MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 – OCTOBER 2010
ROUND 1 VOLUME & SURFACES

ANSWERS

A) ______________________ units^3

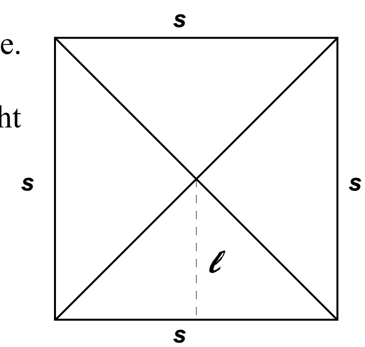
B) ____________ : ____________

C) ______________________

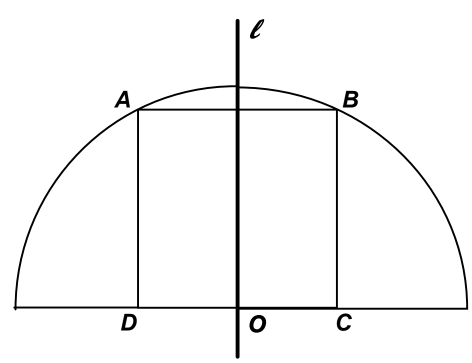
***** NO CALCULATORS ON THIS ROUND ****

A) The lengths of the edges of a rectangular solid are in a 1 : 2 : 3 ratio. If the total surface area of this solid is 198 units^2, compute its volume?

B) At the right is an aerial (top) view of a right pyramid with a square base. The ratio of its lateral surface area to its total surface area is 5 : 8. Compute the ratio of the length of an edge of the base to the slant height of the pyramid, namely $s : l$.



C) In the diagram at the right, a semicircle and a rectangle are rotated around line l which is perpendicular to the diameter of the semi-circle at the center point O, creating a hemisphere and a cylinder. A and B lie on the semicircle and C and D lie on the diameter. If the radius of the semicircle is 10, compute r, the radius of the cylinder, given that <u>the height of the cylinder is an integer</u>, such that the ratio of the volume of the cylinder to the volume of the hemisphere is 9 : 16.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010
ROUND 2 PYTHAGOREAN RELATIONS IN RECTILINEAR FIGURES

ANSWERS

A) ________________________________

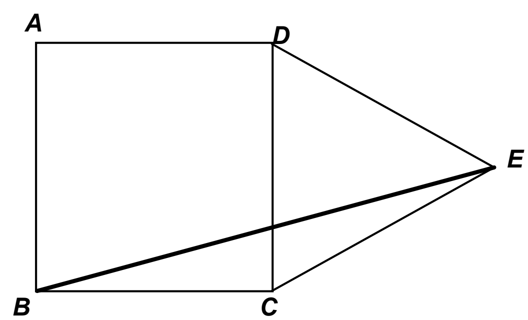
B) ________________________________

C) ________________________________

******* NO CALCULATORS ON THIS ROUND ******

A) In right triangle ABC, where $\angle C$ is <u>not</u> a right angle, $AB = 9$ and $BC = 13$.
Compute <u>all</u> possible values of AC.

B) $ABCD$ is a square, ΔCDE is isosceles, where $\angle E$ is the vertex angle, $AB = 16$ and $CE = 17$.
Express BE as a <u>simplified</u> radical.



C) In right triangle ABC, $\overline{AC}$ is the hypotenuse, $AB = 2\sqrt{6}$ and $BC = 5\sqrt{3}$.

The length of the altitude to the hypotenuse, in simplest form, may be expressed as $\dfrac{x\sqrt{y}}{z}$,

where x, y and z are positive integers..
Compute $x + y + z$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) ______________________________ lbs.

B) ___________________________ units^2

C) _________________________________

***** NO CALCULATORS ON THIS ROUND ****

A) 8 lbs of a mixture of grass seed and lime is 72% lime. A second mixture of grass seed and lime is 57% lime. How many lbs of the second mixture when combined with the first mixture will produce a mixture that is 65% lime?

B) Given: $L = \left\{ (x, y) : \frac{x}{A} + \frac{y}{B} = 1 \right\}$

If $\begin{cases} A + B = 127 \\ A - B = 7 \end{cases}$, compute the area of the region bounded by L, the vertical line $x = 0$ and the horizontal line $y = 0$.

C) Given: $\frac{1}{2} y = \frac{2}{3} x + \frac{3}{5}$

Solving for x in terms of y, we get $x = \frac{Ay + B}{C}$, where $A < 0$ and A, B and C are integers.

Compute $\frac{ABC}{AB + AC}$

Created with nitroPDF professional
download the free trial online at nitropdf.com/professional

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

ANSWERS

A) ______________________________

B) (___________ , ___________)

C) __________________________ mph

***** NO CALCULATORS ON THIS ROUND ****

A) Express the average of $\frac{1}{4}$, $1\frac{1}{5}$ and 1.5 as a simplified ratio of integers.

B) Let $a \oplus b = \dfrac{a+2b}{2a-b}$. Compute the ordered pair (a, b) for which $a \oplus b = 2$ and $a - b = 2$.

C) During rush hour a Microsoft employee averages only 40 mph on the ride from home to office. After a long day at the office, he returns home late at night over the same route. What average speed (in mph) on his return trip insures that his overall average speed is 55 mph, assuming he does not stop and is not stopped for speeding?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

ANSWERS

A) _______________________________

B) _______________________________

C) _______________________________

******* NO CALCULATORS ON THIS ROUND ******

A) Let N be the solution of $x + 2011 = |x|$.
Compute the largest <u>integer</u> less than or equal to N.

B) Solve for x: $\left|x^2 - 3\right| < 2$

C) For how many lattice points in the region $|x| + |y| \le 2010$ is it true that both $|x|$ and $|y|$ are prime factors of 2010?

Recall: $P(x, y)$ is a lattice point if and only if x and y are integers.

Created with nitroPDF professional
download the free trial online at nitropdf.com/professional

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010
ROUND 6 ALG 1: EVALUATIONS

ANSWERS

A) ______________________________ 3

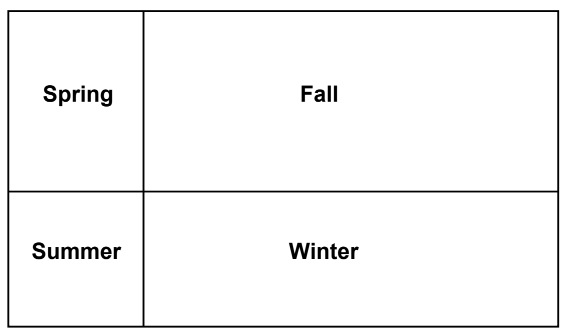
B) ______________________________

C) ______________________________

***** NO CALCULATORS ON THIS ROUND ****

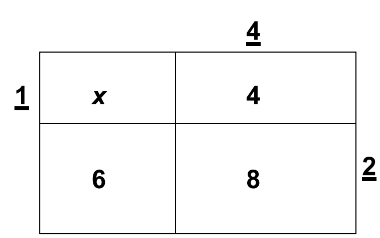
A) My lucky number expressed <u>in base 2</u> is 1101.
Determine <u>the base 3</u> representation of my lucky number.

B) Given: $a*b=a^b$, $a \circ b=(a+b)^2$ and $a\$b=\sqrt{ab}$
Compute: $\left((2*3)\circ 4\right)\$\left((4\circ 3)*2\right)$



C) An ad campaign consists of pictures embedded in a rectangle which is subdivided into 4 smaller rectangles <u>with integer dimensions</u> by a pair of perpendicular lines passing through its interior. Each of the 4 smaller rectangles represents a season. In the diagram at the right, the smaller rectangles are not necessarily drawn to scale.
An ad for Denver, CO might feature skiing in the winter rectangle and make it the largest. An ad for Washington DC might feature cherry blossoms in springtime and make the spring the largest rectangle and so on.

Suppose the areas of the four smaller rectangles are 4, 6, 8 and x in some order. There are exactly <u>two</u> possible values of x.
Arranging the areas of the subregions as indicated in the diagram at the right, x would equal 3. Determine the other possible value of x.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010
ROUND 7 TEAM QUESTIONS
ANSWERS

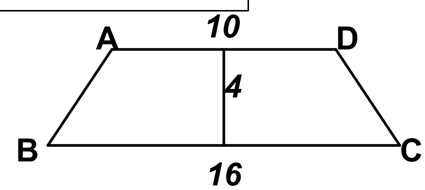
A) ____________________________ D) (___________ , ___________)

B) ____________________________ E) ____________________________

C) (___________ , ___________) F) ____________________________

***** NO CALCULATORS ON THIS ROUND ****

A) The following diagram is the cross-section of the frustum of a right circular cone. Compute the total surface area.



B) Right triangle ABC has legs 6 and 8. The altitude to the hypotenuse, the median to the hypotenuse and the hypotenuse bound a triangular region R. Express the ratio of the area of R to the area of ABC as the quotient of relatively prime integers.

C) Rowing upstream it took 4 hours to paddle 12 miles. Downstream, the same distance took an hour and a half. Under the same conditions, it would take A minutes and B seconds to paddle one mile downstream. If $B < 60$, compute (A, B).

D) $\frac{n}{2}$ is subtracted from the numerator and denominator of $\frac{22}{7}$ producing a positive integer. Let L and S denote the largest and smallest positive integer values of n for which this is possible. Determine the ordered pair (L, S).

E) Solve for x over the real numbers: $\sqrt{5x+9}+\sqrt{8x+17}=2$

F) The MML Contest Director lives in Fremont, NH, formerly called POPLIN.
There are 359 different anagrams (rearrangements) of the letters in the word POPLIN.
[$\frac{6!}{2!}-1=\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1}-1=360-1=359$. Obviously, POPLIN is not an anagram of itself and, therefore 1 was subtracted from the total.]

How many anagrams are there where the Ps are not consecutive <u>and</u> the vowels are not consecutive?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010 ANSWERS

Round 1 Geometry Volumes and Surfaces

A) 162 units^3 B) 6 : 5 C) $5\sqrt{3}$

Round 2 Pythagorean Relations

A) $5\sqrt{10}$, $2\sqrt{22}$ B) $5\sqrt{41}$ C) 43

Round 3 Linear Equations

A) 7 lbs B) 2010 C) 180

Round 4 Fraction & Mixed numbers

A) $\frac{59}{60}$ B) (8, 6) C) 88 mph

Round 5 Absolute value & Inequalities

A) -1006 B) $-\sqrt{5} < x < -1$ or $1 < x < \sqrt{5}$ C) 64

Round 6 Evaluations

A) 111 B) 588 C) 12

Team Round

A) 154π D) (13, 4)

B) $\frac{7}{50}$ E) $-\frac{16}{9}$ (8 is extraneous)

C) (7, 30) F) 167

Created with nitroPDF professional
download the free trial online at nitropdf.com/professional

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010 SOLUTION KEY

Round 1

A) Draw a diagram and label the edges of the solid x, $2x$ and $3x$. Then the total surface area of the solid is given by $2\left(2x^2+3x^2+6x^2\right)=198$ → $x = 3$.

Thus, the volume is $6x^3 = 6(3)^3 =$ **<u>162</u>** units^3.

B) $$\frac{4\left(\frac{1}{2}ls\right)}{4\left(\frac{1}{2}ls\right)+s^2}=\frac{5}{8} \rightarrow \frac{2ls}{2ls+s^2}=\frac{2l}{2l+s}=\frac{5}{8} \rightarrow 16l = 10l + 5s \rightarrow 6l = 5s \rightarrow s : l = \underline{\mathbf{6:5}}$$

C) If h denotes the height of the cylinder, then $r^2 = 100 - h^2$.

Additionally, $\dfrac{\pi r^2 h}{\frac{2}{3}\pi(10)^3}=\dfrac{3r^2h}{2(10)^3}=\dfrac{9}{16}$ → $r^2h = 3(5)^3 = 375$ (***)

Substituting for r^2, $r^2h=\left(100-h^2\right)h=375$ → $h^3-100h+375=0$.

Since $h < 10$ and an integer, we note that $h = 1, 2, 3$ and 4 must be rejected (incorrect units digits) and, trying $h = 5$, we immediately see that it works. [125 – 500 + 375 = 0]

Therefore, substituting in (***), $r^2 = 75$ → $r = \underline{\mathbf{5\sqrt{3}}}$

[Aside: $h^3-100h+375=(h-5)(h^2+5h-375)=0$

The quadratic factor gives additional values of $\dfrac{5\left(1\pm\sqrt{13}\right)}{2}$, but neither is an integer.]

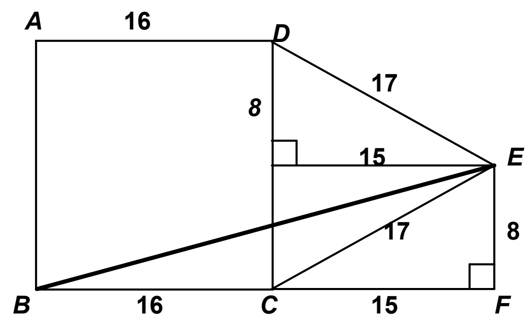
Created with nitroPDF professional
download the free trial online at nitropdf.com/professional

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010 SOLUTION KEY

Round 2

A) $\overline{AB}$ is not the hypotenuse, but either of the other two sides could be.
Thus, either $AC^2 = 81 + 169$ or $169 - 81$ ➔ 250 or 88
$AC = \underline{\mathbf{5\sqrt{10}}}$, $\underline{\mathbf{2\sqrt{22}}}$

B) Method #1:
Noting the 8 – 15 – 17 special right triangle and using right triangle ΔBEF, we have
$BE^2 = 31^2 + 8^2 = 961 + 64 = 1025 = 25(41)$
➔ $BE = \underline{\mathbf{5\sqrt{41}}}$



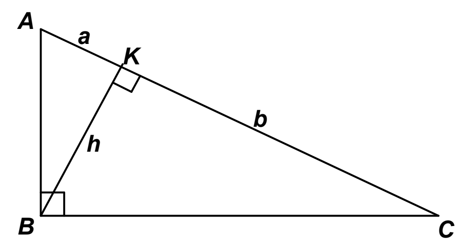
Method #2:
$\cos(\angle ECF) = 15/17$ ➔ $\cos(\angle ECB) = -15/17$
By the law of cosines, $BE^2 = 16^2 + 17^2 - 2(16)(17)(-15/17) = 256 + 289 + 480 = 1025 = 25(41)$.
➔ $BE = \underline{\mathbf{5\sqrt{41}}}$

C) $AC^2 = \left(2\sqrt{6}\right)^2 + \left(5\sqrt{3}\right)^2 = 24 + 75 = 99$ ➔ $AC = 3\sqrt{11}$.

The area of $\Delta ABC = \frac{1}{2}\left(2\sqrt{6}\right)\left(5\sqrt{3}\right) = \frac{1}{2}\left(3\sqrt{11}\right)h$ ➔ $10\sqrt{18} = 3h\sqrt{11}$ ➔ $10\sqrt{2} = h\sqrt{11}$

➔ $h = \frac{10\sqrt{2}}{\sqrt{11}} = \frac{10\sqrt{22}}{11}$ ➔ $x + y + z = \underline{\mathbf{43}}$.

Alternate Solution (Tuan Le)



$AC^2 = \left(2\sqrt{6}\right)^2 + \left(5\sqrt{3}\right)^2 = 24 + 75 = 99$ ➔ $AC = a + b = 3\sqrt{11}$.

$AB^2 = 24 = h^2 + a^2$

$BC^2 = 75 = h^2 + b^2$ (***)

Subtracting, $51 = b^2 - a^2 = (b+a)(b-a) = 3\sqrt{11}(b-a)$

➔ $b - a = \frac{17}{\sqrt{11}}$.

Solving simultaneously for b, $b = \frac{25}{\sqrt{11}}$.

Substituting in (***), $h^2 = 75 - \left(\frac{25}{\sqrt{11}}\right)^2 = 75 - \frac{25^2}{11} = \frac{825 - 625}{11} = \frac{2\left(10^2\right)11}{11^2}$ ➔ $h = \frac{10\sqrt{22}}{11}$

➔ $x + y + z = \underline{\mathbf{43}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010 SOLUTION KEY

Round 3

A) $(0.72)(8) + 0.57x = 0.65(x + 8)$ ➔ $72(8) + 57x = 65x + 65(8)$ ➔ $8x = (72 - 65)8$ ➔ $x = \underline{\mathbf{7}}$

B) Solving the system of equations, $(A, B) = (67, 60)$.
L is a linear function (a line) given in intercept-intercept form.
The x-intercept is $(67, 0)$ and the y-intercept is $(0, 60)$.
Thus, the area of the triangular region is $\frac{1}{2}\cdot 67\cdot 60 = 67(30) = \underline{\mathbf{2010}}$.

C) $30\left(\frac{1}{2}y = \frac{2}{3}x + \frac{3}{5}\right) \to 15y = 20x + 18 \to 20x = 15y - 18 \to x = \frac{15y-18}{20}$ ➔ $(A, B, C) = (15, -18, 20)$
However, since $A < 0$, we must multiply through by -1. ➔ $(A, B, C) = (-15, +18, -20)$
$\frac{ABC}{AB+AC} = \frac{ABC}{A(B+C)} = \frac{BC}{B+C}$ (since $A \neq 0$) ➔ $\frac{18(-20)}{18+(-20)} = 18(10) = \underline{\mathbf{180}}$

Round 4

A) The average is $\frac{\frac{1}{4}+\frac{6}{5}+\frac{3}{2}}{3}\cdot\frac{20}{20} = \frac{5+24+30}{60} = \underline{\mathbf{\frac{59}{60}}}$.

B) $a \oplus b = \frac{a+2b}{2a-b} = 2$ ➔ $4a - 2b = a + 2b$ ➔ $3a = 4b$
Substituting $a = b + 2$, we have $3(b + 2) = 4b$ ➔ $b = 6$ ➔ $(a, b) = \underline{\mathbf{(8, 6)}}$.

C) Let d denote the distance between home and office and r the average return rate in mph.
The overall average is the total distance traveled divided by the total time required.
The total distance traveled is $2d$ and the time required is the sum of the time going and the time returning, i.e. $\frac{d}{40}+\frac{d}{r}$. Therefore, $\frac{2d}{\frac{d}{40}+\frac{d}{r}} = 55 \to \frac{2}{\frac{1}{40}+\frac{1}{r}} = 55 \to \frac{80r}{40+r} = \frac{55}{1} \to \frac{16r}{40+r} = \frac{11}{1}$.
Cross multiplying, $440 + 11r = 16r$ ➔ $5r = 440$ ➔ $r = \underline{\mathbf{88}}$

You might wonder why a simple average $\frac{40+r}{2} = 55 \to r = 70$ is incorrect.
A simple average assumes that you have traveled at each of these speeds for the same time. Clearly, this is not the case, since returning home over the same route at a faster speed will take less time. The overall average of 40mph and 70 mph would be closer to 40 than 70, i.e. less than 55, since you traveled at 40 mph for a longer time. The required average is a weighted average and is given by the formula $\frac{2r_1r_2}{r_1+r_2}$ It's called a harmonic average. Check it out.

$$\frac{2\cdot 40\cdot 88}{40+88} = \frac{80\cdot 88}{128} = \frac{\cancel{10}^{5}\cdot\cancel{88}^{11}}{\cancel{16}} = 55$$

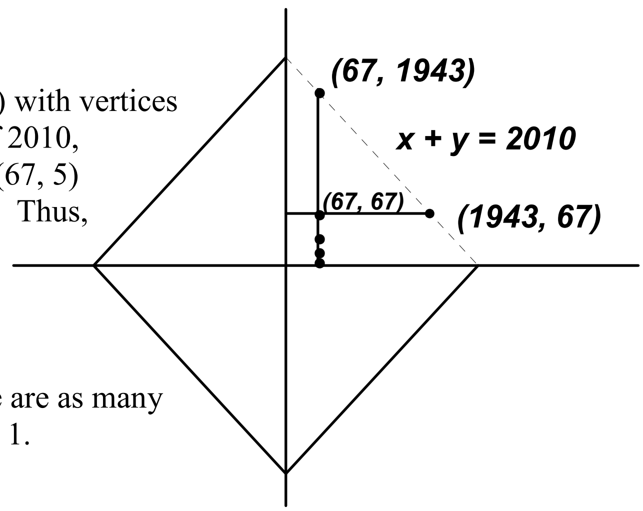
Created with nitroPDF professional
download the free trial online at nitropdf.com/professional

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010 SOLUTION KEY

Round 5

A) Clearly, for $x > 0$ there is no solution.
For $x < 0$, the equation is equivalent to $x + 2011 = -x$.
Thus, $2x = -2011 \rightarrow x = -1005.5 \rightarrow N = \underline{\mathbf{-1006}}$

B) $|x^2 - 3| < 2 \rightarrow -2 < x^2 - 3 < +2 \rightarrow x^2 - 3 > -2$ and $x^2 - 3 < +2$
$\rightarrow x^2 - 1 > 0$ and $x^2 - 5 < 0 \rightarrow$ outside ± 1 and inside $\pm\sqrt{5}$
i.e. $\underline{\mathbf{-\sqrt{5} < x < -1}}$ or $\underline{\mathbf{1 < x < \sqrt{5}}}$



C) The given inequality defines a diamond (actually a square) with vertices at $(\pm 2010, 0)$ and $(0, \pm 2010)$. There are 4 prime factors of 2010, namely 2, 3, 5 and 67. Clearly, the points (67, 2), (67, 3), (67, 5) and (67, 67) are in the first quadrant and inside the region. Thus, there are 4(4) = 16 points in quadrant 1 satisfying the requirements and **64** in total.

Note: If (x, y) is a solution of $|x| + |y| \le 2010$, $(-x, y)$, $(x, -y)$ and $(-x, -y)$ are also solutions; hence, there are as many solutions in quadrants 2, 3 and 4 as there were in quadrant 1.

Round 6

A) The decimal equivalent of my lucky number is computed as follows:
$$1101_2 = 1(2^3) + 1(2^2) + 0(2^1) + 1(2^0) = 8 + 4 + 1 = 13_{10}$$
Converting to base 3, $13 = 9 + 3 + 1 = 1(3^2) + 1(3^1) + 1(3^0) = \underline{\mathbf{111_3}}$

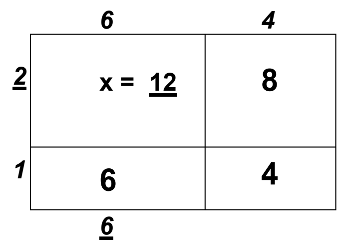
B) $((2*3)\circ 4) = 2^3 \circ 4 = 8 \circ 4 = (8+4)^2 = 144$
$((4 \circ 3)*2) = 7^2 * 2 = 49 * 2 = 49^2$
Thus, $((2*3)\circ 4)\ \$\ ((4\circ 3)*2) = 144\ \$\ 49^2 = \sqrt{144(49^2)} = 12(49) = \underline{\mathbf{588}}$

Created with nitroPDF professional
download the free trial online at nitropdf.com/professional

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010 SOLUTION KEY

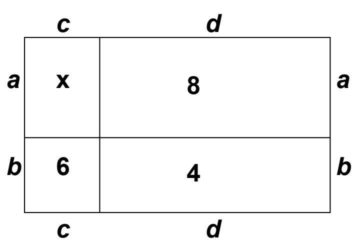
Round 6 - continued

C) The arrangement at the right produces x = **<u>12</u>**.
No other arrangements produce integer dimensions.
If interested, read on to see why this is the case.

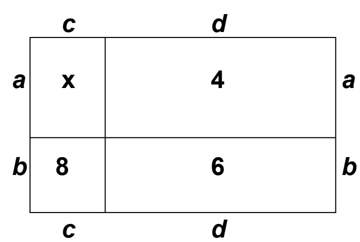


There are 6 possible arrangements, namely the rectangle with area 4, 6 or 8 is opposite the rectangle with area x and the other two may be interchanged.

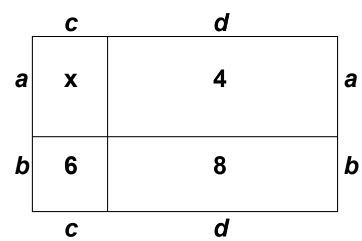
Case 1



Case 2



Case 3



In the following explanation, "|" means <u>such that</u>.

Case 1: $(b, d) \mid bd = 4$
$(2, 2) \rightarrow a = 4, c = 3, x = 12$
$(1, 4) \rightarrow a = 2, c = 6, x =$ **<u>12</u>**
$(4, 1) \rightarrow c$ fractional, i.e. $\frac{6}{4}$ - rejected

Case 2: $(a, d) \mid ad = 4$
(2, 2), (4, 1), (1, 4)
c fractional - all rejected

Case 3: $(a, d) \mid ad = 4$
$(1, 4) \rightarrow b = 2, c = 3, x =$ **<u>3</u>**

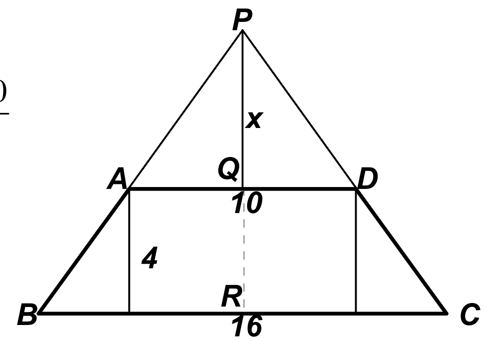
Case 4: (Case 1 - interchange 6 and 8) $x = 3(4)$ or $2(6) = 12$
The only change is the orientation. The order of the areas (x - 6 - 4 - 8) is counterclockwise (CCW) in case 1 and clockwise (CW) in case 4.

In similar fashion, cases 5 and 6 are CW versions of cases 2 and 3 and introduce no additional solutions. Thus, there are only two answers.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010 SOLUTION KEY

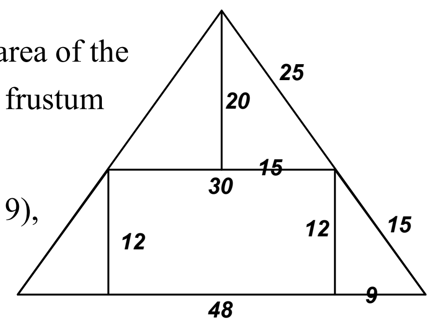
Team Round

A) Extend the non-parallel sides of the trapezoid until they intersect. The triangle formed is a cross-section of the cone from which the frustum was formed. $\Delta PQD \sim \Delta PRC$ ➔ $\dfrac{x}{x+4} = \dfrac{5}{8}$ ➔ $x = \dfrac{20}{3}$



To avoid computations with fractions, scale the diagram by a factor of 3: See diagram below.

Since the lateral surface area of a cone is given by $\pi r l$, where l is the lateral height, the lateral surface area of the frustum is $\pi \cdot 24 \cdot 40 - \pi \cdot 15 \cdot 25 = \pi(960 - 375) = 585\pi$ and, adding the surface area of the circular bases (225π and 576π), we have the total surface area of the frustum is 1386π.



But we scaled the linear dimensions by 3 (and consequently areas by 9), so the adjusted TSA is $1386\pi/9$ = **$\underline{154\pi}$**.

Alternative solution #1: Straightforward - does not scale the numbers, i.e. works with the fractions

Small cone: $l^2 = \dfrac{400}{9} + 25 = \dfrac{400}{9} + \dfrac{225}{9} = \dfrac{625}{9} \rightarrow l = \dfrac{25}{3}$ LA = $\pi(5)\left(\dfrac{25}{3}\right) = \dfrac{125}{3}\pi$

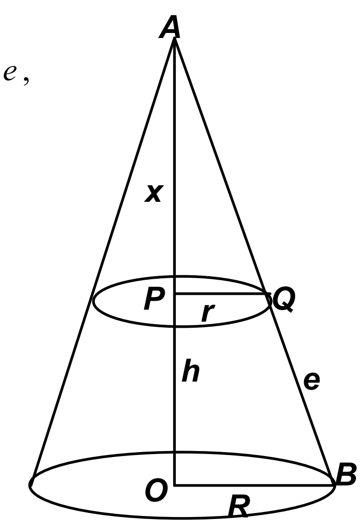
Large cone: $l^2 = \dfrac{1024}{9} + 64 = \dfrac{1024}{9} + \dfrac{576}{9} = \dfrac{1600}{9} \rightarrow l = \dfrac{40}{3}$ LA = $\pi(8)\left(\dfrac{40}{3}\right) = \dfrac{320}{3}\pi$

Total surface area of frustum: $25\pi + 64\pi + \left(\dfrac{320}{3}\pi - \dfrac{125}{3}\pi\right) = 89\pi + \dfrac{195}{3}\pi = (89 + 65)\pi = \underline{\mathbf{154\pi}}$

Alternative solution #2 utilizes this formula: LSA(frustum) = $\pi(R + r)e$, where $e^2 = h^2 + (R - r)^2$

$r = 5$, $R = 8$ and $h = 4$ ➔ $e = 5$

Thus, TSA = $\pi(5)^2 + \pi(8 + 5)(5) + \pi(8)^2 = (25 + 65 + 64)\pi =$ **$\underline{154\pi}$**



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010 SOLUTION KEY

Team Round – continued

B) The hypotenuse has length 10. Let h denote the length of the altitude to the hypotenuse.

Then: Area = $\frac{1}{2}\cdot 6\cdot 8=\frac{1}{2}\cdot 10\cdot h \rightarrow h = 4.8$

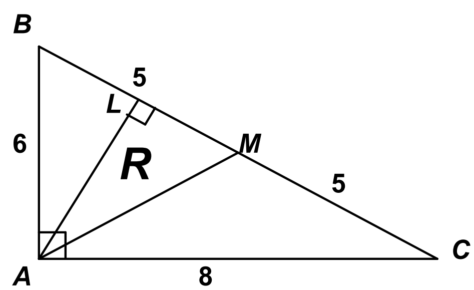
The length of the median is half the hypotenuse $\rightarrow$ $m = 5$

We avoid using the Pythagorean Theorem by looking for a special right triangle.

$(a, b, c) = (?, 4.8, 5) = \frac{1}{10}(?, 48, 50) = \frac{1}{5}(?, 24, 25)$

Special rt. Triangle (7, 24, 25) $\rightarrow$ $a = 7/5$ and the required ratio is

$$\frac{\frac{1}{2}(1.4)(4.8)}{\frac{1}{2}(6)(8)}=\frac{1.4}{10}=\underline{\mathbf{\frac{7}{50}}}$$



Alternate Solution (Tuan Le):

$$\cos B=\frac{AB}{BC}=\frac{6}{10}=\frac{3}{5}=\frac{BL}{AB}=\frac{BL}{6} \rightarrow BL=\frac{18}{5}$$

$$\rightarrow LM=5-\frac{18}{5}=\frac{7}{5}$$

Therefore, $\frac{R}{\text{area}(\Delta ABC)}=\frac{\frac{1}{2}(AL)(LM)}{\frac{1}{2}(AL)(BC)}=\frac{LM}{BC}=\frac{7/5}{10}=\underline{\mathbf{\frac{7}{50}}}$

C) Let r and c denote the row rate and the current respectively (in mi/hr). Then:

$\begin{cases}\text{downstream}: (r+c)1.5=12\\ \text{upstream}: (r-c)4=12\end{cases}$. Subtracting, $2.5r - 5.5c = 0 \rightarrow r = \frac{11c}{5}$ Substituting,

$\left(\frac{11c}{5}-c\right)=3 \rightarrow \frac{6c}{5}=3 \rightarrow c = 2.5, r = 5.5$

To travel 1 mile downstream, $8T = 1 \rightarrow T = 1/8$ hour = **<u>7</u>** minutes **<u>30</u>** seconds $\rightarrow$ **<u>(7, 30)</u>**

D) $\frac{22-\frac{n}{2}}{7-\frac{n}{2}}=c \rightarrow \frac{44-n}{14-n}=c$

The quotient is positive for $n < 14$ and $n > 44$.

For integer values of $n \geq 45$, the quotients are $\frac{1}{31},\frac{2}{32},\frac{3}{33},\ldots$

Clearly none of these are integers. Thus, we restrict our attention to $1 \leq n \leq 13$.

$n = 1, 2, 3, 4$ produce $\frac{43}{13},\frac{42}{12},\frac{41}{11}$ and $\frac{40}{10}=4$

$n = 13$ produces $\frac{31}{1}=31 \rightarrow (L, S) =$ **<u>(13, 4)</u>**.

Created with nitroPDF professional
download the free trial online at nitropdf.com/professional

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010 SOLUTION KEY

Team Round – continued

E) $\sqrt{5x+9}+\sqrt{8x+17}=2 \rightarrow \sqrt{8x+17}=2-\sqrt{5x+9}$

Squaring both sides, $8x+17=4-4\sqrt{5x+9}+5x+9=5x+13-4\sqrt{5x+9}$

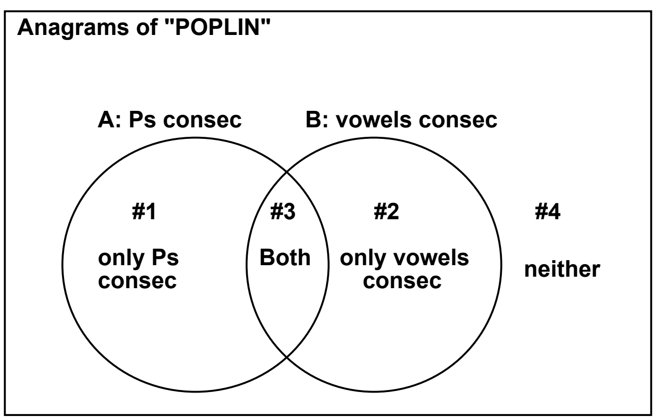
$\rightarrow 3x+4=-4\sqrt{5x+9}$ (***)

$\rightarrow 9x^2+24x+16=16(5x+9)=80x+144 \rightarrow 9x^2-56x-128=(9x+16)(x-8)=0$

$\rightarrow x = \underline{\boldsymbol{-\frac{16}{9}}} \left(\sqrt{\frac{-80}{9}+\frac{81}{9}}=\sqrt{\frac{1}{9}}=\frac{1}{3}, \sqrt{\frac{-128}{9}+\frac{153}{9}}=\sqrt{\frac{25}{9}}=\frac{5}{3}\right)$

(8 is extraneous $7 + 9 \neq 2$ or note from (***) above that $x \le -\frac{4}{3}$, since both sides must be negative.)

F) Consider a Venn Diagram with two circles containing anagrams with consecutive Ps and anagrams with consecutive vowels. The intersection (region #3) contains anagrams satisfying both conditions. POPLIN would reside in region #4, BUT it is not an anagram (of itself) and, therefore, must be excluded entirely.
These 4 regions are mutually exclusive, i.e. any anagram resides in exactly one of these regions, ensuring that no anagram is missed or double counted.



Region #3: Consider the 6 letters as 4 distinct items, namely PP, OI, L and N.
They may be arranged to form 4!(2) = 48 anagrams

Circle A: Consider the 6 letters as 5 items. Namely PP, O, I, L and N
These can be arranged in 5! = 120 ways → Region #1: 120 – 48 = 72 anagrams.

Circle B: Consider the 6 letters as 5 items, OI, P, P, L and N

These can be arranged in $\frac{5!\cdot 2}{2!}$ = 120 ways → Region #2: 120 – 48 = 72 anagrams

Thus, region #4 contains 359 – (72 + 48 + 72) = 359 – 192 = **<u>167</u>** anagrams.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010 SOLUTION KEY

Team Round – continued

F) Alternate solution (Tuan Le) - 3 cases
Case 1: 2Ps are next to each other, but O and I are not
PP, L and N create 4 spaces to place the remaining letters.
{ ___ PP ___ L ___ N ___}
The PP, L and N can be arranged in $3! = 6$ ways and there are $4 \cdot 3 = 12$ ways to place the O and I
➔ $6 \cdot 12 = 72$

Case 2: OI are next to each other, but the Ps are not
Again OI, L and N create 4 spaces to place the remaining letters.
{____ OI ___ L ___ N ___}
Arrange OI next to each other – 2 ways
Arrange OI, L and N – 6 ways
Use 2 of the 4 spaces for the Ps - $\binom{4}{2} = \frac{4!}{2!2!} = 6$
➔ $2(6)(6) = 72$

Case 3: Both OI and PP in adjacent positions
Arrange the 4 items. Only OI can be flipped.
{OI, PP, L, N} ➔ $4! \cdot 2! = 48$
Since the total number of anagrams is $6! - 1 = 359$, the number of anagrams without adjacent Ps and without adjacent vowels is $359 - (2(72) + 48) =$ **<u>167</u>**.