MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2010
ROUND 1 ANALYTIC GEOMETRY: ANYTHING

ANSWERS

A) ________________________________

B) ________________________________

C) (____________ , ____________)

***** NO CALCULATORS IN THIS ROUND *****

A) Determine the coordinates (x, y) of all possible intersection points with the x- and y-axes of $x^2 - (y-1)^2 = 1$.

B) Compute the diameter of a circle concentric with $5x^2 + 5y^2 + 15x = 21$ and tangent to $2x + 4y + 13 = 0$.

C) The points $P(6, 5)$, $Q(11, 7)$ and R lie on a parabola whose vertex is at $V(2, 1)$. The axis of symmetry is parallel to one of the coordinate axes. The focus of the parabola lies on $\overline{QR}$. Compute the (x, y) coordinates of the point R.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010
ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

******* NO CALCULATORS IN THIS ROUND *******

A) Simplify completely. $$\frac{6x^2-5x-6}{10+15x}$$

B) Compute the value of the integer constant A for which a solution of the following equation is 2.

$$\frac{12x^2+12x-45}{9-4x^2}=\frac{4x-A}{7}$$

C) Solve for x. $$\frac{4}{5-\frac{3+x}{3}}=\frac{16}{4+\frac{8}{3-\frac{6}{x}}}$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010
ROUND 3 TRIG: EQUATIONS WITH A RESAONABLE NUMBER OF SOLUTIONS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

***** NO CALCULATORS IN THIS ROUND *****

A) Let x denote a real number (or an angle measure in radians).
Compute the smallest positive value of x for which $(\sin x)^x = 1$

B) If x denotes the unique solution to $2\cot(2x) - \tan(x) = 0$ between $\frac{\pi}{2}$ and π, compute $\cos(x)$.

C) Solve for θ, where $0° < \theta < 360°$: $\dfrac{\sin\theta}{\sqrt{3}+\sqrt{3}\cos\theta} = -1$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010
ROUND 4 ALG 2: QUADRATIC EQUATIONS / THEORY OF QUADRATICS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

******* NO CALCULATORS IN THIS ROUND *******

A) Consider the following quadratic equation: $x^2 + 3x + 2M = 0$

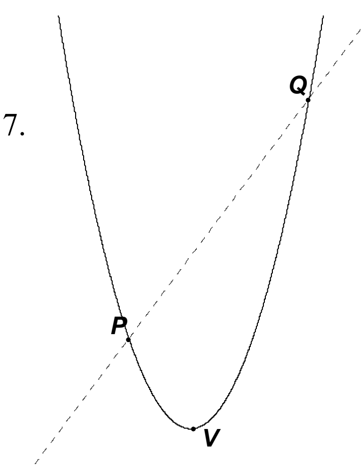
If $M = a$, the constant term is 3 greater than the coefficient of x^2.
If $M = b$, the equation has equal roots.
If $M = c$, the product of the roots is 10.
Compute the product abc.

B) Find all values of the constant k for which the roots of the quadratic equation

$$y^2 + k^2y = 5ky + 6y + 7$$

are numerically equal, but <u>opposite</u> in sign.

C) The line $2x - y + 7 = 0$ intersects $y = Ax^2 + Bx + C$ at $x = -2$ and $x = 7$.
The low point V has coordinates $(1, -3)$
Compute the value of C.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010
ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

ANSWERS

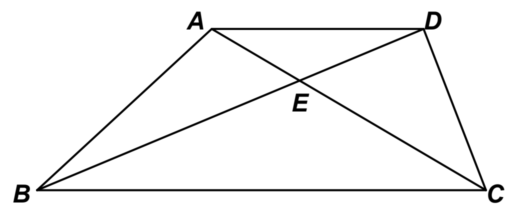
A) _________________________ units^2

B) ______________ : ______________

C) __________ : ________ : _________

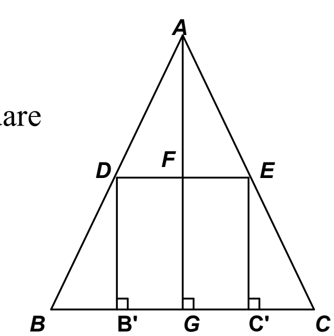
******* NO CALCULATORS IN THIS ROUND *******

A) In trapezoid $ABCD$, $\overline{AD} \parallel \overline{BC}$,
$AE = 4$, $BE = 15$, $CE = 10$ and $DE = 6$.
If the area of ΔBEC is 50 units^2,
what is the area of ΔADE?



B) The ratio of the length of the longest diagonal in a regular hexagon A to the length of shortest diagonal in regular hexagon B is 4 : 3. Compute the ratio of the length of shortest diagonal of hexagon A to the length of the longest diagonal of hexagon B.

C) Given: ΔABC is isosceles, $\overline{DE} \parallel \overline{BC}$, $\dfrac{FG}{AG} = \dfrac{2}{3}$ and $DEC'B'$ is a square
Express $area(\Delta AFD) : area(DEC'B') : area(DECB)$
as a simplified ratio.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010
ROUND 6 ALG 1: ANYTHING

ANSWERS

A) ______________________________

B) ______________________________

C) ______ minutes ______ seconds

******* NO CALCULATORS IN THIS ROUND *******

A) Compute: $$\left(\sqrt{8}-\frac{1}{\sqrt{2}-\frac{1}{\sqrt{2}}}\right)^2$$

B) Write as a single simplified fraction with no negative exponents.

$$\left(\frac{1}{a+b}-\frac{1}{a-b}\right)\left(a^{-1}-b^{-1}\right)$$

Note: For any real number $x \neq 0$, x^{-1} is equivalent to $\frac{1}{x}$.

C) On the interstate Mario traveling 100 mph passed a state trooper with a radar gun parked beside the road. The trooper immediately decided to give chase. 48 seconds after Mario passed the state trooper's parked car, the trooper had gone 1/6 of a mile and had reached his top speed of 121 mph, which he maintained until he overtook Mario. How long after Mario passed the trooper was he apprehended? Express your answer in minutes and seconds, accurate to the nearest second.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010
ROUND 7 TEAM QUESTIONS
ANSWERS

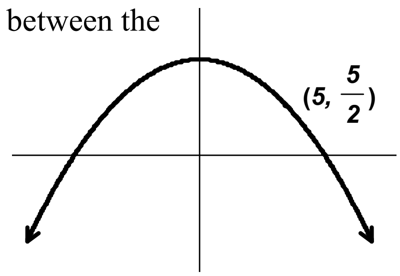
A) ____________________________ D) ____________________________

B) (_____ , _____ , _____) E) ____________ : ____________

C) ____________________________ F) ______________________ yards

******* CALCULATORS ARE PERMITTED IN THIS ROUND *******

A) The maximum height of a parabola above the x-axis is twice the distance between the vertex of the parabola and its focus. If the parabola contains the point (5, 5/2) and has an axis of symmetry on the y-axis, compute the distance between its zeros.

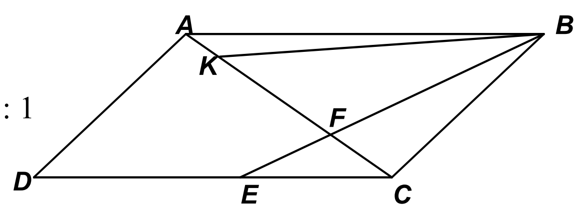


B) A, B, C and D are positive integers.
$(2x+3y+A)(Bx+Cy+D) = C_1x^2 + C_2xy + C_3y^2 + C_4x + C_5y + C_6$
All coefficients in this expansion are either 1 or prime.
Determine all possible ordered triples (C_4, C_5, C_6)

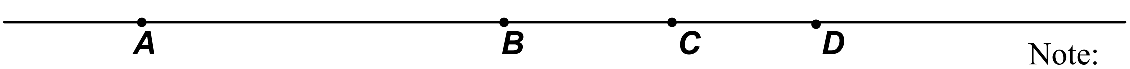
C) Given: $(\cos^4 4x - \sin^4 4x)(1 - 2\sin^2 x) = 0$, where $0 \le x < \pi$.
Let (p, q) denote the smallest and eighth largest solutions respectively over the specified interval. Compute $\frac{q}{p}$.

D) A rectangular pane of stamps would contain 48 fewer stamps if it consisted of three more rows each containing 5 fewer stamps.
There are N^2 stamps on the original pane.
Compute the smallest possible integer value of N.

E) Given: parallelogram $ABCD$, $AK : KF = 2 : 7$, $DE : EC = 2 : 1$
Compute the ratio $\dfrac{area(FADE)}{area(ABCD)}$.



F) A treasure is located at a point along a straight road with landmarks A, B, C and D located (in the given order) as indicated on the map below:



Note:
Relative distances are rarely accurate on these old pirate maps.
The following instructions were included:
(1) Start at A and go 1/2 of the distance to C
(2) Then go 1/3 of the distance to D
(3) Then go 1/4 of the distance to B and dig!
If AB = 120 yards and BC = 80 yards and the treasure is buried midway between A and D, compute the distance from B to D.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010 ANSWERS

Round 1 Analytic Geometry: Anything

A) $\left(\pm\sqrt{2},0\right)$ B) $2\sqrt{5}$ C) $\left(\frac{19}{9},\frac{1}{3}\right)$

Round 2 Alg1: Factoring

A) $\frac{2x-3}{5}$ B) 35 C) 3, 6

Round 3 Trig: Equations

A) $\frac{\pi}{2}$ B) $-\frac{\sqrt{6}}{3}$ C) 240°

Round 4 Alg 2: Quadratic Equations

A) $\frac{45}{4}$ or 11.25 B) -1, 6 C) $-\frac{7}{3}$

Round 5 Geometry: Similarity

A) 8 B) 1 : 1 C) 1 : 8 : 16

Round 6 Alg 1: Anything

A) 2 B) $\frac{2}{a(a+b)}$ C) 4 minutes 8 seconds

Team Round

A) $10\sqrt{2}$ D) 42

B) (5, 7, 2) only E) 11 : 24

C) 12 F) 200 yards

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010 SOLUTION KEY

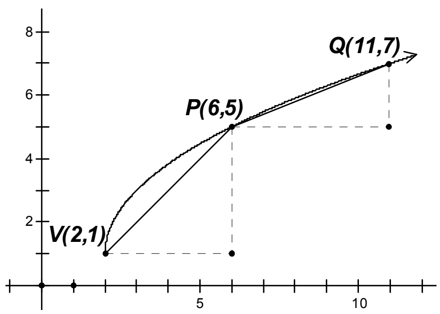
Round 1

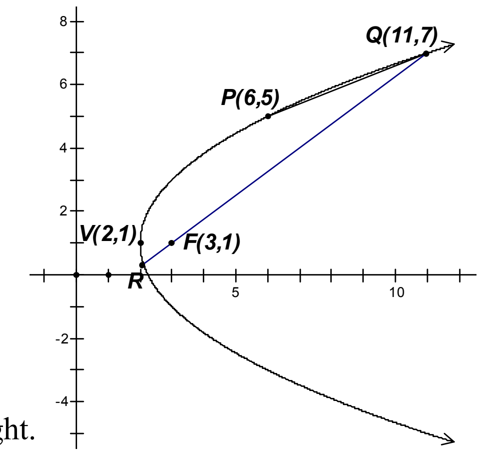
A) Y-intercepts ($x = 0$): $(y-1)^2 = -1 \rightarrow$ no Y-intercepts
X-intercepts ($y = 0$): $x^2 - 1 = 1 \rightarrow x = \pm\sqrt{2} \rightarrow \underline{\mathbf{(\pm\sqrt{2}, 0)}}$

B) Completing the square, $5x^2+5y^2+15x=21 \rightarrow 5\left(x^2+3x+\frac{9}{4}\right)+5y^2=21+5\left(\frac{9}{4}\right) \rightarrow$

$5\left(x+\frac{3}{2}\right)^2+5y^2=21+5\left(\frac{9}{4}\right) \rightarrow$ Center @ $(-3/2, 0)$

The distance from this point to $2x + 4y + 13 = 0$ can be computed by the point to line distance formula, $r = \dfrac{|2(-3/2)+4(0)+13|}{\sqrt{2^2+4^2}} = \dfrac{10}{2\sqrt{5}} = \dfrac{5}{\sqrt{5}} = \sqrt{5} \rightarrow d = \underline{\mathbf{2\sqrt{5}}}$





C) The points $P(6, 5)$ and $Q(11, 7)$ lie on the same side of the axis of symmetry. The parabola must open up or to the right. The slope of $\overline{VP}$ is 1 and the slope of $\overline{PQ}$ is 2/5.

Since the slope is decreasing as we move from left to right, the parabola must open to the right and, therefore has the form $(y-1)^2 = 4p(x-2)$. Substituting $P(6, 5)$, we have $(5-1)^2 = 4p(6-2) \rightarrow p = 1$. Thus, the focus is at $(3, 1)$ and the slope of $\overleftrightarrow{QR}$ is $\dfrac{7-1}{11-3} = \dfrac{3}{4}$

and the equation of $\overleftrightarrow{QR}$ is $3x - 4y = 5$ or $x = \dfrac{4y+5}{3}$.

Substituting, $(y-1)^2 = 4\left(\dfrac{4y+5}{3}-2\right) \rightarrow 3(y-1)^2 = 16y-4 \rightarrow 3y^2-22y+7=0$

$\rightarrow (3y-1)(y-7)=0 \rightarrow y = 1/3 \rightarrow (x, y) = \underline{\left(\mathbf{\dfrac{19}{9}, \dfrac{1}{3}}\right)}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010 SOLUTION KEY

Round 2

A) $\dfrac{6x^2-5x-6}{10+15x} = \dfrac{(2x-3)\cancel{(3x+2)}}{5\cancel{(2+3x)}} = \underline{\dfrac{\mathbf{2x-3}}{\mathbf{5}}}$

B) Given: $\dfrac{12x^2+12x-45}{9-4x^2} = \dfrac{4x-A}{7}$ and $x = 2$ The right hand side is $\dfrac{8-A}{7}$.

The left hand side is $\dfrac{3(2x-3)(2x+5)}{(3+2x)(3-2x)} = \dfrac{-3(2x+5)}{(3+2x)} = \dfrac{-27}{7}$ for $x = 2$.

Equating, A = **<u>35</u>**.

C) Verify that to avoid division by zero, we require that $x \neq$ 0, 2, 12 or $\dfrac{6}{5}$.

$$\frac{4}{5-\frac{3+x}{3}} = \frac{16}{4+\frac{8}{3-\frac{6}{x}}} \rightarrow \frac{4}{\frac{15-(3+x)}{3}} = \frac{16}{4+\frac{8}{\frac{3x-6}{x}}} \rightarrow \frac{4}{\frac{12-x}{3}} = \frac{16}{4+\frac{8x}{3x-6}} \rightarrow \frac{1}{\frac{12-x}{3}} = \frac{4}{4+\frac{8x}{3x-6}}$$

Cross multiplying, $4\left(\dfrac{12-x}{3}\right) = 4+\dfrac{8x}{3x-6}$.

$\rightarrow 16-\dfrac{4x}{3} = 4+\dfrac{8x}{3x-6} \rightarrow 12-\dfrac{4x}{3} = \dfrac{8x}{3x-6} \rightarrow 3-\dfrac{x}{3} = \dfrac{2x}{3(x-2)}$

Multiplying through by $3(x-2)$, $9(x-2)-x(x-2) = 2x$.

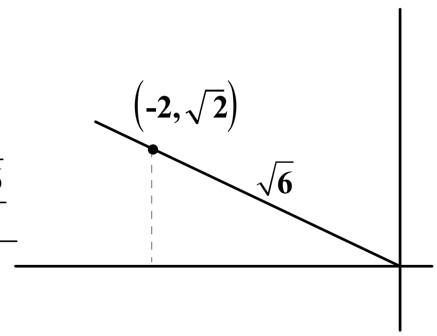
$9x - 18 - x^2 + 2x = 2x \rightarrow x^2 - 9x + 18 = (x-3)(x-6) = 0 \rightarrow x =$ **<u>3, 6</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010 SOLUTION KEY

Round 3

A) $x^0 (x \neq 0)$ and $(1)^x$ both produce 0. By inspection, $x = \pi/2$ solves the equation.
But is it the smallest? Taking the log of both sides, we have $x\log(\sin x) = \log(1) = 0$
Since $x > 0$, $\log(\sin x) = 0$ ➔ $\sin x = 1$ ➔ $x = \pi/2 + 2n\pi$ and $\boldsymbol{\dfrac{\pi}{2}}$ is the smallest solution.

B) $\dfrac{2}{\tan(2x)} - \tan x = \dfrac{2(1-\tan^2 x)}{2\tan x} - \tan x = 0$ ➔ $1 - \tan^2 x - \tan^2 x = 0$
➔ $\tan^2 x = \dfrac{1}{2}$ ➔ $\tan x = -\dfrac{\sqrt{2}}{2}$ (x lies in quadrant 2) ➔ $\cos(x) =$ $\boldsymbol{-\dfrac{\sqrt{6}}{3}}$



C) Potential extraneous answers: ($\cos\theta = -1$) $\theta \neq 180 + 360n$
$\sin\theta = -\sqrt{3}\,(1 + \cos\theta)$ ➔ $\sin^2\theta = 3(1 + 2\cos\theta + \cos^2\theta)$
$1 - \cos^2\theta = 3 + 6\cos\theta + 3\cos^2\theta$ ➔ $4\cos^2\theta + 6\cos\theta + 2 = 0$
➔ $2\cos^2\theta + 3\cos\theta + 1 = (2\cos\theta + 1)(\cos\theta + 1) = 0$
➔ $\cos\theta = -1/2$ ➔ $\theta = 120°, 240°$ or $\cos\theta = -1$ ➔ $\theta = 180°$ (extraneous)

Checking:

$\theta = 120°$: $\dfrac{\sqrt{3}/2}{\sqrt{3}+\sqrt{3}(-1/2)} = \dfrac{1/2}{1-1/2} = 1$ (extraneous)

$\theta = 240°$: $\dfrac{-\sqrt{3}/2}{\sqrt{3}+\sqrt{3}(-1/2)} = \dfrac{-1/2}{1-1/2} = -1$ (ok)

Alternate solution: Using the identity $\dfrac{\sin x}{1+\cos x} = \tan\left(\dfrac{x}{2}\right)$,

$\dfrac{\sin\theta}{\sqrt{3}+\sqrt{3}\cos\theta} = \dfrac{\sin\theta}{\sqrt{3}\left(1+\cos\theta\right)} = \dfrac{\tan(\theta/2)}{\sqrt{3}}$ ➔ $\tan\left(\dfrac{\theta}{2}\right) = -\sqrt{3}$ ➔ $\dfrac{\theta}{2} = \begin{cases}120° \\ 300°\end{cases} + 360n$ ➔ $\theta = 240°$ only

Round 4

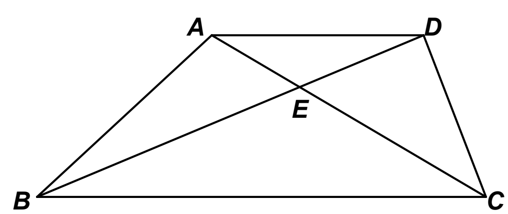
A) $2a = 3 + 1$ ➔ $a = 2$
Equal root ➔ discriminant = 0 ➔ $3^2 - 4(1)(2b) = 0$ ➔ $b = 9/8$
$2c = 10$ ➔ $c = 5$
Thus, $abc = 10\left(\dfrac{9}{8}\right) =$ $\boldsymbol{\dfrac{45}{4}}$ or **11.25**

Round 4 - continued

B) Rewrite equation as $y^2 + (k^2 - 5k - 6)y - 7 = 0$
To have roots that are numerically equal and opposite in sign B must be 0.
[Then the roots of $Ax^2 + Bx + C = 0$ would be $\dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \dfrac{\pm\sqrt{-4AC}}{2A}$]
Therefore, $k^2 - 5k - 6 = (k + 1)(k - 6) = 0 \rightarrow k = -1, 6$

C) Vertex at $(1, -3)$ and a vertical axis of symmetry $\rightarrow$ equation of the parabola must be of the form $(y+3) = a(x-1)^2$
Substituting in the equation of the line $(2x - y + 7 = 0)$, $x = -2 \rightarrow y = 3$.
Substituting in the equation of the parabola, $6 = 9a \rightarrow a = 2/3$
Expanding, $y = -3 + \frac{2}{3}(x-1)^2 \rightarrow C = -3 + \frac{2}{3} = \underline{\boldsymbol{-\frac{7}{3}}}$

Alternate solution (longer, but more straightforward):
$P(-2, 3)$, $Q(7, 21)$ and $V(1, -3)$
Substituting in the quadratic $y = Ax^2 + Bx + C$, $\begin{cases} (1) \quad 49A + 7B + C = 21 \\ (2) \quad 4A - 2B + C = 3 \\ (3) \quad A + B + C = -3 \end{cases}$
$(1) - (2) \rightarrow 45A + 9B = 18 \rightarrow 5A + B = 2$
$(2) - (3) \rightarrow 3A - 3B = 6 \rightarrow A - B = 2$
Adding, $6A = 4 \rightarrow (A, B) = \left(\frac{2}{3}, -\frac{4}{3}\right)$ Substituting in (3),
$C = -3 - \frac{2}{3} + \frac{4}{3} = \underline{\boldsymbol{-\frac{7}{3}}}$



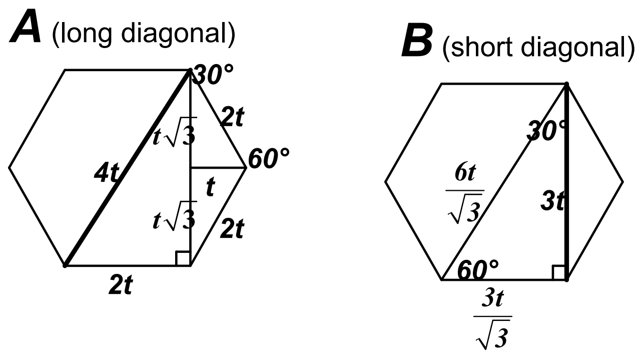
Round 5

A) Let K denote the area of ΔADE.
$\Delta ADE \sim \Delta CBE \rightarrow \dfrac{AE}{CE} = \dfrac{4}{10} = \dfrac{2}{5} \rightarrow \dfrac{area(\Delta ADE)}{area(\Delta CBE)} = \left(\dfrac{2}{5}\right)^2 \rightarrow \dfrac{4}{25} = \dfrac{K}{50} \rightarrow K = \underline{\mathbf{8}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2010 SOLUTION KEY

Round 5 - continued

B)



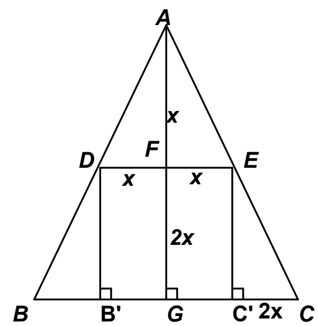
Study the diagrams at the right:

$A_{\text{short}} : B_{\text{long}} = 2\sqrt{3}t : \dfrac{6t}{\sqrt{3}}$ ➔ **1 : 1**

C) $\dfrac{FG}{AG} = \dfrac{2}{3}$ ➔ $\dfrac{AF}{AG} = \dfrac{1}{3}$ ➔ $\dfrac{area(\Delta ADE)}{area(\Delta ABC)} = \dfrac{1}{9}$ ➔ $\dfrac{area(\Delta AFD)}{area(\Delta ABC)} = \dfrac{1}{18}$

$\dfrac{area(DECB)}{area(\Delta ABC)} = \dfrac{8}{9}$

$\dfrac{area(DEC'B')}{area(\Delta ABC)} = \dfrac{4x^2}{\frac{1}{2}\cdot 6x \cdot 4x} = \dfrac{4}{9}$ ➔ $\dfrac{1}{18} : \dfrac{4}{9} : \dfrac{8}{9}$ ➔ **1 : 8 : 16**

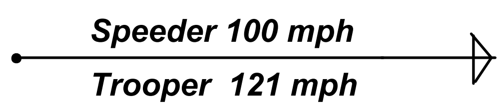


Round 6

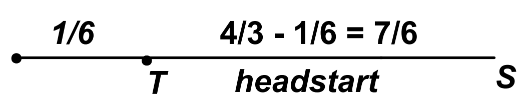
A) $\left(\sqrt{8} - \dfrac{1}{\sqrt{2} - \frac{1}{\sqrt{2}}}\right)^2 = \left(2\sqrt{2} - \dfrac{1}{\frac{2-1}{\sqrt{2}}}\right)^2 = \left(\sqrt{2}\right)^2 =$ **2**

B) $\left(\dfrac{1}{a+b} - \dfrac{1}{a-b}\right)\left(a^{-1} - b^{-1}\right) = \left(\dfrac{(a-b)-(a+b)}{(a+b)(a-b)}\right)\left(\dfrac{1}{a} - \dfrac{1}{b}\right) = \left(\dfrac{-2b}{(a+b)(a-b)}\right)\left(\dfrac{b-a}{ab}\right)$

$= \left(\dfrac{+2b}{(a+b)(b-a)}\right)\left(\dfrac{b-a}{ab}\right) = \mathbf{\dfrac{2}{a(a+b)}}$



C) $\dfrac{48}{60} = \dfrac{4}{5}$ mi/min = $\dfrac{4}{300}$ mi/hour · 100 mi/hr ➔ $\dfrac{4}{3}$ mi.



Let t denote the fraction of an hour required to catch the speeder (after the trooper reached his cruising speed.)

$121t = 100t + \left(\dfrac{4}{3} - \dfrac{1}{6}\right)$ ➔ $21t = \dfrac{7}{6}$ ➔ $\dfrac{1}{18}$ hour · 60 = $\dfrac{10}{3}$ minute

= 3 minutes 20 sec.

Therefore, it took exactly **4 minutes 8 seconds** to overtake Mario.

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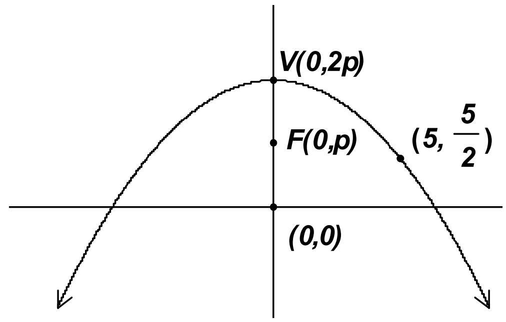
Team Round



A) $x^2 = -4p(y - 2p)$, where $p > 0$.

Substituting (5, 5/2), $25 = -4p(5/2) + 8p^2$

➔ $8p^2 - 10p - 25 = (4p + 5)(2p - 5)$ ➔ $p = 5/2$

Thus, $x^2 = -10(y - 5)$ ➔ $x^2 = 50$

➔ span = $\underline{\mathbf{10\sqrt{2}}}$

B) Since $C_1 = 2B$ and C_1 is prime, B must be 1. Likewise $C_3 = 3C$ ➔ $C = 1$.

$$(2x+3y+A)(Bx+Cy+D)=(2x+3y+A)(x+y+D)=2x^2+5xy+3y^2+(2D+A)x+(3D+A)y+AD$$

Since AD is prime, we must examine two cases:
1) $A = 1$ and D is prime
2) D = 1 and A is prime

The first case requires $(C_4, C_5) = (2D + 1, 3D + 1)$. $D = 1$ fails, but $D = 2$ ➔ (5, 7)
Any other prime values of D will be odd and this forces C_5 to be an even composite number.
The second case requires $(C_4, C_5) = (A + 2, A + 3)$. Both $A = 1$ and 2 fail. Likewise, any other prime values of A will be odd and this forces C_5 to be an even composite number.
Therefore, the only ordered triple is **<u>(5, 7, 2)</u>**

C) $(\cos^4 4x-\sin^4 4x)(1-2\sin^2 x) = (\cos^2 4x+\sin^2 4x)(\cos^2 4x-\sin^2 4x)(1-2\sin^2 x)$

Since the first factor is always equal to 1, it can be ignored and the original equation simplifies to $(\cos 8x)(\cos 2x) = 0$

$8x = \frac{\pi}{2}+n\pi$ ➔ $x = \frac{(2n+1)\pi}{16}$ ➔ $\frac{\pi}{16},\frac{3\pi}{16},\frac{5\pi}{16},\frac{7\pi}{16},\frac{9\pi}{16},\frac{11\pi}{16},\frac{13\pi}{16},\frac{15\pi}{16}$

$2x = \frac{\pi}{2}+n\pi$ ➔ $x = \frac{(2n+1)\pi}{4}$ ➔ $\frac{\pi}{4},\frac{3\pi}{4}$

Thus, $p = \frac{\pi}{16}$ and $q = \frac{3\pi}{4}$ ➔ $\frac{q}{p} = \frac{3\pi}{4}\cdot\frac{16}{\pi} = \underline{\mathbf{12}}$

Team Round - continued

D) Suppose the original pane had R rows and C columns. Then:

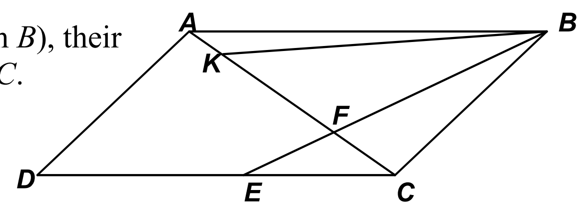
$RC = (R + 3)(C - 5) + 48 \rightarrow 0 = 3C - 5R + 33 \rightarrow C = \frac{5R-33}{3} = R - 11 + \frac{2R}{3}$

The smallest value of R that returns a positive integer value for C is 9. [$(R, C) = (9, 4)$]
Using slope, we create a table of values until the product RC is a perfect square.

R	9	12	15	18	21	24	27	30	33	36
C	4	9	14	19	24	29	34	39	44	49

Using this lookup table, the last ordered pair gives us $RC = (36)(49) = (6\cdot 7)^2 = 42^2 \rightarrow N =$ **<u>42</u>**
The values satisfying this relationship get quite large very quickly.
The next three values satisfying $RC = N^2$ may be determined with a calculator or spreadsheet. They are: $(324)(529) = 18^2\cdot 23^2 = (414)^2$ $\quad (2025)(3364) = 45^2\cdot 58^2 = (2610)^2$ and $(19881)(33124) = 141^2\cdot 182^2 = (25662)^2$

E) Since ΔBAK, ΔBKF and ΔBFC have a common altitude (from B), their areas are in the same ratio as their bases, namely $AK : KF : FC$.



Let $AK = 2x$, $KF = 7x$ and $FC = y$. Let $DE = 2a$ and $CE = a$.
Since $\Delta ABF \sim \Delta CEF$, their areas are in a 9 : 1 ratio and

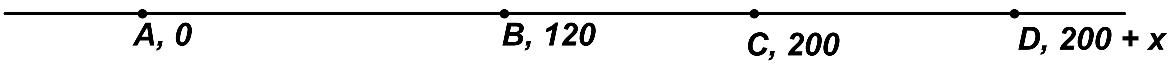
$\frac{CF}{AF} = \frac{CE}{AB} \rightarrow \frac{y}{9x} = \frac{1a}{3a} \rightarrow y = 3x.$

Let area(ΔBAK) = $2N$, area(ΔBKF) = $7N$ and area(ΔBFC) = $3N$. Then area(ΔCEF) = N and area (ΔADC) = $12N$.

Thus, $\frac{area(FADE)}{area(ABCD)} = \frac{12N - N}{24N} = \mathbf{\frac{11}{24}}$

Team Round - continued

F) Assign coordinates to the 4 points.



(1) → start at 0, end at 100 (1/2 between 0 and 200)

(2) → start at 100, end at $\dfrac{400+x}{3}$ (1/3 of the way to 200 + x)

$$100+\frac{1}{3}\left((200+x)-100\right)=\frac{400+x}{3}$$ or by weighted average $$\frac{100(\mathbf{2})+(200+x)(\mathbf{1})}{1+2}$$

(3) → start at $\dfrac{400+x}{3}$, end at $\dfrac{520+x}{4}$ (1/4 of the way to 120)

$$\frac{400+x}{3}-\frac{1}{4}\left(\frac{400+x}{3}-120\right)=\frac{400+x}{3}-\frac{1}{4}\left(\frac{40+x}{3}\right)=\frac{1600+4x-40-x}{12}=\frac{1560+3x}{12}=\frac{520+x}{4}$$

By weighted average $$\frac{120(\mathbf{1})+\left(\dfrac{400+x}{3}\right)(\mathbf{3})}{1+3}=\frac{520+x}{4}$$

Midway between A and D → $\dfrac{520+x}{4}=\dfrac{\mathbf{200+x}}{\mathbf{2}}$ → 520 + x = 400 + 2x → x = 120

→ BD = 320 – 120 = **<u>200</u>** yards

Addendum to the original contest
5C – prime omitted – question adjusted after the contest so answer as intended (1 : 8 : 16)
$area(\Delta AFD) : \underline{area(DEC'B)} : area(DECB)$
Answer to original question: 1 : 12 : 16

Round 6 – 6A changed after contest and note added to 6B
Negative exponents should be avoided in algebra contest at this time of year.

6A – question actually asked $\left(\sqrt{8} - \dfrac{1}{\sqrt{2} - \dfrac{1}{\sqrt{2}}}\right)^{-2}$ Ans: 1/2

6B - added to the original question:

Note: For any real number $x \neq 0$, x^{-1} is equivalent to $\dfrac{1}{x}$.