MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2010
ROUND 1 ALG 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS

ANSWERS

A) ________________________________

B) ________________________________

C) (_________ , _________)

***** NO CALCULATORS IN THIS ROUND *****

A) Given: $\begin{cases} y = 17 - 2x \\ y = 3x + 2 \\ y = k \end{cases}$ Determine the value of k for which this system of lines determines a <u>minimum</u> number of points of intersection.

B) Find <u>all</u> ordered pairs (x, y) satisfying $\begin{cases} x - 2y = 1 \\ |x| + |y| = 1 \end{cases}$.

C) Connecting the points $A(3, 7)$, $B(-1, 2)$, $C(3x, -x)$ and $D(10, 1)$ you have an outline of the deck in my backyard. It's a nondescript convex quadrilateral and finding its area is baffling my builders. If $x > 0$ and A and C are opposite vertices, compute the coordinates of C so the area of my deck is 52 square units.

Note: The area of a triangle with vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) can be computed as $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2010
ROUND 2 ALG1: EXPONENTS AND RADICALS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

******* NO CALCULATORS IN THIS ROUND *******

A) Given $\sqrt{2^4\left(5^{-2}-13^{-2}\right)}=\dfrac{A}{B}$, a reduced fraction. Compute $A+B$.

Note: a^{-n} is equivalent to $\dfrac{1}{a^n}$

B) Compute $\left(\dfrac{2\sqrt{5}-\sqrt{10}}{\sqrt{10}-\sqrt{5}}\right)^2$

C) Given: $\sqrt{\dfrac{9}{16}-\dfrac{9}{25}}=\dfrac{3}{4}-\dfrac{3}{x}$ for some integer value of x.

Compute $\sqrt{\dfrac{2^x\cdot 4^{3x+4}}{8^{2x+2}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2010
ROUND 3 TRIGONOMETRY: ANYTHING

ANSWERS

A) ______________________________ °

B) ______________________________

C) ______________________________

***** NO CALCULATORS IN THIS ROUND *****

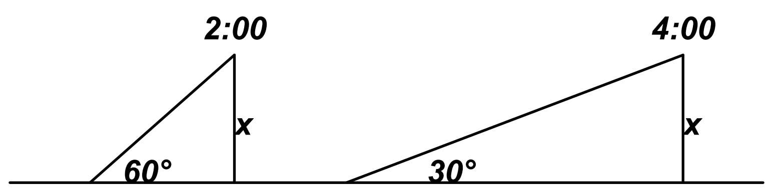
A) Solve for θ, $0° \le \theta < 360°$.

$$\sin\theta\cos\theta\tan(90°-\theta)\cot(90°+\theta)\sec(270°+\theta)\csc(270°-\theta)=\cot(180°-\theta)$$

B) Robert Wadlow, the world's tallest man, stands x feet tall. Your task is to compute x given the following information:

At 4:00 PM, Robert's shadow is 10 feet longer than his shadow at 2:00 PM.
At 4:00 PM the sun makes a 30° with the ground, but it made a 60° at 2:00 PM.

Recall: An exact answer is required. The given information is summarized in the diagram.



C) Given: $\begin{cases} \sin A = \dfrac{k}{\sqrt{41}} \\ \cos A = \dfrac{k+1}{\sqrt{41}} \end{cases}$ and $k > 0$. Compute $\sec\left(A-\dfrac{5\pi}{2}\right)\cdot\cos\left(A-5\pi\right)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2010
ROUND 4 ALG 1: ANYTHING

ANSWERS

A) ________________________________

B) __________________________ years

C) ________________________________

******* NO CALCULATORS IN THIS ROUND *******

A) Determine all possible positive integral values of x for which $236_8 = 134_x$.
The subscripts denote the base in which the numeral is written.

B) Alice and her mother have the same birthday (MM/DD).
Alice is 18 years younger than her mother.
When her mother was as old as Alice is now, her mother was twice as old as Alice was then.
In how many years from now will the ratio of Alice's age to her mother's age be 7 : 10?

C) Compute the two rational roots of the following quadratic:

$$4A(48A+1) = 5(16-201A).$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2010
ROUND 5 PLANE GEOMETRY: ANYTHING

ANSWERS

A) ________________________________

B) _______________ : _______________

C) ________________________________

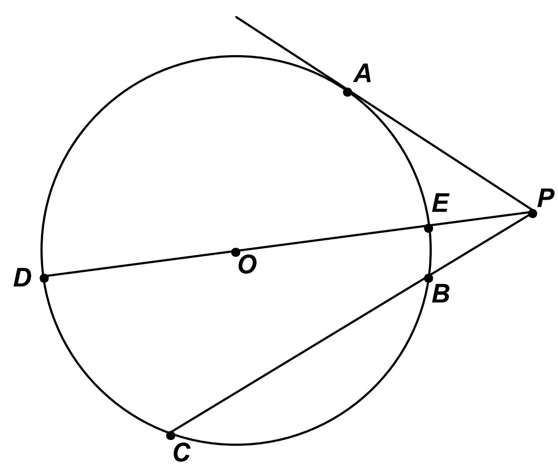
***** NO CALCULATORS IN THIS ROUND *****

A) Ten coplanar lines determine N points of intersection.
Compute the maximum value of N.

B) An equilateral triangle and a regular hexagon have equal perimeters.
What is the ratio of the area of the triangle to the hexagon?

C) Given: $\overrightarrow{PA}$ is tangent to circle O,
$PA = 14$,
$CB = 3PB$ and $EP = 2\sqrt{PB}$

Compute the area of circle O.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2010
ROUND 6 ALG 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) ______________________

B) ______________________

C) ______________________

******* NO CALCULATORS IN THIS ROUND *******

A) One container has 5 blue chips and 4 red chips, while another container has 4 blue chips and 5 red chips. One chip is chosen at random from each container. What is the probability that one chip is red and the other is blue?

B) Three mathletes A, B and C are simultaneously (but independently) trying to solve a difficult math problem. The probability of A correctly solving the problem is $\frac{1}{5}$ and the probability of B correctly solving the same problem is $\frac{1}{4}$.
Let P denote the probability of the problem being solved by at least one of the mathletes.
Let Q denote the probability that the problem is solved by exactly one of the mathletes.
If $P : Q = 8 : 5$, compute the probability that C solves the problem correctly.

C) The 5th term in the expansion of $\left(\frac{1}{2}x^2 + Ax^{-1}\right)^7$ is $\frac{70}{81}x^T$.
Compute <u>all</u> possible ordered pairs (A, T).
Note: The 1st term in the expansion is $\frac{x^{14}}{128}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2010
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ____________________________ D) (__________ , __________)

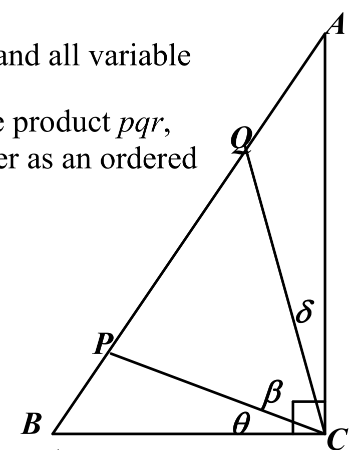
B) (______ , ______ , ______) E) ____________________________

C) ____________________________ F) ____________________________

******* NO CALCULATORS ARE PERMITTED IN THIS ROUND *******

A) Let functions *f* and *g* be defined by $f(x) = -k\,|\,x\,| + A$ and $g(x) = 4k\,|\,x\,| - 5B$.
If $k > 1$, $A = \begin{vmatrix} 1 & k \\ -k & 1 \end{vmatrix}$, $B = \begin{vmatrix} k & 1 \\ 1 & k \end{vmatrix}$ and the functions *f* and *g* have the same zeros, compute the area of the region bounded by *f* and *g*.

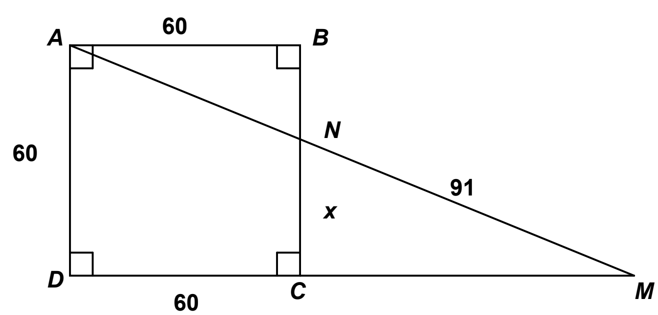
B) Suppose $\dfrac{\sqrt[4]{2 \cdot 3^a} \cdot \sqrt[6]{3 \cdot 2^b}}{\sqrt[3]{12}} = \sqrt[r]{2^p \cdot 3^q}$, each radical is in simplest form and all variable denote positive integers. Compute the number of distinct values of the product *pqr*, the maximum product and the minimum product. Express your answer as an ordered triple (count, max, min).



C) Points *P* and *Q* lie on the hypotenuse of right triangle *ABC*.
If $\sin\theta = \dfrac{3}{5}$ and $\sin\delta = \dfrac{7}{25}$, compute the value of $\sin\beta$.

D) Two bottles of equal volume contain rubbing alcohol and water. The ratios of alcohol to water in the two bottles are 3 : 1 and *A* : *B*, where *A* and *B* are relatively prime integers. The contents of the two bottles are mixed and the new ratio of alcohol to water is 27:13. Determine the ordered pair (*A*, *B*).

E) Given square *ABCD* below with sides as indicated. Compute *x*, the length of $\overline{NC}$. Diagram is <u>not</u> necessarily drawn to scale.



F) Let *N* be the positive difference between the two largest coefficients in the expansion of $(3a + 2b)^{11}$.
Determine the prime factorization of *N*.
When writing the product, <u>list the primes from smallest to largest</u>.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2010 ANSWERS

Round 1 Alg 2: Simultaneous Equations and Determinants

A) 11 B) $(1,0), \left(-\frac{1}{3}, -\frac{2}{3}\right)$ C) $(9, -3)$

Round 2 Alg 1: Exponents and Radicals

A) 113 B) 2 C) 64

Round 3 Trigonometry: Anything

A) $135°, 315°$ B) $5\sqrt{3}$ feet C) $-\frac{5}{4}$ (or -1.25)

Round 4 Alg 1: Anything

A) 11 B) 6 C) $-\frac{16}{3}, \frac{5}{64}$

Round 5 Plane Geometry: Anything

A) 45 B) 2 : 3 C) 252π

Round 6 Alg 2: Probability and the Binomial Theorem

A) $\frac{41}{81}$ B) $\frac{4}{5}$ (.8 or 0.8) C) $\left(\pm\frac{2}{3}, 2\right)$

Team Round

A) $\frac{500}{3}$ (or equivalent) D) (3, 2)

B) (9, 420, 12) E) 35

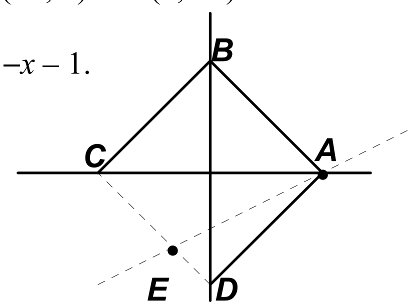
C) $\frac{3}{5}$ (.6 or 0.6) F) $2^5 \cdot 3^7 \cdot 11^1$ (the last exponent is not required)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2010 SOLUTION KEY

Round 1

A) If the three lines were parallel, then no points of intersection would be determined. This is not possible, since the first two lines have different slopes and the third line is horizontal. The (unique) minimum will occur when the horizontal line passes through the point of intersection of the first two lines. $17 - 2x = 3x + 2 \rightarrow x = 3 \rightarrow y = 11 \rightarrow k =$ <u>**11**</u>.

B) The 1st condition describes a line $y = \dfrac{-x+1}{2}$ (slope of $-1/2$ and an x-intercept of $(1, 0)$).
The 2nd condition describes square connecting (in order): $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$.
Clearly, <u>**(1, 0)**</u> is a solution.
The equation of the side of the square in quadrant 3 lies on the line $y = -x - 1$.
$\dfrac{x-1}{2} = -x - 1 \rightarrow x - 1 = -2x - 2 \rightarrow 3x = -1 \rightarrow \underline{\left(-\dfrac{1}{3}, -\dfrac{2}{3}\right)}$.



C) $Area(ABCD) = Area(\Delta ABD) + Area(\Delta BCD)$
Using the determinant method,

$$Area(ABCD) = \frac{1}{2}\begin{vmatrix} 3 & 7 & 1 \\ -1 & 2 & 1 \\ 10 & 1 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} -1 & 2 & 1 \\ 3x & -x & 1 \\ 10 & 1 & 1 \end{vmatrix} =$$

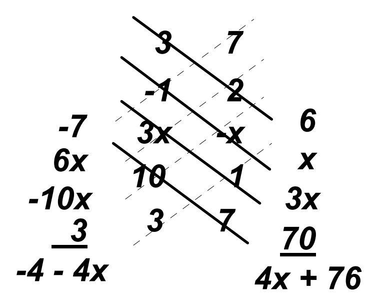
$$\frac{1}{2}\big((6-1+70)-(-7+20+3)\big)+\frac{1}{2}\big((x+3x+20)-(6x-10x-1)\big)=52$$

$\rightarrow \dfrac{1}{2}\big(59+(8x+21)\big)=52 \rightarrow 8x + 80 = 104 \rightarrow x = 3 \rightarrow$ <u>**(9, –3)**</u>.

Alternative #1:

The area is given by $\dfrac{1}{2}\begin{vmatrix} 3 & 7 \\ -1 & 2 \\ 3x & -x \\ 10 & 1 \\ 3 & 7 \end{vmatrix}$. This "determinant" is evaluated by the *lattice* method. Take the absolute value of the difference between

the sum of the diagonal down products and
the sum of the diagonal up products.



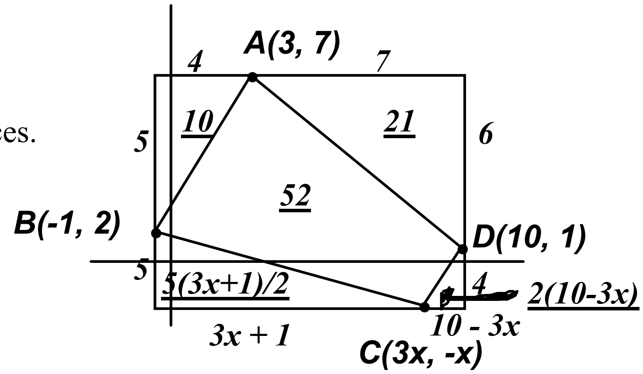
Thus, the area is $\dfrac{1}{2}\big(4x+76-(-4-4x)\big)=52 \rightarrow x = 3 \rightarrow$ <u>**(9, –3)**</u>.

This technique works for finding the area of any convex polygonal region, given given a clockwise or counterclockwise listing of the coordinates of its vertices, repeating the starting coordinates. See if it works for some concave regions! Does it work for all polygonal regions – convex or concave?
The simplicity and generality of this method is astounding.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2010 SOLUTION KEY

Round 1 - continued

Alternative #2 (Utilizing only basic area formulas)
Study the diagram at the right. Draw a pair of verticals and a pair of horizontals through opposite pairs of vertices.

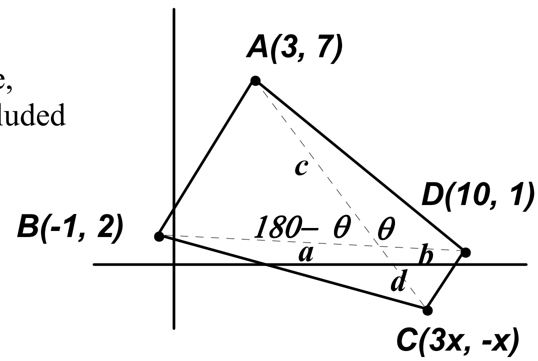


$$10+21+\frac{5(3x+1)}{2}+2(10-3x)+52=10(11)$$

➔ $\frac{5(3x+1)}{2}+2(10-3x)=110-83=27$

➔ $15x+5+40-12x=54$ ➔ $3x=9$ ➔ $x=3$ ➔ **<u>(9, –3)</u>**.

Alternate #3



Here is a (more involved) solution which utilizes the triangle area formula for any quadrilateral, convex or concave, namely half the product of the diagonals times the sine of the included angle. We start with an area formula for the triangle $\boxed{\frac{1}{2}ab\sin\theta}$, where θ denotes the included angle.

Since $\sin(180-\theta)=\sin\theta$, we have Area($ABCD$) =

$$\frac{1}{2}\sin\theta\left(ac+ad+bd+bc\right)=\frac{1}{2}\sin\theta\left(a(c+d)+b(c+d)\right)=$$

$$\frac{1}{2}\sin\theta\left((a+b)(c+d)\right)=\frac{1}{2}\sin\theta\cdot AC\cdot BD.$$ Let's show that (9, –3) produces an area of 52.

Using the distance formula, $AC=\sqrt{(-6)^2+10^2}=\sqrt{136}=2\sqrt{34}$ and $BD=\sqrt{11^2+(-1)^2}=\sqrt{122}$

The slope of $\overleftrightarrow{AC}=-\frac{5}{3}$ and the slope of $\overleftrightarrow{BD}=-\frac{1}{11}$.

We need the angle between the two lines. If θ denotes the acute angle, then $\tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|$.

In this case, $\tan\theta=\frac{-\frac{1}{11}+\frac{5}{3}}{1+\frac{1}{11}\cdot\frac{5}{3}}=\frac{-3+55}{33+5}=\frac{52}{38}=\frac{26}{19}$ ➔ $\sin\theta=\frac{26}{\sqrt{1037}}$ and the area is

$\frac{1}{2}\cdot\frac{26}{\sqrt{1037}}\cdot2\sqrt{34}\cdot\sqrt{122}$ But does this equal 52 ?????

$\frac{1}{2}\cdot\frac{26}{\sqrt{1037}}\cdot2\sqrt{34}\cdot\sqrt{122}=\frac{1}{2}\cdot\frac{26}{\sqrt{61}\cdot\sqrt{17}}\cdot2\left(\sqrt{2}\cdot\sqrt{17}\right)\cdot\left(\sqrt{2}\cdot\sqrt{61}\right)=13(4)=$ **<u>52</u>**.

Amazing (albeit tedious) solution. The basic area formulas above are worth remembering.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2010 SOLUTION KEY

Round 2

A) $\sqrt{2^{4}\left(5^{-2}-13^{-2}\right)}=\sqrt{2^4\left(\frac{1}{5^2}-\frac{1}{13^2}\right)}=\sqrt{2^4\left(\frac{13^2-5^2}{5^2\cdot 13^2}\right)}=\sqrt{2^4\left(\frac{12^2}{5^2\cdot 13^2}\right)}=\frac{4\cdot 12}{5\cdot 13}=\frac{48}{65}$

$\Rightarrow$ $A+B=$ **<u>113</u>**.

B) $\left(\frac{2\sqrt{5}-\sqrt{10}}{\sqrt{10}-\sqrt{5}}\right)^2=\frac{20-4\sqrt{50}+10}{10-2\sqrt{50}+5}=\frac{30-4\sqrt{50}}{15-2\sqrt{50}}=\frac{30-20\sqrt{2}}{15-10\sqrt{2}}=\frac{10\left(3-2\sqrt{2}\right)}{5\left(3-2\sqrt{2}\right)}=$ **<u>2</u>**.

C) First and foremost - $\sqrt{\frac{9}{16}-\frac{9}{25}}\neq\frac{3}{4}-\frac{3}{5}$!!! Squaring both sides,

$\sqrt{\frac{9}{16}-\frac{9}{25}}=\frac{3}{4}-\frac{3}{x}$ $\Rightarrow$ $\frac{9}{16}-\frac{9}{25}=\left(\frac{3}{4}-\frac{3}{x}\right)^2=\frac{9}{16}-2\left(\frac{3}{4}\right)\left(\frac{3}{x}\right)+\frac{9}{x^2}=\frac{9}{16}-\frac{9}{2x}+\frac{9}{x^2}$

$\Rightarrow$ $\frac{9(25-16)}{16(25)}=\frac{9}{16}-\frac{9}{2x}+\frac{9}{x^2}$ $\Rightarrow$ $\frac{9}{16(25)}=\frac{1}{16}-\frac{1}{2x}+\frac{1}{x^2}$ $\Rightarrow$

$\frac{9}{16(25)}-\frac{25}{16(25)}=-\frac{1}{25}=-\frac{1}{2x}+\frac{1}{x^2}$.

Multiplying through by $-50x^2$, $2x^2=25x-50$ $\Rightarrow$ $2x^2-25x+50=(2x-5)(x-10)=0$

$\Rightarrow$ $x=10$.

$\sqrt{\frac{2^x\cdot 4^{3x+4}}{8^{2x+2}}}=\sqrt{\frac{2^x\cdot 2^{6x+8}}{2^{6+6x}}}=\sqrt{2^{x+2}}$ Since $x=10$, we have $\sqrt{2^{12}}=2^6=$ **<u>64</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 3

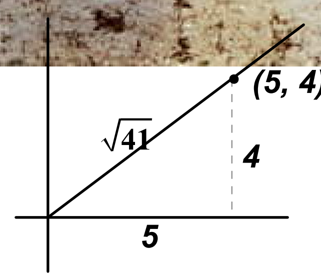
A) Rather than evaluating by direct substitution, let's simplify the expression and show that, if the expression is defined, the product P is invariant, i.e. has the same value, namely 1.
Using reduction formulas, θ is a reference value (angle) in quadrant 1 and when we see $(90 - \theta)$ – think Q1, $(90 + \theta)$ – think Q2, $(270 - \theta)$ – think Q3, $(270 + \theta)$ – think Q4
The mneumonic ASTC identifies the <u>sign</u> of the trig functions in quadrants 1 – 4.

Simplifying the lefthand side, we have $\sin\theta\cos\theta(\cot\theta)(-\tan\theta)(\csc\theta)(-\sec\theta)$
Regrouping, $(\sin\theta\cdot\csc\theta)(\cos\theta\cdot-\sec\theta)(\cot\theta\cdot-\tan\theta) = 1\cdot-1\cdot-1 = \underline{\mathbf{1}}$.
Thus, $\cot(180 - \theta) = -\cot(\theta) = 1$ ➔ θ belongs to 45° family in quadrants 2 and 4 ➔ <u>**135°, 315°**</u>.

As a 13 year old Boy Scout – 7' 4"

B) Let s denote the length of Shorty's shadow at 2:00. Then:
$\frac{x}{s} = \tan 60^\circ$ ➔ $x = s\sqrt{3}$.
At 4:00 $\frac{x}{s+10} = \tan 30^\circ = \frac{1}{\sqrt{3}}$ ➔ $x\sqrt{3} = s + 10$.
Multiplying by $\sqrt{3}$, $3x = s\sqrt{3} + 10\sqrt{3}$ ➔ $3x = x + 10\sqrt{3}$
➔ $x = \underline{\mathbf{5\sqrt{3}}}$.
Note: Wadlow was a real person and his actual maximum adult height was 8 feet 11.1 inch.
He died at the age of 22 and is acknowledged to be the modern world's tallest man.
Goto http://www.maniacworld.com/worlds_tallest_man.htm
to learn more or Google <u>Robert Wadlow</u>.

C) Since $\begin{cases}\sin A = \frac{k}{\sqrt{41}} \\ \cos A = \frac{k+1}{\sqrt{41}}\end{cases}$, $\frac{k^2}{41} + \frac{(k+1)^2}{41} = 1$.



$2k^2 + 2k = 40$ ➔ $k^2 - k - 20 = (k+5)(k-4) = 0$. $k > 0$ ➔ $k = 4$.
Simplifying, $\sec\left(A - \frac{5\pi}{2}\right)\cdot\cos(A - 5\pi) =$

$$\sec\left(A - \frac{\pi}{2}\right)\cdot\cos(A-\pi) = \sec\left(\frac{\pi}{2} - A\right)\cdot\cos(\pi - A) = \csc A\cdot-\cos A = -\frac{\cos A}{\sin A} = -\cot A \text{ ➔ } \underline{-\frac{5}{4}}.$$

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 4

A) $2(8^2)+3(8)+6=1x^2+3x+4$ ➔ $158 = x^2 + 3x + 4$ ➔ $x^2 + 3x - 154 = (x-11)(x+14) = 0$
➔ $x =$ **<u>11</u>**, $\cancel{-14}$.

B) Mom and Alice are now $x + 18$ and x years old respectively. Clearly, 18 years ago, Mom was as old as Alice is now and Alice was only $x - 18$ years old.
$x = 2(x - 18)$ ➔ $x = 36$. Thus, in n years the required ratio is satisfied, namely $\frac{36+n}{54+n}=\frac{7}{10}$
➔ $360 + 10n = 378 + 7n$ ➔ $n =$ **<u>6</u>**.

C) $4A(48A+1)=5(16-201A)$ ➔ $192A^2+4A=80-1005A$ ➔ $192A^2+1009A-80=0$
No doubt the coefficients are intimidating, BUT…
Since the roots are <u>rational</u>, this quadratic must factor.
How do we avoid tedious trial and error?
The key observations are:

$192=3\cdot 2^6$, $80=5\cdot 2^4$ (each with exactly one odd factor) and
the coefficient of the middle term of the trinomial is **odd**.

Since the <u>OI</u> in FOIL produces the middle term, we must have an odd outer product and an even inner product (or vice versa). Thus, the factorization must be $(3A + 16)(64A - 5) = 0$
➔ $A = \underline{\mathbf{-\frac{16}{3}, \frac{5}{64}}}$.

Round 5

A) Two coplanar lines determine at most one point of intersection.
A third line will determine at most two more, if it crosses both of the existing lines.
Likewise, each subsequent line adds a maximum number of new intersection points if it crosses each of the existing lines.
Thus, for 10 lines $N_{\max} = 1 + 2 + 3 + 4 + \ldots + 9 = 9(10)/2 =$ **<u>45</u>**.

B) Let p = perimeter.

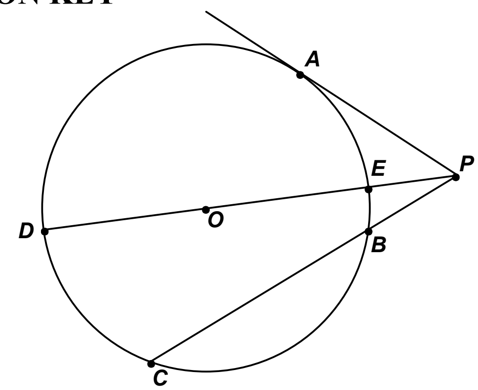
Area of triangle = ½(base)(height)
$$=\tfrac{1}{2}\left(\tfrac{p}{3}\right)\left(\tfrac{p}{3}\right)\tfrac{\sqrt{3}}{2}$$
$$=\frac{p^2\sqrt{3}}{36}$$

Area of hexagon = 6(1/2)(base)(height)
$$=6\left(\tfrac{1}{2}\right)\left(\tfrac{p}{6}\right)\left(\tfrac{p}{6}\right)\tfrac{\sqrt{3}}{2}$$
$$=\frac{p^2\sqrt{3}}{24}$$

Ratio of triangle to hexagon is **<u>2 : 3</u>**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2010 SOLUTION KEY

Round 5 - continued



C) Let $PB = x$ and $BC = 3x$. According to the tangent-secant relation, $PA^2 = PB(PC)$, i.e. outer(outer + inner)
$x(4x) = 14^2 \rightarrow x = 7 \rightarrow PE = 2\sqrt{7}$.
Let $OE = OD = r$. Then $14^2 = 2\sqrt{7}\left(2\sqrt{7}+2r\right)$.

$\rightarrow 196 = 28 + 4\sqrt{7}r \rightarrow r = \dfrac{168}{4\sqrt{7}} = \dfrac{42}{\sqrt{7}} = 6\sqrt{7}$.

Thus, the area of circle O is $\pi\left(6\sqrt{7}\right)^2$ = $\underline{\mathbf{252\pi}}$.

Round 6

A) The desired outcome could be either R_1B_2 or B_1R_2.
OR → ADD
Thus, the probability is $\dfrac{4}{9}\cdot\dfrac{4}{9} + \dfrac{5}{9}\cdot\dfrac{5}{9} = \dfrac{16+25}{81} = \underline{\mathbf{\dfrac{41}{81}}}$.

B) $P = 1 -$ P(none of them solve the problem) = 1– P(not A)·P(not B)·P(not C) = $1-\dfrac{4}{5}\cdot\dfrac{3}{4}\cdot(1-x)$

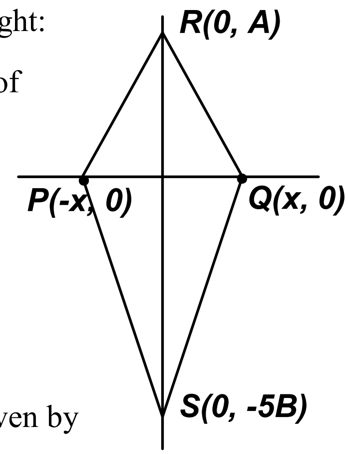
$= \dfrac{2+3x}{5} = \dfrac{8+12x}{20}$.
Q = P(A not$B$ not$C$ or not$A$ B not$C$ or not$A$ not$B$ C) =
$\dfrac{1}{5}\cdot\dfrac{3}{4}\cdot(1-x) + \dfrac{4}{5}\cdot\dfrac{1}{4}\cdot(1-x) + \dfrac{4}{5}\cdot\dfrac{3}{4}\cdot x = \dfrac{7}{20}(1-x) + \dfrac{12x}{20} = \dfrac{7+5x}{20}$.
Thus, $P : Q = 8 : 5 \rightarrow \dfrac{8+12x}{7+5x} = \dfrac{8}{5} \rightarrow 40 + 60x = 56 + 40x \rightarrow 20x = 16 \rightarrow x = \underline{\mathbf{\dfrac{4}{5}}}$.

C) The 5th term is $\binom{7}{4}\left(\dfrac{1}{2}x^2\right)^3\left(Ax^{-1}\right)^4 = \dfrac{7\cdot6\cdot5}{1\cdot2\cdot3}\cdot\dfrac{1}{8}\cdot A^4\cdot x^6\cdot x^{-4} = \dfrac{35A^4}{8}x^2 = \dfrac{70}{81}x^T$

$\rightarrow T = 2$ and $A^4 = \dfrac{16}{81} \rightarrow (A, T) = \underline{\left(\mathbf{\pm\dfrac{2}{3}, 2}\right)}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2010 SOLUTION KEY

Team Round



A) The graph of the bounded region is represented in the diagram at the right: $Q(x, 0)$ is $\left(\frac{A}{k},0\right)$ from f's point of view and $\left(\frac{5D}{4k},0\right)$ from g's point of view. Evaluating the determinants, equating and canceling out the common factors of k in the denominator,

$\frac{1+k^2}{k}=\frac{5\left(k^2-1\right)}{4k}$ ➔ $4 + 4k^2 = 5k^2 - 5$ ➔ $k^2 = 9$ ➔ $k = 3$.

$P\left(-\frac{10}{3},0\right)$, $Q\left(\frac{10}{3},0\right)$, $R(0,10)$ and $S(0,-40)$

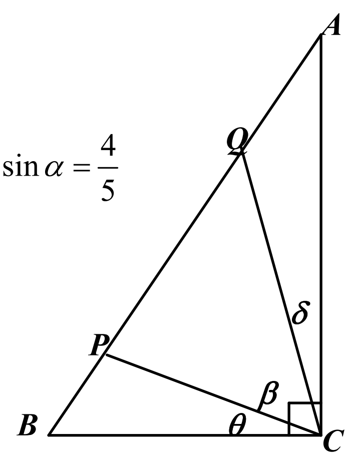
$PRQS$ is a quadrilateral w/perpendicular diagonals; hence its area is given by

$\frac{1}{2}d_1d_2$ ➔ $\frac{1}{2}PQ\cdot RS = \frac{1}{2}\left(\frac{20}{3}\right)(10-(-40))=\frac{10}{3}\cdot 50 = \underline{\boldsymbol{\frac{500}{3}}}$.

B) $\frac{\sqrt[4]{2\cdot 3^a}\cdot\sqrt[6]{3\cdot 2^b}}{\sqrt[3]{12}}=\frac{\left(2\cdot 3^a\right)^{\frac{1}{4}}\left(3\cdot 2^b\right)^{\frac{1}{6}}}{\left(2^2\cdot 3\right)^{\frac{1}{3}}}=\frac{\left(2\cdot 3^a\right)^{\frac{3}{12}}\left(3\cdot 2^b\right)^{\frac{2}{12}}}{\left(2^2\cdot 3\right)^{\frac{4}{12}}}=\left(\frac{2^3 3^{3a} 3^2 2^{2b}}{2^8 3^4}\right)^{\frac{1}{12}}=\left(2^{2b-5}3^{3a-2}\right)^{\frac{1}{12}}$

$=\sqrt[12]{2^{2b-5}\cdot 3^{3a-2}}$.

Since this expression must equal $\sqrt[r]{2^p 3^q}$, we know that $r = 12$, $p = 2b - 5 < 12$, $q = 3a - 2 < 12$. Since each of the original radicals was also in simplest form, $a = 1, 2, 3$ and $b = 1, 2, 3, 4, 5$. Thus, $p = 1, 3$ or 5 and $q = 1, 4, 7$.
Since $pqr = 12\,pq$, the minimum value occurs for $(p, q) = (1,1)$ and the maximum occurs for $(p, q) = (5, 7)$. (count, Max, min) = **__(9, 420, 12)__**.



C) Think θ: 3 – 4 – 5 and δ: 7 – 24 – 25
Let $\alpha = \theta + \delta$. Then

$\cos\alpha=\cos\left(\theta+\delta\right)=\cos\theta\cos\delta-\sin\theta\sin\delta=\frac{4}{5}\cdot\frac{24}{25}-\frac{3}{5}\cdot\frac{7}{25}=\frac{75}{125}=\frac{3}{5}$ ➔ $\sin\alpha=\frac{4}{5}$

$\sin\left(\beta+(\theta+\delta)\right)=\sin\left(\beta+\alpha\right)=\sin\beta\cos\alpha+\cos\beta\sin\alpha=\sin(90°)=1$

If $\sin\beta=k$, then $\cos\beta=\sqrt{1-k^2}$ ➔ $k\left(\frac{3}{5}\right)+\sqrt{1-k^2}\left(\frac{4}{5}\right)=1$

➔ $4\sqrt{1-k^2}=5-3k$ ➔ $16\left(1-k^2\right)=25-30k+9k^2$ ➔

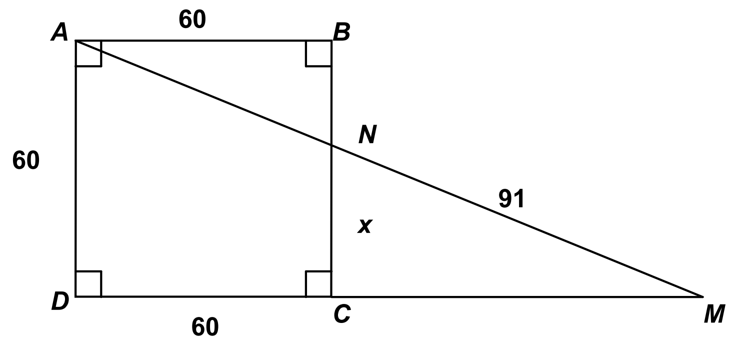
$25k^2-30k+9=(5k-3)^2=0$ ➔ $k=\sin\beta=\underline{\boldsymbol{\frac{3}{5}}}$.

Does this imply that $\overrightarrow{CP}$ actually bisects $\angle QCB$?

Team Round - continued

D) Bottle #1 contains 4 parts and bottle #2 contains ($A + B$) parts.
To insure equal volumes – multiply the bottle #1 ratio by ($A + B$) → $3(A + B) : (A + B)$ and the bottle #2 ratio by 4 → $4A : 4B$.

The alcohol : water ratio, after mixing, is $(7A + 3B) : (A + 5B) = 27 : 13$
→ $91A + 39B = 27A + 135B$ → $64A = 96B$ → $2A = 3B$.
The only relatively prime pair of values satisfying this equality is $(A, B) =$ **<u>(3, 2)</u>**.



E) Since $\Delta NBA \sim \Delta NCM$, we require $\dfrac{NC}{CM} = \dfrac{NB}{BA}$

Solution #1: (Banking on a hunch)
A triangle with sides 5 – 12 – 13 is a right triangle and since 91 = 7(13), we might guess that $NC = x = 7(5) = 35$, $CM = 7(12) = 84$, $\dfrac{NC}{CM} = \dfrac{5}{12}$

$NB = 60 - 35 = 25$ and $\dfrac{NB}{BA} = \dfrac{25}{60} = \dfrac{5}{12}$, so our hunch was correct, $x =$ **<u>35</u>**.

In the absence of a hunch, here are two analytical solutions.
Solution #2: (Thanks to Andrew Geng – Westford Academy/M.I.T.)
Let $y = 60 - x$. By applying the Pythagorean Theorem and similarity relations:

$$\frac{60^2 + y^2}{91^2} = \frac{y^2}{x^2} \rightarrow \left(60^2 + y^2\right)x^2 = 91^2 y^2 \quad (***)$$

Rewriting $60^2 + y^2$ as $\left(60 - y\right)^2 + 120y$ and noting that $60 - y = x$, we have $60^2 + y^2 = x^2 + 120y$.
This substitution will make it possible to reduce equation (***) to a quadratic.

$$\left(x^2 + 120y\right)x^2 = 91^2 y^2 \rightarrow x^4 + 120x^2 y - 91^2 y^2 = 0$$

The quadratic formula can be used to solve for x^2 (in terms of y) or alternately, the left side can be factored by noticing that 91^2 is the product of 7^2 and 13^2, which differ by 120.

$$x^2 = \frac{1}{2}\left(-120y \pm \sqrt{120^2 y^2 + 4 \cdot 91^2 y^2}\right) = -60y \pm \sqrt{60^2 y^2 + 91^2 y^2}$$

$$= -60y \pm \sqrt{11881y^2} = -60y \pm 109y = -49y, -169y$$

Since both x and y must be positive, $-169y$ can be discarded. Applying the substitution $y = 60 - x$ and using the quadratic formula (or factoring again) finishes the problem:

$$x^2 = 49(60 - x) \rightarrow x^2 + 49x - 49 \cdot 60 = 0$$

$$x = \frac{1}{2}\left(-49 \pm \sqrt{49^2 + 4 \cdot 49 \cdot 60}\right) = \frac{1}{2}\left(-49 \pm 7\sqrt{49 + 240}\right) = \frac{7}{2}\left(-7 \pm \sqrt{289}\right)$$

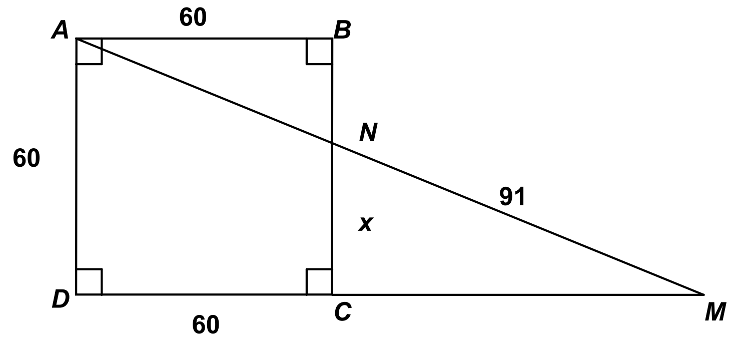
$$= \frac{7}{2}\left(-7 \pm 17\right) = \underline{\mathbf{35}}, \cancel{-84}$$

Team Round - continued

E) Solution #3 (Norm Swanson – Hamilton-Wenham/M.I.T.)

Since $\Delta ABN \sim \Delta MCN$, let $BN = kx$, then

$x + kx = 60 \rightarrow x = \dfrac{60}{k+1}$ and $CM = \dfrac{60}{k}$



By the Pythagorean Theorem, we have

$\left(\dfrac{60}{k}\right)^2 + \left(\dfrac{60}{k+1}\right)^2 = 91^2$.

Since 91 = 7(13), let $r = \dfrac{60}{7}$ and rewrite the equation as $\left(\dfrac{r}{k}\right)^2 + \left(\dfrac{r}{k+1}\right)^2 = 13^2$. Recalling that 5-12-13 is a pythagorean triple, we try $\dfrac{r}{k} = 12$ and $\dfrac{r}{k+1} = 5$ (the larger the denominator, the smaller the fraction).

Since $x = \dfrac{60}{k+1}$, solving for $k + 1$ will give us x.

$\dfrac{r}{k+1} = 5 \rightarrow \dfrac{60/7}{k+1} = 5 \rightarrow k+1 = \dfrac{12}{7}$. Thus, $x = \dfrac{60}{12/7} =$ **<u>35</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2010 SOLUTION KEY

Team Round - continued

F) Which coefficients are the largest coefficients is not so obvious.
How do we avoid calculating all the coefficients in order to decide?

The combinatorial coefficients are: **1 11 55 165 330 462** 462 330 165 55 55 1

The last six coefficients are identical to the first six since $\binom{11}{i} = \binom{11}{j}$ whenever $i,\ j \geq 0$ and $i + j = 11$.

Think of successive combinatorial coefficients in terms of multiplicative factors of

11, 5, 3, 2, 7/5, 1, etc.

Successive coefficients in the expansion of $(3a+2b)^{11}$ introduce one more factor of 2 and one less factor of 3, i.e. a multiplicative factor of 2/3.

Successive coefficients will continue to increase as long as the multiplicative factor is greater than 1.

Combining these multipliers we can determine which coefficient is the largest, without a great deal of tedious arithmetic.

The first coefficient is $\binom{11}{0}3^0 2^{11} = 2^{11}$.

The composite multipliers are:

11(2/3) = 22/3, 5(2/3) = 10/3, 3(2/3) = 2, 2(2/3) = 4/3, (7/5)(2/3) = 14/15

Specifically, the 5th term is 4/3 times the 4th, but the 6th term is only 14/15 times the 5th.
Therefore, the 5th and 6th terms are the two largest coefficients.

$c_5 = \binom{11}{4}(3)^7(2)^4 = 330 \cdot (3)^7(2)^4$ and $c_6 = \binom{11}{5}(3)^6(2)^5 = 462 \cdot (3)^6(2)^5$

→ $c_5 - c_6 = 2^4 \cdot 3^6 (330 \cdot 3 - 462 \cdot 2) = 2^4 \cdot 3^6 (990 - 924) = 2^4 \cdot 3^6 \cdot 66 = \underline{\mathbf{2^5 \cdot 3^7 \cdot 11}}$

Notice we did not bother to find the numerical value of either c_5 or c_6 or their difference.