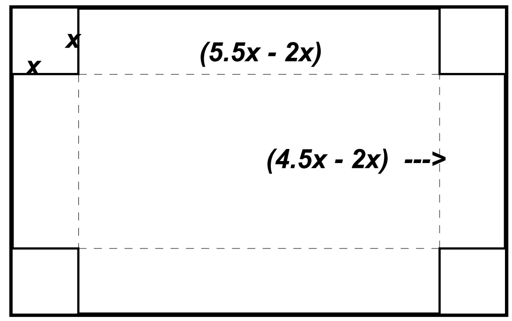
C) (671, 319)   F) $\left(58\frac{2}{3}, 70\frac{2}{3}\right)$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009 SOLUTION KEY



Round 1

A) $\dfrac{V_{sphere}}{V_{cone}} = \dfrac{\frac{4}{3}\pi(5)^3}{\frac{\pi}{3}(3)^2(4+5)} = \dfrac{4(125)}{81}$ ➔ **<u>500 : 81</u>**



B) The volume of the box is $x\left(\dfrac{9x}{2}-2x\right)\left(\dfrac{11x}{2}-2x\right)$

$= x\cdot\dfrac{5x}{2}\cdot\dfrac{7x}{2} = \dfrac{35x^3}{4} = 560 = 35(16)$ ➔ $x^3 = 64$ ➔ $x = 4$.

Thus, the dimensions of the sheet of cardboard are 18 x 22 ➔ Per = 2(18 + 22) = **<u>80</u>**

C) Let the side of the base be $2t$.

Let V_1 and v_1 denote the volumes of the original and smaller pyramids respectively.

Then the perimeter of the base is $8t$ and altitude of the pyramid is $3t$.

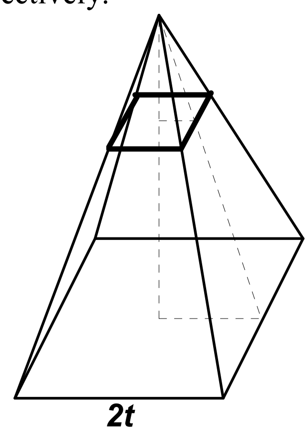
$V = \frac{1}{3}(2t)^2\cdot 3t = 108$ ➔ $4t^3 = 108$ ➔ $t = 3$

Computing the slant height (l) of the original pyramid,

$3^2 + 9^2 = 90$ ➔ $l = 3\sqrt{10}$

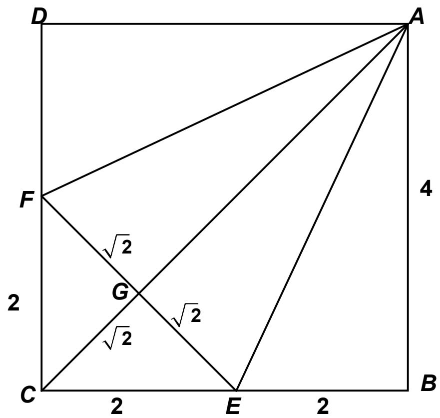
Thus, the linear dimensions of the pyramids are in 3 : 1 ratio and their volumes are is a 27 : 1 ratio.

$$\frac{V_1}{v_1} = \frac{108}{27} = \underline{\mathbf{4}}$$



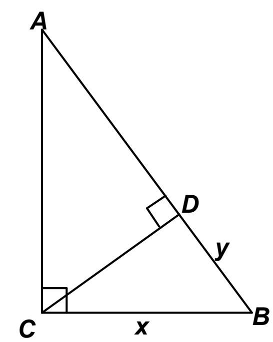
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009 SOLUTION KEY

Round 2



A) Solution #1 (arithmetic only)
Square – 3 right triangles!
The area of ΔABE is 16 – (4 + 4 + 2) = 16
Thus, the required ratio of 6 : 16 = **<u>3 : 8</u>**

Solution #2 (algebraic)
In isosceles triangle *CFE*, $CG = GF = GE = \sqrt{2}$
Since the diagonal $AC = 4\sqrt{2}$, $AG = 3\sqrt{2}$
Area of $\Delta AEF = \frac{1}{2}\left(2\sqrt{2}\right)\left(3\sqrt{2}\right) = 6$ ➔ 6 : 16 = **<u>3 : 8</u>**



B) $AC = 10$. Using the Pythagorean Theorem in ΔABC and ΔBCD,

$$\begin{cases} (1)\ x^2 + 100 = (y+8)^2 \\ (2)\ x^2 - y^2 = 36 \end{cases}$$

Expanding (1), we have $x^2 - y^2 = 16y + 64 - 100$
Substituting for $x^2 - y^2$, $16y = 72$ ➔ $y = 9/2$
Then $x^2 = 36 + \frac{81}{4} = \frac{225}{4}$

➔ $x = 15/2$ and the required ordered pair is $\underline{\left(\mathbf{\frac{15}{2}, \frac{9}{2}}\right)}$

Alternative method: Note $\Delta ADC \sim \Delta CDB$ ➔ $\frac{AD}{CD} = \frac{DC}{DB} = \frac{AC}{CB}$ ➔ $\frac{8}{6} = \frac{6}{y} = \frac{10}{x}$

➔ $(x, y) = \underline{\left(\mathbf{\frac{15}{2}, \frac{9}{2}}\right)}$

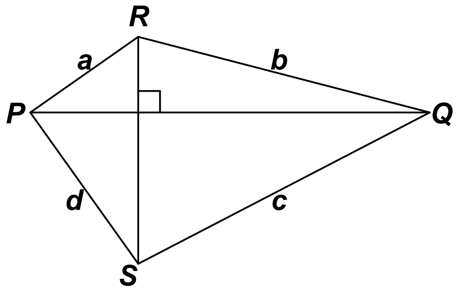
C) Let *x* and *y* denote the lengths of the original right triangle.

$$\text{Then} \begin{cases} (1)\ x^2 + y^2 = 65^2 \\ (2)\ (x+4)^2 + (y-8)^2 = 65^2 \end{cases}$$

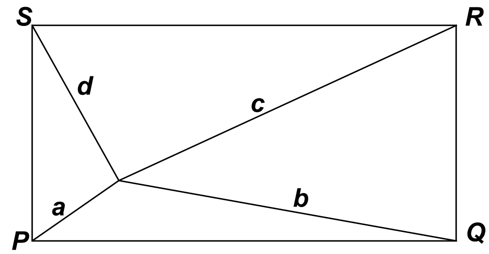
Expanding and subtracting (2) – (1), $8x + 16 - 16y + 64 = 0$ ➔ $x = 2y - 10$
Substituting in (1), $5y^2 - 40y = 65^2 - 100$ ➔ $y^2 + 8y = 13(65) - 20$
➔ $y^2 - 8y - 825 = (y - 33)(y + 25) = 0$ ➔ $y = 33$, $x = 56$ ➔ Per = 89 + 65 = **<u>154</u>**

Round 2 Pythagorean Theorem - CAVEATS
Here are the two nice relations derived from applying the Pythagorean Theorem mentioned last year. If you tried justifying them to yourself, compare the following with your results.
These two results may be used <u>in any future contests</u>.
Ignore them at your own peril!

#1

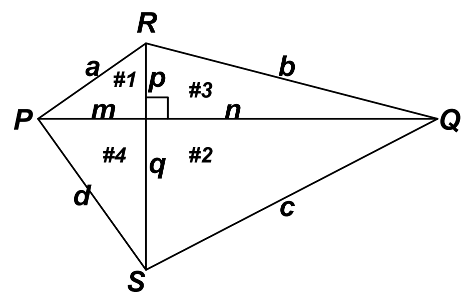


#2



#1: If $\overline{PQ} \perp \overline{RS}$, then $a^2 + c^2 = b^2 + d^2$
In **any quadrilateral with perpendicular diagonals**, the sums of the squares of the opposite sides are equal.

(Of course, this is true in any square, in any rhombus and in any kite, but it's not very useful in these cases. It is, however, very useful in those cases when neither diagonal bisects the other diagonal! Such is the case in the diagram below. It could be a trapezoid, but in general, it's just got 4 sides.)

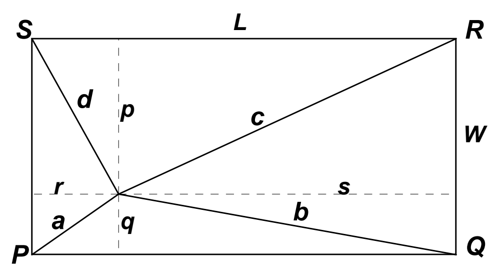


The proof:
Rt Δs 1, 2: $a^2 = m^2 + p^2,\ c^2 = q^2 + n^2$
Rt Δs 3, 4: $b^2 = p^2 + n^2,\ d^2 = m^2 + q^2$

Adding, we have the required result:
$a^2 + c^2 = m^2 + p^2 + q^2 + n^2 = b^2 + d^2$

Q.E.D. (or … That's all folks!)

#2 IF *PQRS* is a rectangle, $a^2 + c^2 = b^2 + d^2$
From any <u>interior</u> point in a rectangle, the sums of the squares of the distances to opposite vertices are equal.



The proof:
$a^2 = r^2 + q^2$, $b^2 = q^2 + s^2$, $c^2 = p^2 + s^2$ and $d^2 = r^2 + s^2$
Adding $a^2 + c^2 = r^2 + q^2 + p^2 + s^2 = b^2 + d^2$

Q.E.D. (or … That's all folks!)***

Hmmmm? – I wonder if it's true for <u>exterior</u> points, or points <u>on</u> the rectangle? How about other quadrilaterals?
Something to ponder in your spare time.

*** This abbreviation is for the Latin "quod erat demonstratum" which translates literally to "which was to be proven". It was commonly used at the end of proofs in surviving Latin translations of Euclid's 13 volume geometric masterpiece 'The Elements'. (ISBN: 978-0-7607-6312-4)

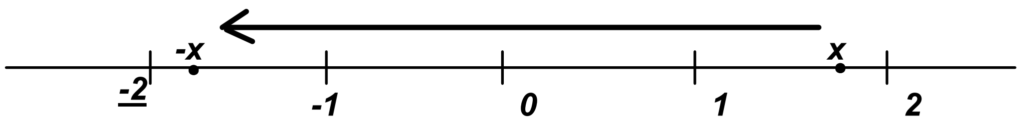
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009 SOLUTION KEY

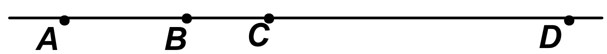
Round 3

A) $x = 1 + \frac{1}{2}(-x)$ → 2x = 2 – x → x = 2/3 → 1/x = **<u>3/2</u>**

B) $100\left(.06x + .15(10 - x) = \frac{3x}{4}\right)$ → $6x + 150 - 15x = 75x$ → $84x = 150$ → $x = 1.\underbrace{♣ ♦ ♥ ♠ \cdots}_{\text{actual digits unimportant}}$

Thus, [–x] = [–1.♣ ♦ ♥ ♠ ⋯] = **<u>–2</u>**





C) Since $\frac{AC}{BD} = \frac{4}{9}$, let $AC = 4x$, $BD = 9x$ and the overlap $BC = k$. Then:
$AB = 4x - k$, $CD = 9x - k$ → $13x - k = 203$
$\frac{AB}{CD} = \frac{4x-k}{9x-k} = \frac{10}{27}$ → $18x = 17k$
Substituting for x in the first equation, $13\left(\frac{17}{18}k\right) - k = 203$ → $(13\cdot17 - 18)k = 18\cdot203$
→ $203k = 18\cdot203$ → k = **<u>18</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009 SOLUTION KEY

Round 4

A) Since $R \cdot T = D$, i.e. (Rate)(Time) = Distance, the <u>average</u> rate is the total distance traveled divided by the total time it took to travel that distance. $r_{ave} = \dfrac{2+10+40}{3+4+1.5} = \dfrac{52}{8.5} = \dfrac{104}{17}$.

$6 < \dfrac{104}{17} = 6\dfrac{2}{17} < 7$ ➔ (A, C, B) = <u>**(6, 6, 7)**</u>

B) $\begin{cases} x = \dfrac{4}{5}y \\ y = \dfrac{1}{2}w \end{cases}$ ➔ $x = \dfrac{4}{15}w$ ➔ $x^2 = \dfrac{16}{225}w^2$ Since $k\% = \dfrac{k}{100}$, we have $\dfrac{k}{100} = \dfrac{16}{225}$.

$\dfrac{k}{4} = \dfrac{16}{9}$ ➔ $k = \dfrac{64}{9} = 7\dfrac{1}{9} = 7.111\cdots$ ➔ <u>**7.1**</u>

C) $\left.\begin{matrix} (1)\ a+b=2c \\ (2)\ a+c=3b \end{matrix}\right\}$ ➔ $b-c = 2c-3b$ ➔ $4b = 3c$

It's reasonable to assume that the value of $\dfrac{b+c}{a}$ is <u>unique</u>, i.e. is the same for all values of a, b and c that satisfy the initial conditions. Thus, take $b = 3$, $c = 4$ ➔ $a = 5$ ➔ $\dfrac{3+4}{5} = \mathbf{\dfrac{7}{5}}$

Of course <u>two</u> equations in <u>three</u> unknowns do not nail down a unique solution.
We need to also show that the value of the required fraction is invariant, i.e. the same for all choices of a, b and c.

Substituting $b = \dfrac{3}{4}c$ in (1), we have $a + \dfrac{3}{4}c = 2c$ ➔ $c = \dfrac{5}{4}c$

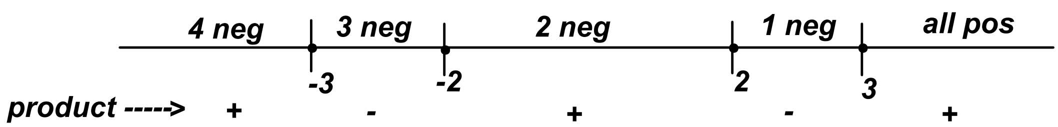
Now, $\dfrac{b+c}{a} = \dfrac{\frac{3}{4}c + c}{\frac{5}{4}c} = \dfrac{3c+4c}{5c} = \dfrac{7}{5}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009 SOLUTION KEY

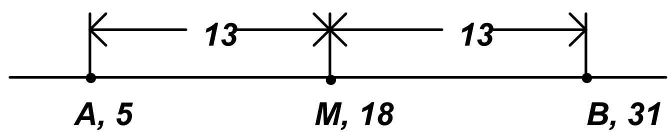
Round 5

A) $\left(x^2-4\right)\left(x^2-9\right)=(x+2)(x-2)(x+3)(x-3)$

Test each of the 5 sections on the number line determined by the critical values, ±2, ±3
The number of negative factors determines the sign of the product.



B) $16 \le 3k+2 \le 96$ ➔ $14 \le 3k \le 94$ ➔ $15 < 3k \le 93$
➔ $5 \le k \le 31$



This set of values is shown in the diagram at the right.
Since <u>distance</u> between two points on the number line is the <u>absolute value of the difference</u> of the coordinates of the points involved, we note the midpoint of the interval has coordinate 18 and the distance to each endpoint is 13.
Therefore, the equivalent absolute value representation is $|k-18| \le 13$ ➔ (a, b) = **<u>(18, 13)</u>**

C) If the units digit of A were 5 or any even digit, then A would not be prime.
The only digits that can be used to form A and B are 1, 3, 7 and 9.
Thus, there are 4 possible ordered pairs (A, B): (13, 31), (17, 71), (37, 73) and (79, 97)
The values of $|A-B|$ are: 18, 54, 36 and 18 ➔ $C = 18$ and $D = 54$
The interval $72 < x < 216$ contains 216 – 72 – 1 = **<u>143</u>** integers

Round 6

A) $$2^{-1}-\sqrt{\frac{25}{9}-\frac{64}{25}}+(3\cdot 5)^{-1}=\frac{1}{2}-\sqrt{\frac{25^2-9(64)}{9(25)}}+\frac{1}{15}=\frac{1}{2}-\sqrt{\frac{625-576}{9(25)}}+\frac{1}{15}=\frac{1}{2}-\frac{7}{15}+\frac{1}{15}=\frac{1}{2}-\frac{2}{5}=\underline{\mathbf{\frac{1}{10}}}$$

B) $$\begin{cases} a=\frac{2}{3}b \\ b=\frac{4}{5}c \end{cases} \;➔\; a=\frac{2}{3}\cdot\frac{4}{5}\cdot c=\frac{8}{15}c$$

Substituting and multiplying through by 15, $8c + 12c + 15c = 35c = 15(70)$ ➔ c = **<u>30</u>**

C) Recall: $$a*b=\begin{cases} a+ab, \text{ when } b \text{ is a proper fraction} \\ b-ab, \text{ when } b \text{ is an improper faction} \end{cases}$$

Since $\left(6*\frac{2}{3}\right)=(6+4)=10$ and $\left(\frac{3}{4}*\frac{3}{2}\right)=\left(\frac{3}{2}-\frac{3}{4}\cdot\frac{3}{2}\right)=\frac{3}{2}-\frac{9}{8}=\frac{3}{8}$, we have

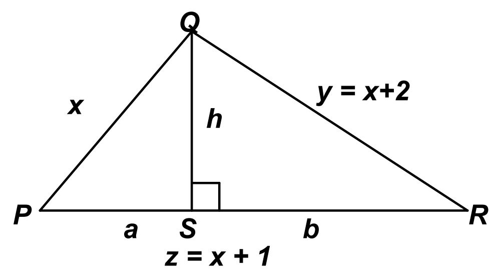
$$\left(6*\frac{2}{3}\right)*\left(\frac{3}{4}*\frac{3}{2}\right)=\left(10*\frac{3}{8}\right)=10+10\left(\frac{3}{8}\right)=10+\frac{15}{4}=\underline{\mathbf{\frac{55}{4}}}\quad\left(\underline{\mathbf{13\frac{3}{4}}}\text{ or }\underline{\mathbf{13.75}}\right)$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2009 SOLUTION KEY

Team Round

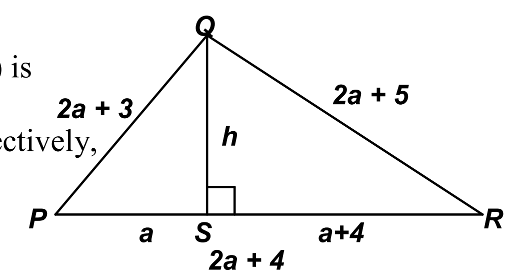
A) 3 faces: corner cubes (at the 8 vertices) → 8
2 faces: edge cubes (12 edges) → 7 x 8: 22, 7 x 9: 24, 8 x 9: 26 → 72
1 face: center cubes (6 faces) → 2[(5)(6) + (5)(7) + (6)(7)] →214
0 faces: interior cubes only 5(6)(7) → 210
This totals 504 unit cubes in total and there should be 7(8)(9) = 504 cubes.
Thus, (8 + 72) : (210 + 214) → **<u>10 : 53</u>**

B) In right triangles PQS and RQS, we have $\begin{cases} (1)\ a^2 + h^2 = x^2 \\ (2)\ b^2 + h^2 = y^2 \\ (3)\ z = a + b \end{cases}$



Subtracting (2) – (1),
$b^2 - a^2 = (b + a)(b - a) = y^2 - x^2 = (x + 2)^2 - x^2 = 4(x + 1)$
But $a + b = x + 1$!!!
Canceling, $b - a = 4$ → $b = a + 4$
Substituting, $2a + 4 = x + 1$ → $x = 2a + 3$
It remains to find h in terms of a.
In ΔPQS, $a^2 + h^2 = x^2$ → $a^2 + h^2 = (2a + 3)^2$ → $h^2 = 3(a^2 + 4a + 3)$ → $h = \sqrt{3(a+1)(a+3)}$

Clearly, we want values of a for which $3(a+1)(a+3)$ is a perfect square, i.e $(a + 1)$ is a perfect square and $(a + 3)$ is 3 times a perfect square or vice versa.
The first and third term are generated by $a = 5$ and 95 respectively, so we restrict our search to integer values of a between 6 and 94 inclusive.
$a = 24$ → $\sqrt{3 \cdot 25 \cdot 27} = 45$ BINGO!
Thus, the second term is **<u>(51, 52, 53)</u>**.



Check: ΔPQS: (24, 45, 51) → 3(8, 15, 17) and ΔRQS: (28, 45, 53) → $28^2 + 45^2 = 2809 = 53^2$

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Team Round

C) Solving for x, $x = \dfrac{451+7y}{4} = 112 + y + \dfrac{3(1+y)}{4}$.

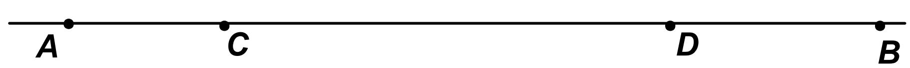
Thus, y must be picked so that $\dfrac{3(1+y)}{4}$ is an integer.

This occurs when $y = 3$ and correspondingly $x = 112 + 3 + 3 = 118$.
Since the given equation is a straight line with slope 4/7, an increase of 7 in the value of x corresponds to an increase of 4 in the value of y. Thus, a general solution is $(118 + 7t, 3 + 4t)$ and $x + y = 121 + 11t = 11(11 + t)$ The expression is a multiple of 3 for $t = 1, 4, 7, 10, \ldots$, i.e for t of the form $3k + 1$. $11(12 + 3k) < 1000$ ➔ $k < 26.3$ ➔ $k = 26$ ➔ $t = 79$ ➔ $(x, y) =$ **<u>(671, 319)</u>**

Alternate solution:
The largest possible value of $x + y$ is 999.
If we try 999, we notice that $4(999 - y) - 7y = 451$ or $4(999) - 451 = 11y$ requires divisibility by 11.

$4(999) - 451 = 3996 - 451 = 3545$ which fails [since $(5 + 5) - (3 + 4) = 3$]
$4(996) - 451 = 3533$ which fails [since $(5 + 3) - (3 + 3) = 2$]
$4(993) - 451 = 3521$ which fails [since $(5 + 1) - (3 + 2) = 1$]
$4(990) - 451 = 3509$ BINGO! [since $(5 + 9) - (3 + 0) = 11$, a multiple of 11]

Thus, $\begin{cases} 4x - 7y = 451 \\ x + y = 990 \end{cases}$ ➔ $(x, y) =$ **<u>(671, 319)</u>**



D) Let x denote Barbara's original rate (in mph).
Then $AB = AD + DB = (x + 10)(1) + x(1) = 2x + 10$
Also $AB = AC + CB = (x + 10 - k)(2/3) + (x + 10)(2/3)$
Equating these expressions for AB, $2x + 10 = \dfrac{2(2x + 20 - k)}{3}$ ➔ $2x = 10 - 2k$ ➔ $x = 5 - k$

Alice's reduced rate ➔ $\dfrac{x + 10 - k}{x + 10} = \dfrac{3}{4}$ ➔ $4x + 40 - 4k = 3k + 30$ ➔ $x = 4k - 10$

$5 - k = 4k - 10$ ➔ $k = 3$, $x = 2$, $AB = 14$, $AD = 12$, $DB = 2$, $AC = (2 + 10 - 3)(2/3) = 6$
➔ $CD = 14 - 2 - 6 =$ **<u>6</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
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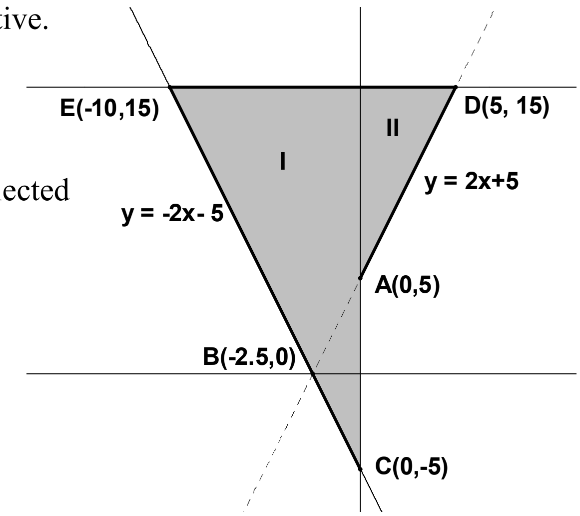
E) $y = 2x + 5$ is a line with slope 2 and y-intercept at (0, 5) and x-intercept at (–2.5, 0) – the dotted line

$\frac{|x|}{x} = \pm 1$ depending on whether x is positive or negative.

The following diagram shows the region in question.



For positive x, y was positive and it stays positive.
For negative x, y changed sign, i.e. the graph was reflected across the x-axis.

The area of region I is $\frac{1}{2}(10)(20) = 100$

The area of region II is $\frac{1}{2}(5)(10) = 25$

Thus, the total area is **<u>125</u>**

F) (10)(5)(3) = 150 coins available.

I have 50 P quarters.

I have at least $\frac{3}{4}50 = 37.5$ ➔at least 38 D quarters.

I have at most $\frac{1}{8}\cdot 50\ = 6.5$ ➔ at most 6 S quarters

Thus, minimum and maximum number of coins I have are 50 + 38 + 0 = 88 and 50 + 50 + 6 = 106 and $(m, M) = \left(\frac{88}{150}, \frac{106}{150}\right) = \left(\frac{176}{300}, \frac{212}{300}\right) = \left(\frac{176}{3}\%, \frac{212}{3}\%\right) = \underline{\left(\mathbf{58\frac{2}{3}\%, 70\frac{2}{3}\%}\right)}$