MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2010
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

***** NO CALCULATORS IN THIS ROUND *****

ANSWERS

A) ______________________________

B) ______________________________

C) (_____ , _____ , _____ , _____)

A) Given: $f:\left\{\left(x,f(x)\right) \mid f(x)=\dfrac{3}{x+2}\right\}$
Find the (x, y) coordinates of <u>all</u> points of intersection between f and f^{-1}.

B) In the equation $2x^4 - 7x^3 - 2x^2 + 13x + 6 = 0$, the four roots are A, B, C and D.
Compute $ABC + ABD + ACD + BCD$.

C) Let $f(x) = Ax^3 + Bx^2 + Cx + D$.
If $f(-a) = -f(a)$, $f(-3) + 2f(3) + f(5) = 3$ and one zero of $Ax^3 + Bx^2 + Cx + D = 0$ is 4, determine the ordered quadruple (A, B, C, D).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2010
ROUND 2 ARITHMETIC / NUMBER THEORY

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Let P equal the product of $A = 3{,}481{,}760{,}415{,}523$ and $B = 28{,}576{,}423{,}814$.
Determine the number of digits in the product P.

B) Find <u>all</u> positive integers with the property that the sum of all its divisors is exactly the same as the number of divisors of 360.

C) The 258^{th} natural number that is not divisible by either 3 or 7 is k.
Compute the <u>sum</u> of the prime factors of k.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2010
ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) $Cos^{-1}\left(-\frac{1}{2}\right) - Sin^{-1}\left(-\frac{1}{2}\right) = k°$. Compute k.

Do not include the degree symbol in your answer.

B) Compute the two smallest positive values of x (in radians) that satisfy $\frac{1-2\sin^2 x}{\sin x \cos x} = 2\sqrt{3}$.

C) Given: $\begin{cases} y = \sin^3 t \\ x = \cos^3 t \end{cases}$, where $0 \le t \le 2\pi$.

Compute all real values of y for which $x = \frac{64}{125}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2010
ROUND 4 ALG 1: WORD PROBLEMS

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) _______________________________

B) _______________________________

C) _______________________________

A) On a 26-question test five points were deducted for each wrong answer and eight points were credited for each correct answer. If all the questions were answered, but the score was zero, how many questions were answered correctly?

B) A hall has x rows with $x + 1$ chairs in each row. Increasing the number of chairs in each row by three and increasing the number of rows by four would increase the number of seats by one hundred. Find the original number of chairs in each row.

C) Tom and his sister Sherry are two of the oldest living tortoises. You've probably seen them in the Slowski's TV ad Comcast vs. FIOS. Presently, Tom is 18 years older than his sister. Four score and seven years ago**, their ages were two-digit numbers with the digits reversed and the sum of their ages was 110. How old is Tom now?

** a score is 20 years (from Abraham Lincoln's Gettysburg Address)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2010
ROUND 5 PLANE GEOMETRY: CIRCLES

******* NO CALCULATORS IN THIS ROUND *******

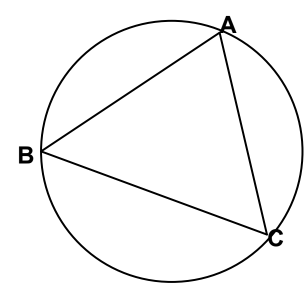
ANSWERS

A) ______________________________ °

B) ______________________________

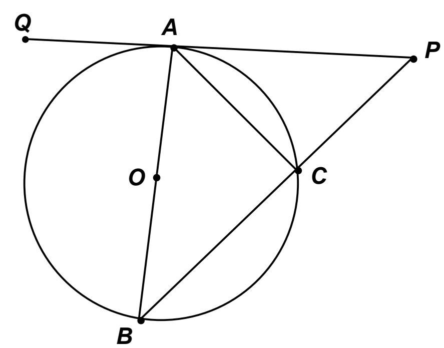
C) ______________________________

A) Given: $m\angle A = 8x - 2$, $m\angle B = 4x + 2$ and minor arc $\overset{\frown}{AC} = 9x - 3$
Find $m\angle C$.

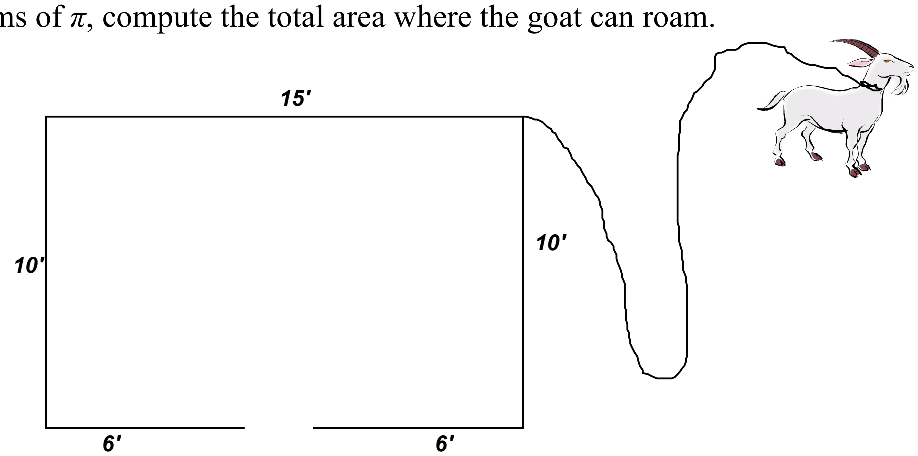


B) $\overline{PQ}$ is tangent to circle O at point A,
$\overline{AOB}$ is a chord of circle O, $AP = 12$ and
$BP : BC = 9 : 5$

Compute AC.



C) A goat is tied with a 20' rope to a 15' x 10' shed as shown. The shed has an <u>open</u> 3' doorway. In terms of π, compute the total area where the goat can roam.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2010
ROUND 6 ALG 2: SEQUENCES AND SERIES

***** NO CALCULATORS IN THIS ROUND *****

ANSWERS

A) _______________________________

B) (_____ , _____ , _____ , _____)

C) _______________________________

A) The sequence $\{a_n\}$ has $a_1 = \frac{1\cdot 2}{3\cdot 4}$, $a_2 = \frac{3\cdot 4}{5\cdot 6}$, $a_3 = \frac{5\cdot 6}{7\cdot 8}$. Compute $a_{12}\cdot a_{13}$.

B) Consider the following sequence of ordered pairs:

$$t_1 = \left(3\cdot 5,\ 2\cdot 3^2\right),\ t_2 = \left(5\cdot 7,\ 2^2\cdot 3\right),\ t_3 = \left(7\cdot 9,\ 2^3\cdot 1\right),\ t_4 = \left(9\cdot 11,\ 2^4\cdot \frac{1}{3}\right), \ldots$$

The 15th term can be written in the form $\left(A\cdot B, 2^x\cdot 3^y\right)$.
Compute the ordered quadruple (A, B, x, y).

C) AB, $3AB$, $18A$ form an increasing geometric progression.
A^3, $A + B + 1$, B form a decreasing arithmetic progression.
If A, B, C and D are real numbers and $(A+Bi)^3 = C + Di$, where $i = \sqrt{-1}$, compute $\frac{C}{D}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2010
ROUND 7 TEAM QUESTIONS
ANSWERS

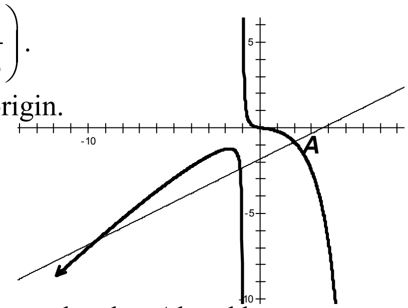
A) (__________ , __________)   D) ______________________

B) A   B   C   E) (____ , ____ , ____ , ____)

C) __________________________ F) __________________________

******* CALCULATORS ARE PERMITTED IN THIS ROUND *******

A) The graph of $f(x)=\dfrac{x^3+x}{x^2-5x-6}$ and $g(x)=\dfrac{3x-11}{6}$ intersect at $A\left(2,-\dfrac{5}{6}\right)$.
Compute the coordinates (x, y) of the point of intersection furthest from the origin.

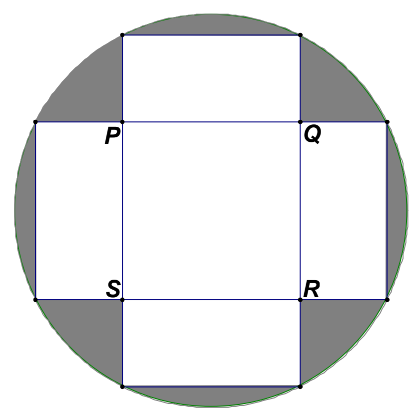


B) Which is larger $A=\sqrt[2008]{2008!}$, $B=\sqrt[2009]{2009!}$, $C=\sqrt[2010]{2010!}$

C) Compute $\cos\left(2\,Arc\cos\left(-\dfrac{3}{5}\right)+Arc\sin\left(-\dfrac{7}{25}\right)\right)$.

D) Sue is currently 24 years old. Eight years ago, the sum of the ages of her younger brother Al and her older sister Pam was 24. If all of these ages must be positive integers, compute the number of possible values for Al's current age.

E) Two 2 x k rectangles are inscribed in a circle, where $\boldsymbol{k > 2}$.
Their intersection consists of 4 points P, Q, R and S which are vertices of a square. The maximum value of k for which the area of the shaded region is exactly half the area of the circle may be written in the form $\dfrac{A+B\sqrt{C-\pi(\pi+D)}}{\pi}$
Determine the ordered quadruple (A, B, C, D).



F) Assume the continued fraction $\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}$ converges to $\sqrt{2}-1$.

Here are successive evaluations of this continued fraction: $\mathbf{\dfrac{1}{2}}, \dfrac{1}{2+\frac{1}{2}}=\mathbf{\dfrac{2}{5}}, \dfrac{1}{2+\frac{1}{2+\frac{1}{2}}}=\mathbf{\dfrac{5}{12}},\dots$

That is, we have a sequence $(a_1,a_2,a_3,\dots)=\left(\dfrac{1}{2},\dfrac{2}{5},\dfrac{5}{12},\dots\right)$. Adding 1, we have approximations of $\sqrt{2}$.

Thus, $(A_1,A_2,A_3)=\left(\dfrac{3}{2},\dfrac{7}{5},\dfrac{17}{12}\right)$ are the first three approximations of $\sqrt{2}$ generated by this continued fraction. Compute A_{10}.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2010 ANSWERS

Round 1 Alg 2: Algebraic Functions

A) $(-3, -3), (1, 1)$ B) $-\frac{13}{2}$ C) $\left(\frac{1}{8}, 0, -2, 0\right)$

Round 2 Arithmetic/ Number Theory

A) 23 B) 14, 15 and 23 C) 52

Round 3 Trig Identities and/or Inverse Functions

A) 150 B) $\frac{\pi}{12}, \frac{7\pi}{12}$ C) $\pm\frac{27}{125}$ (or ± 0.216)

Round 4 Alg 1: Word Problems

A) 10 B) 13 C) 151

Round 5 Geometry: Circles

A) 96° B) $4\sqrt{5}$ C) 339.25π (or $\frac{1357\pi}{4}$)

Round 6 Alg 2: Sequences and Series

A) $\frac{46}{63}$ B) $(31, 33, 15, -12)$ C) -1

Team Round

A) $\left(\frac{-16-\sqrt{157}}{3}, \frac{-27-\sqrt{157}}{6}\right)$ D) 7 [namely, 9 through 15 inclusive]

B) C E) $(16, 2, 64, 8)$

C) $-\frac{336}{625}$ (or -0.5376) F) $\frac{8119}{5741}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2010 SOLUTION KEY

Round 1

A) To determine f^{-1}, let $y = f(x)$, switch x and y and solve the resulting equation for y.

$$x = \frac{3}{y+2} \Rightarrow xy + 2x = 3 \Rightarrow y = \frac{3-2x}{x} \Rightarrow f^{-1}(x) = \frac{3-2x}{x}$$

Equating, $\frac{3}{x+2} = \frac{3-2x}{x} \Rightarrow 3x = (3-2x)(x+2) = 3x + 6 - 2x^2 - 4x$

$\Rightarrow 2x^2 + 4x - 6 = 2\left(x^2 + 2x - 3\right) = 2(x+3)(x-1) = 0 \Rightarrow x = -3, 1 \Rightarrow$ **(–3, –3), (1, 1)**

Alternate (better) solution:
$y = f(x)$ and $y = f^{-1}(x)$ always intersect along $y = x$. Thus, we may find the points of intersection without explicitly finding the inverse function! We must solve $\frac{3}{x+2} = x$.

Cross multiplying, $x(x + 2) - 3 = x^2 + 2x - 3 = (x + 3)(x - 1) = 0 \Rightarrow$ **(–3, –3), (1, 1)**

B) Can we avoid having to find the numerical value of the individual roots and evaluating the given tedious expression? YES. The answer is actually determined by just two of the coefficients!!
Since the lead coefficient is 2, the polynomial has factors of 2 and $(x - A)$, $(x - B)$, $(x - C)$, $(x - D)$.
Thus, $2x^4 - 7x^3 - 2x^2 + 13x + 6 = 0 = 2(x - A)(x - B)(x - C)(x - D)$.
Multiplying out these factors, there will be 16 terms, but many of them can be combined.
Convince yourself that:
Only 1 term contains x^4, 4 terms contain x^3, 6 terms contain x^2, 4 terms contain x and 1 term is a constant.
Specifically, the expansion is:

$$2\left(x^4 - (A+B+C+D)x^3 + (\underbrace{AB + AC + \ldots + CD}_{6\ pairs})x^2 - (\boldsymbol{ABC} + \boldsymbol{ABD} + \boldsymbol{ACD} + \boldsymbol{BCD})x + ABCD\right) = 0$$

Thus, $-2\left(ABC + ABD + ACD + BCD\right) = 13 \Rightarrow$ **–13/2**

Note: The required quantity was just _the opposite of the x-coefficient divided by the lead coefficient_.
The actual factorization is $(2x + 1)(x + 1)(x - 2)(x - 3) \Rightarrow$ roots: –1/2, –1, 2 and 3.

$$\left(-\frac{1}{2}\cdot -1\cdot 2\right) + \left(-\frac{1}{2}\cdot -1\cdot 3\right) + \left(-\frac{1}{2}\cdot 2\cdot 3\right) + \left(-1\cdot 2\cdot 3\right) = 1 + \frac{3}{2} - 3 - 6 = \frac{5-18}{2} = \boldsymbol{-\frac{13}{2}}$$

C) $f(-a) = -f(a) \Rightarrow f$ is an odd function $\Rightarrow B = D = 0$
4 is a zero $\Rightarrow f(4) = 64A + 4C = 0$ or $C = -16A$ (Condition #1)
$f(-3) + 2f(3) + f(5) = 3 \Rightarrow f(3) + f(5) = 3 \Rightarrow (27A + 3C) + (125A + 5C) = 3$

$\Rightarrow 152A + 8C = 3$ (Condition #2) Solving the system of equations, we have $(A, C) = \left(\frac{1}{8}, -2\right) \Rightarrow \boldsymbol{\left(\frac{1}{8}, 0, -2, 0\right)}$.

Alternate solution:

f is an odd function $\Rightarrow \begin{cases} \text{(1) Since 4 is a zero of the function, } -4 \text{ must be as well.} \\ \text{(2) } D \text{ must be zero} \\ \text{(3) 0 must also be a zero of the function} \end{cases}$

Thus, $f(x) = ax(x+4)(x-4) = a\left(x^3 - 16x\right)$ and $\begin{cases} f(-3) + 2f(3) + f(5) = 3 \\ f(-3) = -f(3) \end{cases} \Rightarrow f(5) = 3 - f(3)$

Substituting, $(125 - 80)a = 3 - (27 - 48)a \Rightarrow 24a = 3 \Rightarrow a = 1/8$ and the result follows.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2010 SOLUTION KEY

Round 2

A) The first number has 13 digits and the second number has 11 digits.
The product of an a-digit integer and a b-digit integer will have $a + b$ or $(a + b - 1)$ digits.
[Ex: (2-digit)(3-digit) - min (10)(100) = 1000 (4 digits)
- max $(99)(999) = (100 - 1)(1000 - 1) = 10^5 - 10^3 - 10^2 + 1$
100,000 – 1,000 – 100 + 1 = 98,901 (5 digits)]
Thus, the product P has either 23 or 24 digits. Since (348…)(285…) = 99180…, the product will not have the extra digit. The number of digits in P is **<u>23</u>**.
Using scientific notation, we can trap AB as follows:
$AB > 3.4(10^{12}) \text{ x } 2.8(10^{10}) = 9.52(10^{22})$ → 23 digits (lower bound)
$AB < 3.49(10^{12}) \text{ x } 2.86(10^{10}) = 9.9148(10^{22})$ → 23 digits (upper bound)
The actual numerical value is approx. $9.949626125 \text{ x } 10^{22}$.

B) $360 = 2^3 \cdot 3^2 \cdot 5$ → # divisors = (3 + 1)(2 + 1)(1 + 1) = 24
Clearly, 23, a prime with divisors of 1 and 23 satisfies this requirement.
All integers smaller than 11 or bigger than 23 will fail to satisfy this requirement.
We are left to test integers in the range [11, 22].
14 → 1 + 2 + 7 + 14 = 24
15 → 1 + 3 + 5 + 15 = 24
There are no others. It is left to you to verify this.

In all of these cases, the sum of the divisors can be computed by simply adding them all up. However, where there are several divisors, using the shortcut is preferable. Here it is.

If $N = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots$, where the p's denote the primes in the prime factorization of N,

then the sum of the divisors is given by $\dfrac{p_1^{e_1+1} - 1}{p_1 - 1} \cdot \dfrac{p_2^{e_2+1} - 1}{p_2 - 1} \cdot \ldots$

That is, for <u>each</u> prime factor,
- increase the power by 1
- raise the prime to this new power
- subtract 1 from the above result and from the prime itself
- divide the smaller difference into the larger difference

Then take the product of each of these quotients.

For example, to determine the sum of the divisors of 360 would require adding up 24 integers, a tedious task! The shortcut starts with the prime factorization given above: $360 = 2^3 \cdot 3^2 \cdot 5$

Then we compute $\left(\dfrac{2^4 - 1}{2 - 1}\right)\left(\dfrac{3^3 - 1}{3 - 1}\right)\left(\dfrac{5^2 - 1}{5 - 1}\right) = 15(13)(6) = 15(78) = 780 + 390 = 1170$

Check this result out by brute force or by writing a short computer program.

Round 2 – continued

C) Consider the following sets of 7 consecutive natural numbers

$\{1,2,\underline{3},4,5,\underline{6},\underline{7}\}$, $\{8,\underline{9},10,11,\underline{12},13,\underline{14}\}$, $\{\underline{15},16,17,\underline{18},19,20,\underline{21}\}$, $\{22,23,\underline{24},25,26,\underline{27},\underline{28}\},\ldots$

Each set contains exactly 4 natural numbers not divisible by either 3 or 7.

Thus, $4n \leq 258$ ➔ $n = 64$.

The largest number in the 64^{th} set will be 7(64) = 448 and we have counted 4(64) = 256 natural numbers not divisible by either 3 or 7. Examining 449, 450, 451, … for divisibility by 3, we see that 450 is a multiple of 3. Therefore, 451 is the natural number satisfying our requirements. Since $451 = 11^1 \cdot 41^1$ and 11 and 41 are prime, 451 has a total of 4 factors and only these two are prime. The required sum is **<u>52</u>**.

Round 3

A) The domain of Sin^{-1} and Cos^{-1} are $[-90°, 90°]$ and $[0°, 180°]$ respectively.

Thus, $Cos^{-1}\left(-\frac{1}{2}\right)$ denotes an angle in quadrant 2 whose cosine is $-\frac{1}{2}$, i.e. 120° and

$Sin^{-1}\left(-\frac{1}{2}\right)$ denotes an angle in quadrant 4 whose sine is $-\frac{1}{2}$, i.e. –30°.

Therefore, k = 120 – (–30) = **<u>150</u>**.

B) $\frac{1-2\sin^2 x}{\sin x \cos x} = 2\sqrt{3} \leftrightarrow \frac{1-2\sin^2 x}{2\sin x \cos x} = \frac{\cos 2x}{\sin 2x} = \cot 2x = \sqrt{3}$

➔ $2x = \frac{\pi}{6} + n\pi$ ➔ $x = \frac{\pi}{12} + \frac{n\pi}{2} = \frac{(6n+1)\pi}{12}$ ➔ $\underline{\boldsymbol{\frac{\pi}{12}, \frac{7\pi}{12}}}$

C) $x^{2/3} + y^{2/3} = \left(\cos^3 t\right)^{2/3} + \left(\sin^3 t\right)^{2/3} = \cos^2 t + \sin^2 t = 1$ ➔ $y = \left(1 - x^{2/3}\right)^{3/2}$

$x = \frac{64}{125}$ ➔ $y = \left(1 - \left(\left(\frac{64}{125}\right)^{\frac{1}{3}}\right)^2\right)^{\frac{3}{2}} = \left(1 - \frac{16}{25}\right)^{\frac{3}{2}} = \left(\frac{9}{25}\right)^{\frac{3}{2}} = \underline{\pm\boldsymbol{\frac{27}{125}}}$ (or **<u>±0.216</u>**)

Round 4

A) $-5W + 8C = 0$, $W + C = 26$ ➔ $5W + 5C = 130$

Adding, we have $13C = 130$ ➔ C = **<u>10</u>**

B) $x(x + 1) + 100 = (x + 4)(x + 1 + 3) = x^2 + 8x + 16$

➔ $x + 100 = 8x + 16$ ➔ $x = 12$ ➔ original # chairs/row = 12 + 1 = **<u>13</u>**

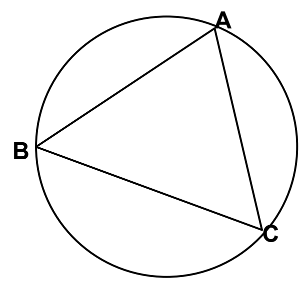
Round 4 - continued

C) Let (Tom, Sherry) = $(T, S) = (x + 18, x)$ denote their current ages.
87 years ago, $(T, S) = (x - 69, x - 87)$.
Let A and B be the digits used to denote their ages 87 years ago.

Now we have: $\begin{cases} x-69=10A+B \\ x-87=10B-A \end{cases}$ ➔ $9A - 9B = 18$ ➔ $A - B = 2$

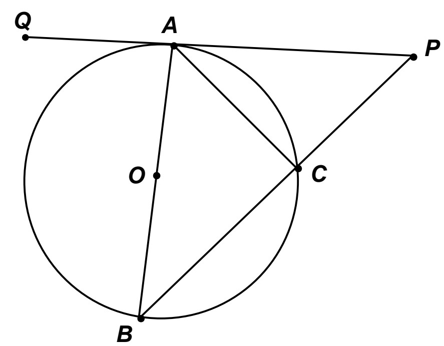
$(A, B) = (3, 1), (4, 2), (5, 3)$ – all rejected, but $(6, 4)$ ➔ $64 + 46 = 110$
Therefore, $x - 69 = 64$ ➔ $x = 133$ ➔ $T =$ **151**

Alternate (faster if not better) solution:
Examining two-digit integers with digits reversed, we quickly determine that Tom and Sherry must be 64 and 46 years old respectively (64 + 46 = 110). Adding 87 to Tom's age (the older sibling), we get his current age of **151**.



Round 5

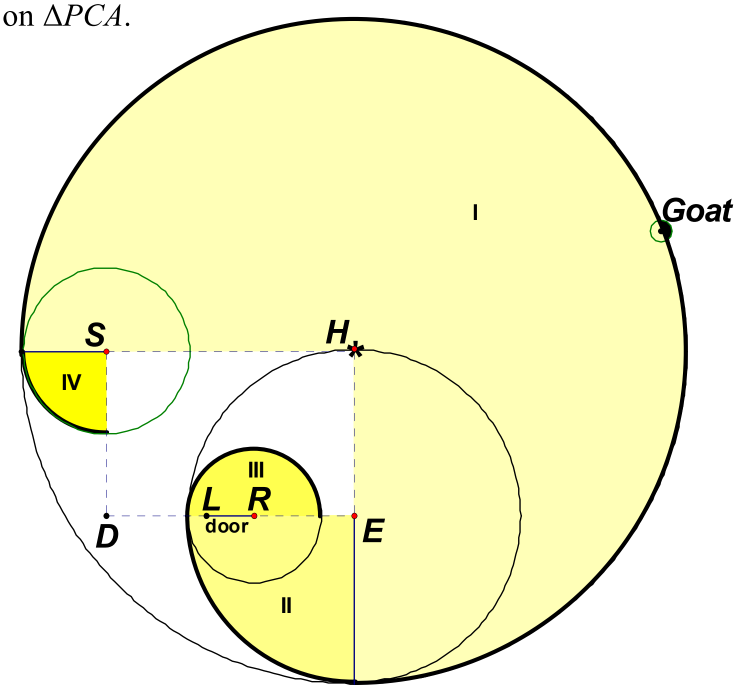
A) Since $\angle B$ is an inscribed angle,
$m\angle B = 4x + 2 = \frac{1}{2}(9x - 3)$ ➔ $x = 7$
and $m\angle C = 180 - (m\angle A + m\angle B) = 180 - 54 - 30 =$ **96**



B) Since $BP : BC = 9 : 5$, let $PC = 4x$ and $BC = 5x$.
Using the tangent-secant relationship, $(5x)(9x) = 12^2$.
$36x^2 = 144$ ➔ $x = 2$. A chord through the center of a circle must be a diameter; therefore ΔBAP is a right triangle.
$BP = 18$, $AP = 12$ ➔ $AB^2 = 324 - 144 = 180$
Since $\angle ACB$ is inscribed in a semicircle, it must be a right angle.
Thus, $AC^2 = AB^2 - BC^2 = 180 - 100 = 80$ ➔ $AC =$ $\mathbf{4\sqrt{5}}$
[Easier, use the Pythagorean Theorem directly on ΔPCA.
$AC^2 = 12^2 - 8^2 = 80$ ➔ $AC =$ $\mathbf{4\sqrt{5}}$.]

C) The region accessible to the goat is:
I: ¾ of a circle w/radius 20' (center at H)
II: ¼ of a circle w/radius 10' (center at E)
III: ½ of a circle w/radius 4' (center at R)
IV: ¼ of a circle w/radius 5' (center at S)

$$\frac{3}{4}400\pi + \frac{1}{4}100\pi + \frac{1}{2}16\pi + \frac{1}{4}25\pi$$
$= \pi(300 + 25 + 8 + 6.25)$
$=$ $\mathbf{339.25\pi}$ or $\mathbf{\frac{1357\pi}{4}}$



Round 6 - continued

A) Either by formula $a_n = \dfrac{(2n-1)(2n)}{(2n+1)(2n+2)} = \dfrac{n(2n-1)}{(n+1)(2n+1)}$ or by brute force we have

$a_{12} \cdot a_{13} = \dfrac{12 \cdot 23}{13 \cdot 25} \cdot \dfrac{13 \cdot 25}{14 \cdot 27} = \dfrac{12 \cdot 23}{14 \cdot 27} = \underline{\mathbf{\dfrac{46}{63}}}$

B) Given $a_1 = \left(3 \cdot 5,\ 2^1 \cdot 3^2\right)$, $a_2 = \left(5 \cdot 7,\ 2^2 \cdot 3^1\right)$, $a_3 = \left(7 \cdot 9,\ 2^3 \cdot 3^0\right)$, $a_4 = \left(9 \cdot 11,\ 2^4 \cdot 3^{-1}\right)$, ... ,

the general term is $a_n = \left((2n+1)(2n+3), 2^n \cdot 3^{3-n}\right)$.

Thus, $a_{15} = \left(31 \cdot 33, 2^{15} \cdot 3^{-12}\right)$ → (A, B, x, y) = **<u>(31, 33, 15, –12)</u>**.

C) $\dfrac{AB}{3AB} = \dfrac{3AB}{18A}$ → $A, B \neq 0$. Canceling, $\dfrac{1}{3} = \dfrac{B}{6}$ → $B = 2$

$A^3 - A - 2 - 1 = A + 3 - 2$ → $A^3 - 2A - 4 = (A-2)(A^2 + 2A + 2) = (A-2)\left((A+1)^2 + 1\right) = 0$

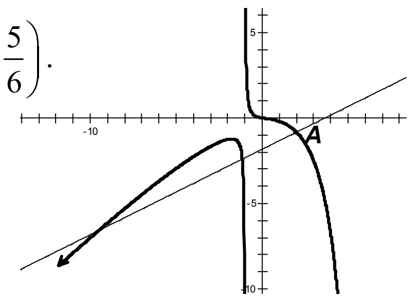
The only real solution is $A = 2$.

$(2 + 2i)^3 = 2^3(1 + i)^3 = 8(1 + i)^2(1 + i) = 8(2i)(1 + i) = -16 + 16i = C + Di$ → $\dfrac{C}{D} = \dfrac{-16}{16} = \underline{\mathbf{-1}}$

Alternate solution:
Once $A = B = 2$, $A + Bi$ in polar form is (2, 45°). Cubing this produces (8, 135°), so the real and imaginary parts are equal, but opposite in sign and the results above follow.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2010 SOLUTION KEY

Team Round



A) The graph of $f(x)=\dfrac{x^3+x}{x^2-5x-6}$ and $g(x)=\dfrac{3x-11}{6}$ intersect at $\left(2,-\dfrac{5}{6}\right)$.

Compute the coordinates (x, y) of the point of intersection furthest from the origin.

$$\frac{x^3+x}{x^2-5x-6}=\frac{3x-11}{6} \rightarrow 6\left(x^3+x\right)=\left(x^2-5x-6\right)(3x-11)$$

$$\rightarrow 6x^3+6x=3x^3-26x^2+37x+66 \rightarrow 6x^3+26x^2-31x-66=0$$

We know $x = 2$ is a solution and by synthetic substitution we have

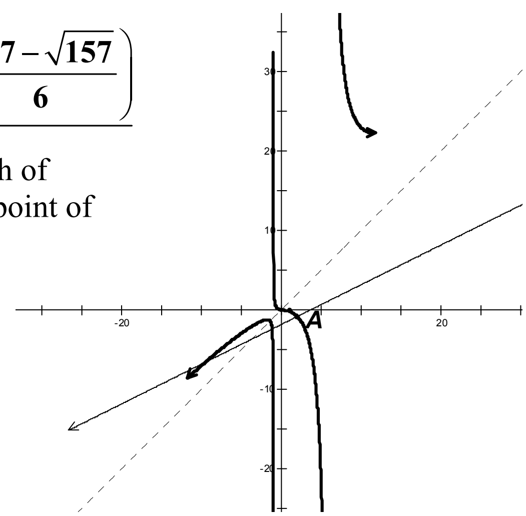
$$6x^3+26x^2-31x-66=(x-2)\left(3x^2+32x+33\right)=0$$

Applying the quadratic formula, $x=\dfrac{-32\pm\sqrt{32^2-12(33)}}{6}=\dfrac{-32\pm\sqrt{4(256-99)}}{6}=\dfrac{-16\pm\sqrt{157}}{3}$

The abscissa (i.e. the x-coordinate) of the point furthest from the origin is $\dfrac{-16-\sqrt{157}}{3}$.

Substituting in the linear function, we easily determine that the ordinate (i.e. the y-coordinate) is

$$\frac{3\left(\dfrac{-16-\sqrt{157}}{3}\right)-11}{6}=\frac{-27-\sqrt{157}}{6} \rightarrow \boldsymbol{\left(\frac{-16-\sqrt{157}}{3},\frac{-27-\sqrt{157}}{6}\right)}$$



Actually, if the window were expanded, there is a third branch of $y = f(x)$ for $x > 6$. How do we know that there is not another point of intersection which is even further from the origin?
The dotted line represents $y = x$.

$f(x)=\dfrac{x^3+x}{x^2-5x-6}$ may be rewritten as $\dfrac{x+\frac{1}{x}}{1-\frac{5}{x}-\frac{6}{x^2}}$.

As x gets larger $(\rightarrow+\infty)$, the fractions in the numerator and denominator approach 0 and $f(x)$ is approximated by

$y=\dfrac{x+0}{1-0-0}=x$. In other words, this branch is asymptotic to $y = x$.

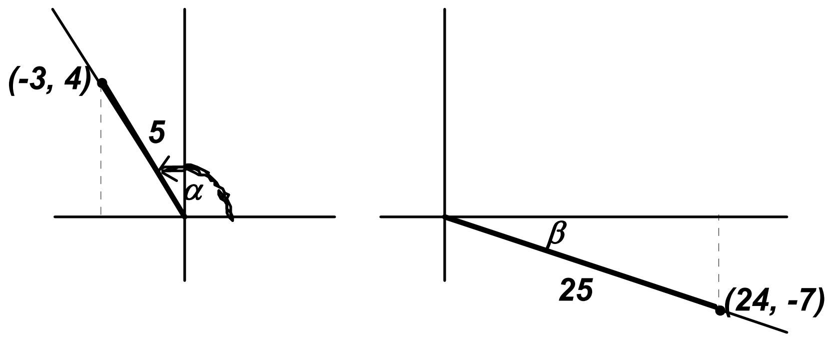
When is $\dfrac{x^3+x}{x^2-2x-6}>x$? $\dfrac{x^3+x-x\left(x^2-2x-6\right)}{x^2-2x-6}=\dfrac{2x^2+7x}{x^2-2x-6}=\dfrac{x(2x+7)}{(x-1)^2-7}>0$

The 4 critical points (-3.5, $1-\sqrt{7}$, 0 and $1+\sqrt{7}\approx 3.6$) divide the number line into 5 regions and the quotient is positive to the extreme left, extreme right and in the middle. Thus, for $x > 6$, $f(x) > x$ and $y = f(x)$ approaches $y = x$ (from above) and, therefore, will never cross $y = g(x)$.]

Team Round - continued

B) Raising A and B to the 2008*2009 power gives us $2008!^{2009}$ and $2009!^{2008}$ respectively.
Dividing by $2008!^{2008}$, we have 2008! and 2009^{2008}.
Observe that 2008! = 2008·2007· … ·2·1 (2008 factors), but $2009^{2008} = \underbrace{2009 \cdot 2009 \cdot \ldots \cdot 2009}_{\text{2008 factors}}$
Thus, $B = \sqrt[2009]{2009!}$ is larger. Similarly $C > B$ and we have **C** is the largest.

C) Let $\alpha = Arc\cos\left(-\frac{3}{5}\right)$ and $\beta = Arc\sin\left(-\frac{7}{25}\right)$. As indicated in the diagram below,
$90 < \alpha < 180$ (quadrant 2) and $-90 < \beta < 0$ (quadrant 4).



$$\cos\left(2Arc\cos\left(-\frac{3}{5}\right) + Arc\sin\left(-\frac{7}{25}\right)\right) = \cos(2\alpha+\beta) = \cos 2\alpha \cos\beta - \sin 2\alpha \sin\beta$$

$$= \left(1 - 2\sin^2\beta\right)\cos\beta - \left(2\sin\alpha\cos\beta\right)\sin\beta = \left(1 - 2\left(\frac{4}{5}\right)^2\right)\cdot\frac{24}{25} - 2\cdot\frac{4}{5}\cdot -\frac{3}{5}\cdot -\frac{7}{25}$$

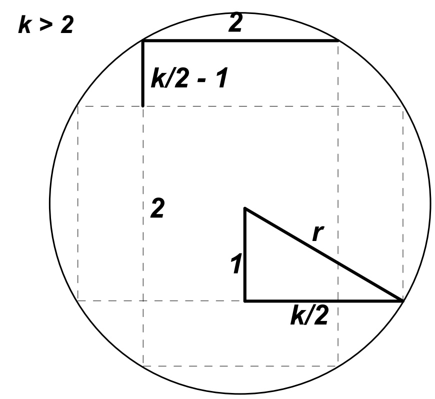
$$= \left(1 - \frac{32}{25}\right)\cdot\frac{24}{25} - \frac{24\cdot 7}{25^2} = -\frac{2\cdot 24\cdot 7}{25^2} = -\frac{336}{625}$$ (or **–0.5376**)

Team Round – continued

D) Let (A, 24, P) denote the current ages of Al, Sue and Pam respectively.
8 years ago, their ages were (A – 8, 16, P – 8) and (A – 8) + (P – 8) = 24 or $A + P = 40$.
Clearly, if A – 8 denotes a positive integer, the minimum value of A is 9.
Since Al is a younger brother, $A < 24$, but the second condition that Pam is an older sister puts a more restrictive condition on the maximum value of A.
Substituting $P = 40 – A$, we have $40 – A > 24$ or $A < 16$ and the maximum value of A is 15.
Thus, there are **7** possible values of A (15 – 9 + 1).

E) The area bounded by the overlapping rectangles: $4\left(2\left(\frac{k}{2}-1\right)\right)+4=4k-4=4(k-1)$

The area of the circle: $\pi r^2=\pi\left(1+\frac{k^2}{4}\right)$



$$4(k-1)=\frac{\pi r^2}{2} \Rightarrow 8(k-1)=\pi\left(1+\frac{k^2}{4}\right)$$

$$\Rightarrow \pi k^2-32k+4(\pi+8)=0 \Rightarrow k=\frac{32\pm\sqrt{32^2-16\pi(\pi+8)}}{2\pi}$$

$$=\frac{32\pm4\sqrt{64-\pi(\pi+8)}}{2\pi}=\frac{16\pm2\sqrt{64-\pi(\pi+8)}}{\pi}$$

The "+" sign gives the maximum value of k and we have
(A, B, C, D) = **(16, 2, 64, 8)**

For curiosity sake, $k \approx 8.521121922\cdots$ produces an area that is approximately half that of the circle. Check it out!
Using the "–" sign, $k \approx 1.664794436\cdots$ and must be rejected.
(You may want to verify that if $k < 2$, then $k=\frac{16-2\sqrt{64-\pi(\pi+8)}}{\pi+8}\approx 0.4694217542\cdots$.)

Team Round - continued

F) Note: $a_n = \frac{1}{2+a_{n-1}}$ So, rather than thinking of a_4 as $\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}}$,

we look at $a_4 = \frac{1}{2+a_3} = \cfrac{1}{2+\frac{5}{12}} = \frac{12}{29}$

Lining up the a-sequence evidence $\begin{matrix} 1 & 2 & 5 & 12 & X & Z \\ 2 & 5 & 12 & 29 & Y & T \end{matrix}$,we notice that $Y = Z$ and $T = X + 2Y$.

Therefore, the a-sequence continues $\frac{29}{2(29)+12} = \frac{29}{70}, \frac{70}{169}, \frac{169}{408}, \frac{408}{985}, \frac{985}{2378}, \frac{2378}{5741}$.

$a_{10} = \frac{2378}{5741}$ ➔ $\boldsymbol{A_{10}} = 1 + \frac{2378}{5741} = \mathbf{\frac{8119}{5741}}$

Note $\dfrac{1}{\sqrt{2}+1} = \sqrt{2}-1$

Therefore, the denominator of the continued fraction is equivalent to $\sqrt{2}+1$.
Then subtracting 1 from the terms in the sequence for the denominator gives the sequence for $\sqrt{2}$, byt with a first term of 1 instead of 3/2.
Letting $r = \sqrt{2}$, we have:

$$1 < r < \frac{3}{2},\ \frac{7}{5} < r < \frac{3}{2},\ \frac{7}{5} < r < \frac{17}{12},\ \frac{41}{29} < r < \frac{17}{12}$$

Here's another:
The fractions for $1+\sqrt{2}$ are: 2/1, 5/2, 12/5, 29/12, 70,79, …

Let $r = \sqrt{2}$, the formula $\dfrac{\left((1+r)^n - (1-r)^n\right)}{2r}$ gives the values 1, 2, 5, 12, 29, 70, …

I got this idea from the formula for the Fibonacci numbers.

Suppose we call this sequence $a(n)$ and we compute $\dfrac{a(n+1)}{a(n)} - 1$

What would the limit be?
Could it be $\sqrt{2}$???