MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2010
ROUND 1 COMPLEX NUMBERS (No Trig)

ANSWERS

A) ______________________________

B) (____________ , ______________)

C) _______________________ sq. units

****** NO CALCULATORS ON THIS ROUND ******

A) Compute: $\left(\dfrac{1-i}{1+i}\right)^{2010}$

B) Find <u>the</u> ordered pair (x, y) of real numbers that satisfy the equation

$$(x^2 - x - 5) + i(y^2 - 7y + 3) = 1 - 7i$$

and for which $y - x$ is <u>as large as possible</u>.

C) The complex numbers $(1 + i)$, $(-1 + i)$, $(-1 - i)$ and $(1 - i)$ form a square when plotted in the complex plane. If each of these numbers is multiplied by $(1 + i)$, a new figure is formed. Compute the area of the new figure.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2010
ROUND 2 ALGEBRA 1: ANYTHING

ANSWERS

A) _______________________ feet

B) _______________________

C) $ _______________________

****** NO CALCULATORS ON THIS ROUND ******

A) Mr. Slowski, the Comcast turtle, recently competed in the Reptilian Olympics. He completed the quarter mile course in an hour. At this rate, compute the distance (in feet) he could travel in 48 minutes. [Recall: 1 mile = 5280 feet]

B) When a two-digit number is divided by the sum of its digits, the quotient is 7. When the same number is multiplied by the sum of its digits, the product is 567. Find this number.

C) The list price for the home theater of my dreams is $6000. Over a three week period, Merchant A gave a discount of $16\frac{2}{3}\%$, then $12\frac{1}{2}\%$, and finally, 4%, each discount off the already discounted price of the previous week. Similarly, Merchant B gave a discount of 8%, then 10%, and finally, 15%. Unbelievably, these offers differ by less than 1%. Compute the positive difference between the discounted sales prices offered by these two vendors.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2010
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

ANSWERS

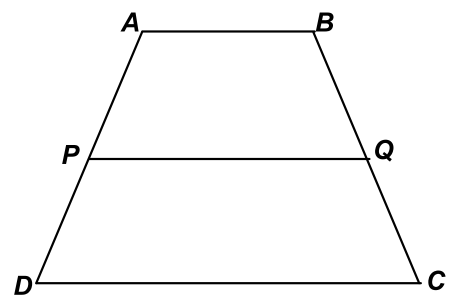
A) _______________ : _______________

B) _________________________ sq. units

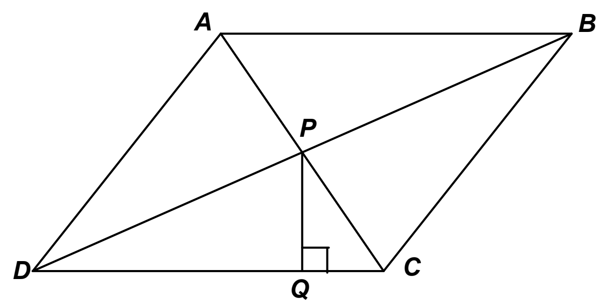
C) _________________________________

****** NO CALCULATORS ON THIS ROUND ******

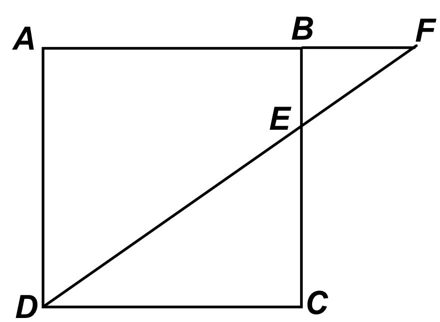
A) $\overline{PQ}$ is a median in trapezoid $ABCD$.
$AB = 12$ and $DC = 20$.
Compute the ratio of the area of trapezoid $ABQP$ to the area of trapezoid $PQCD$.



B) The perimeter of rhombus $ABCD$ is 100 units.
If $PQ = 12$ units and $QC = 9$ units, compute the area of rhombus $ABCD$.



C) $ABCD$ is a square, $\overline{AB}$ is extended to F, $\overline{DF}$ intersects $\overline{BC}$ at E, $BE : EC = 1 : 2$ and the area of ΔBEF is 24.
Compute the area of $ABCD$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2010
ROUND 4 ALG 1: FACTORING AND ITS APPLICATIONS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

****** NO CALCULATORS ON THIS ROUND ******

A) For positive integers a, b and n, $x^2 - x - n = (x+a)(x-b)$.
If $n < 50$, compute the largest possible value of n.

B) Let $P = 280x^3y^2$.
Compute Q, if the greatest common factor of P and Q is $28x^2y^2$ and the least common multiple of P and Q is $3080x^3y^3z$.

C) Factor completely. $8A^2 - 7AB + 13B^2 - 3W^2 - 4B^2 - 4A^2 + 19AB - 13W^2$

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2010
ROUND 5 TRIG: FUNCTIONS OF SPECIAL ANGLES

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

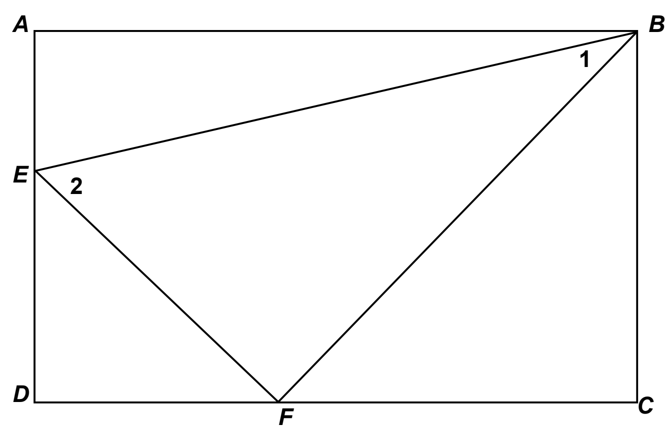
****** NO CALCULATORS ON THIS ROUND ******

A) If possible, compute the exact value of $\dfrac{\cot(45°)+2\sin(210°)}{1+\tan(22.5°)}$; otherwise, specify the value as undefined.

B) Compute: $\left(\sin 510° \cos 240° \cot^3 315° \csc\dfrac{11\pi}{6}\sec\left(\dfrac{-7\pi}{3}\right)\right)^5$

C) Given: $ABCD$ is a rectangle,
$BE = 1$, $DE = DF$, $CF = CB$
$\overline{EF} \perp \overline{FB}$ and $m\angle 2 = 2 \cdot m\angle 1$

Compute $\cos(\angle BED)$.



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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2010
ROUND 6 PLANE GEOMETRY: ANGLES, TRIANGLES AND PARALLELS

ANSWERS

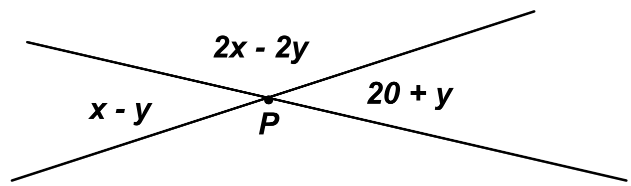
A) ______________________________ °

B) (____________ , ______________)

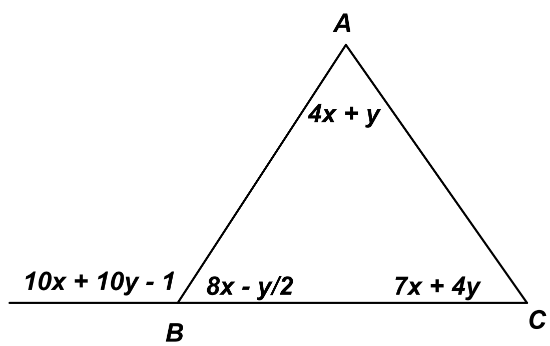
C) ________________________________

****** NO CALCULATORS ON THIS ROUND ******

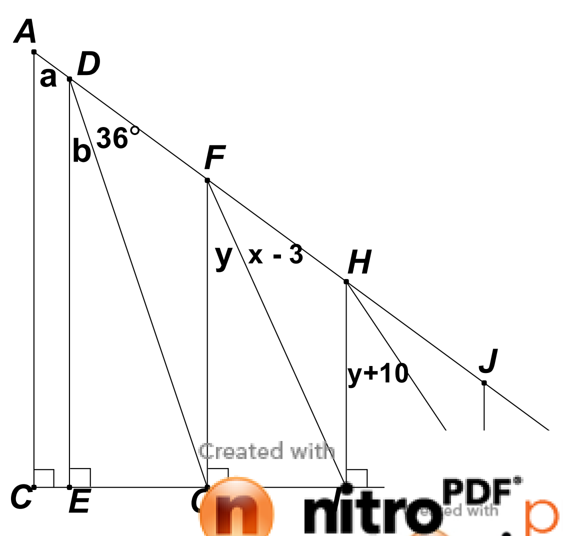
A) Compute the degree measure of the larger of the vertical angles with vertex at P in the diagram at the right.



B) Solve for (x, y) given $AB = AC$.



C) In right triangle ABC, $\angle GFI$, $\angle HFI$, $\angle IHK$, $\angle IKH$ and $\angle JBK$ have measures as indicated in terms of x and y, $m\angle GDF = 36°$ and five perpendiculars to $\overline{BC}$ are marked.
If $m\angle CAD = a$ and $m\angle EDG = b$, compute $a + b$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2010
ROUND 7 TEAM QUESTIONS
ANSWERS

A) (____________ ,____________) D) (______ , ______ , ______ , ______)

B) ____________________________ E) (____________ ,____________)

C) (____________ ,____________) F) ____________________________

****** NO CALCULATORS ON THIS ROUND ******

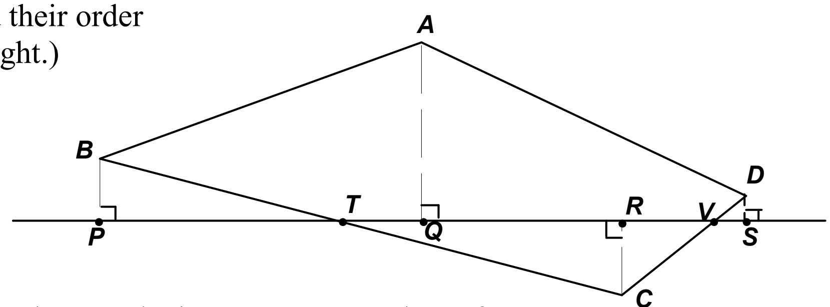
A) Let $z = a + bi$. Compute the ordered pair (a, b), if $\begin{cases} \dfrac{1}{z} = \bar{z} \\ a + b = 1.4 \\ a > b \end{cases}$.

B) If $\dfrac{1}{2x^3} - \dfrac{1}{x^2} - \dfrac{1}{2x} + 1 = 0$, find <u>all</u> possible values of $(x^2 + 1)^2$.

C) Given: $BP : CR : DS = 4 : 5 : 2$; and $\overline{BP}, \overline{AQ}, \overline{CR}$ and $\overline{DS} \perp \overleftrightarrow{TV}$.
Compute the unique ordered pair (a, b) for which the following statement is true:

$$Area(ABCD) = Area(ABPQ) + Area(ADSQ) - a \cdot Area(\Delta BPT) - b \cdot Area(\Delta DSV)$$

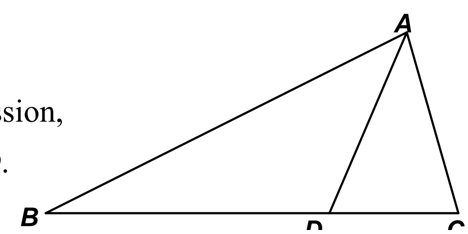
(P, T, Q, R, V and S are collinear and their order is as indicated in the diagram at the right.)



D) $x^{14k} - x^{8k} - x^{6k} + 1$ is factored completely over the integers, as a product of binomials and trinomials, where each lead coefficient is +1. The <u>sum</u> of these factors can be written in the form $Ax^{4k} + Bx^{2k} + Cx^k + D$. Determine the ordered quadruple (A, B, C, D).

E) Given: $\sin 54° = \dfrac{\sqrt{5}+1}{4}$. In simplified form, $\sin 144° \sin 72° = \dfrac{\sqrt{A}}{B}$. Determine (A, B).

F) Given: $AB = BC$, $AD = AC$ and
m$\angle BAD$, m$\angle ADC$, m$\angle ADB$ form an increasing arithmetic progression,
where $(m\angle ADB - m\angle ADC)^2 = m\angle ADC + 60°$. Compute m$\angle BAD$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2010 ANSWERS

Round 1 Algebra 2: Complex Numbers (No Trig)

A) -1 B) $(-2, 5)$ C) 8

Round 2 Algebra 1: Anything

A) 1056 B) 63 C) (A, \$22.80)

Round 3 Plane Geometry: Area of Rectilinear Figures

A) 7 : 9 B) 600 C) 288

Round 4 Algebra 1: Factoring and its Applications

A) 42 B) $308x^2y^3z$ C) $(2A + 3B - 4W)(2A + 3B + 4W)$

Round 5 Trig: Functions of Special Angles

A) 0 B) -1 C) $\dfrac{\sqrt{2}-\sqrt{6}}{4}$

Round 6 Plane Geometry: Angles, Triangles and Parallels

A) 120 B) (9, 2) C) 78

Team Round

A) $\left(\dfrac{4}{5}, \dfrac{3}{5}\right)$ D) (1, 3, 4, 4)

B) 4 or $\dfrac{25}{16}$ E) (5, 4)

C) $\left(-\dfrac{9}{16}, -\dfrac{21}{4}\right)$ F) 72

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2010 SOLUTION KEY

Round 1

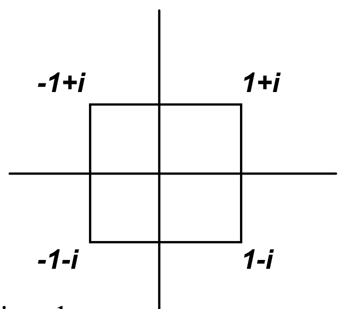
A) $\left(\dfrac{1-i}{1+i}\right)^{2010} = \left(\dfrac{1-i}{1+i}\cdot\dfrac{1-i}{1-i}\right)^{2010} = \left(\dfrac{(1-i)^2}{1-i^2}\right)^{2010} = \left(\dfrac{-2i}{2}\right)^{2010} = (-1)^{2010}\cdot i^{2008}\cdot i^2 = 1\cdot 1\cdot -1 = \underline{\mathbf{-1}}$

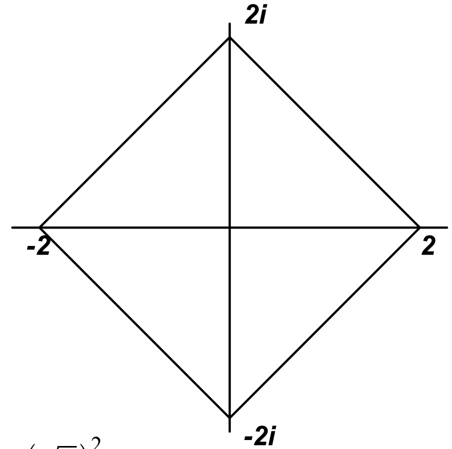
If you know DeMoirve's theorem, you might want to use it to formulate an alternative solution for comparison.

B) Equating the real and imaginary coefficients, $\begin{cases} x^2-x-5=1 \\ y^2-7y+3=-7 \end{cases}$ ➔

$\begin{cases} x^2-x-6=(x-3)(x+2)=0 \\ y^2-7y+10=(y-2)(y-5)=0 \end{cases}$ ➔ $x = 3, -2$ and $y = 2, 5$ ➔ (3, 5), (–2, 5), (3, 2), (–2, 2) ➔ **<u>(–2, 5)</u>**

C) $(1+i)\cdot\begin{cases}(1+i)\\(-1+i)\\(-1-i)\\(1-i)\end{cases} = \begin{cases}2i\\-2\\-2i\\2\end{cases}$





The new figure is a square with side $2\sqrt{2}$, so the area is **<u>8</u>**.

Alternate solution: The area of the original square is $2^2 = 4$, and multiplying the vertices by $(1 + i)$ rotates the square 45° and expands each side by a factor of $|1 + i| = \sqrt{2}$. Therefore, the new square will have area $4\left(\sqrt{2}\right)^2$ = **<u>8</u>**.

Round 2

A) $\dfrac{5280/4 \text{ feet}}{60 \ \cancel{\text{min}}}\cdot 48\ \cancel{\text{min}} = \dfrac{\frac{5280}{4}(4)}{5} = \dfrac{5280}{5} = \underline{\mathbf{1056}}$ feet

B) $\begin{cases} (1)\ \ 10t+u=7(t+u) \\ (2)\ \ (10t+u)(t+u)=567 \end{cases}$

(1) ➔ $t = 2u$

Substituting for $10t + u$ in (2), $7(t+u)^2 = 567$ ➔ $(t+u)^2 = 81$ ➔ $t + u = 3u = 9$

➔ $u = 3$, $t = 6$ ➔ **<u>63</u>**

C) Discounts of 16 2/3% ➔ 1/6 off, 12 1/2% ➔ 1/8 off and 4% ➔ 1/25 off

Merchant A's price $\dfrac{5}{6}\cdot\dfrac{7}{8}\cdot\dfrac{24}{25} = \dfrac{7}{10} = 70\%$ of list (or 30% off list).

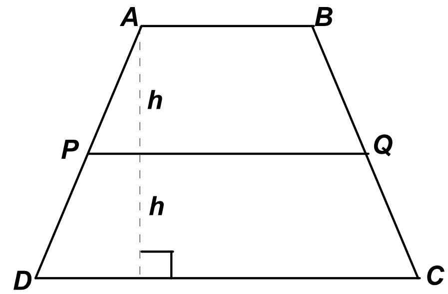
Merchant B's price: (.92)(.9)(.85) = .7038 = 70.38% of list (or 29.62% off list).

Thus, merchant A has the best price by 0.38% (less than 1%)

➔ $\dfrac{0.38}{100}(6000) = 0.38(60) = 22.8$ ➔ **<u>$22.80</u>** .

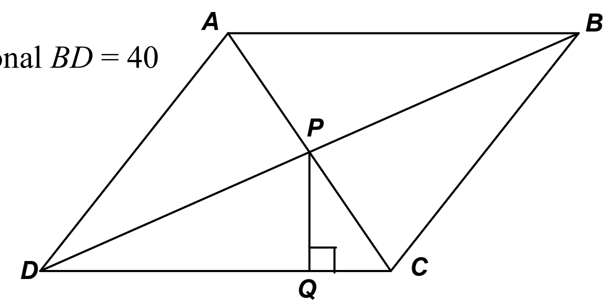
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2010 SOLUTION KEY

Round 3

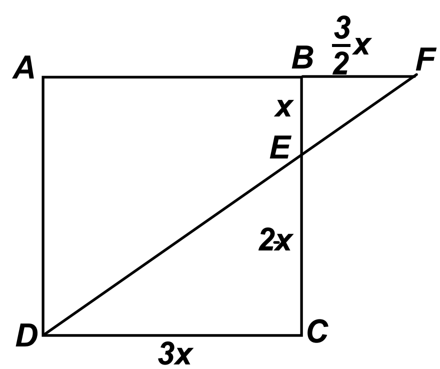


A) As a median, $PQ = \dfrac{12+20}{2} = 16$ and the altitudes of trapezoids $ABQP$ and $PQCD$ are equal in length. Thus, the required ratio is

$$\frac{\frac{1}{2}h(12+16)}{\frac{1}{2}h(16+20)} = \frac{28}{36} = \underline{\mathbf{\frac{7}{9}}}$$



B) $PQ = 12$, $QC = 9$ ➔ $PC = 15$ ➔ diagonal $AC = 30$
Perimeter = 100 ➔ $DC = 25$, $DQ = 16$, $DP = 20$ ➔ diagonal $BD = 40$
Thus the area of the rhombus = $\dfrac{1}{2}d_1 d_2 = \dfrac{1}{2}\cdot 30\cdot 40 = \underline{\mathbf{600}}$



C) $\Delta BEF \sim \Delta CED$, $\dfrac{BE}{CE} = \dfrac{1}{2}$ and $CD = 3x$ ➔ $BF = \dfrac{3}{2}x$

Therefore, $\dfrac{1}{2}x\cdot\dfrac{3}{2}x = 24$ ➔ $x^2 = 32$

➔ area($ABCD$) = $3x\cdot 3x = 9x^2 = 9(32) = \underline{\mathbf{288}}$

<u>Alternate solution (MaryBeth McGinn / Tuan Le):</u>

$\dfrac{area(\Delta BEF)}{area(\Delta DEC)} = \dfrac{BE^2}{EC^2} = \dfrac{1}{4}$ ➔ area(ΔDEC) = 96

$\dfrac{area(\Delta BEF)}{area(\Delta ADF)} = \dfrac{BE^2}{AD^2} = \dfrac{1}{9}$ ➔ $\dfrac{area(\Delta BEF)}{area(BADE)} = \dfrac{1}{8}$ ➔ area($BADE$) = 192 ➔ area($ABCD$) = $\underline{\mathbf{288}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2010 SOLUTION KEY

Round 4

A) Since the middle coefficient is -1, we start with $(a, b) = (1, 2)$.
$x^2 - x - n = (x + 1)(x - 2)$ which gives $n = 2$.
Thus, the absolute value of the difference between a and b must always be 1.
2(3) → 6 3(4) → 12 4(5) → 20 5(6) → 30 6(7) → **<u>42</u>** 7(8) → 56 (too big)

B) Given: $P = 280x^3y^2$, GCF$(P, Q) = 28x^2y^2$ and LCM$(P, Q) = 3080x^3y^3z$.
Note: Given any two integers m and n, $\boxed{mn = \text{GCF}(m, n) \cdot \text{LCM}(m, n)}$.
Ex: GCF(24, 30) = 6 and LCM(24, 30) = 120 and 24(30) = 6(120) = 720
The same principle applies to literal expressions.

$PQ = \text{GCF}(P, Q)\cdot \text{LCM}(P, Q)$ → $280x^3y^2(Q) = (28x^2y^2)(3080x^3y^3z)$
→ $x^3y^2Q = 308x^5y^5z$ → $Q = \underline{\mathbf{308x^2y^3z}}$.

C) Combining like terms, $8A^2 - 7AB + 13B^2 - 3W^2 - 4B^2 - 4A^2 + 19AB - 13W^2$
$= 4A^2 + 12AB + 9B^2 - 16W^2 = (2A + 3B)^2 - (4W)^2$.
As the difference of perfect squares this factors to **<u>$(2A + 3B - 4W)(2A + 3B + 4W)$</u>**

Round 5

A) The numerator $\cot(45°) + 2\sin(210°)$ evaluates to $1 + 2\left(-\frac{1}{2}\right) = 0$.

Without bothering to evaluate, we note that the denominator is nonzero, since the tangent of a first quadrant angle is positive. Thus, the expression evaluates to **<u>0</u>**.

B) $$\left(\sin 510°\cos 240°\cot^3 315°\csc\frac{11\pi}{6}\sec\left(\frac{-7\pi}{3}\right)\right)^5 = \left(\sin 150°\cos 240°\cot^3 315°\csc\frac{11\pi}{6}\sec\left(\frac{5\pi}{3}\right)\right)^5$$
$$= \left(\sin 30°\cdot -\cos 60°\cdot -\cot^3 45°\cdot -\csc 30°\cdot \sec 60°\right)^5 =$$
$$\left(\sin 30°\cdot -\csc 30°\cdot -\cos 60°\cdot \sec 60°\cdot -\cot^3 45°\right)^5 = \left((-1)(-1)(-1)^3\right) = \underline{\mathbf{-1}}$$

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Round 5 - continued

C) The given information is marked in the diagram below and, taking note of the embedded 30 - 60 - 90 triangle, we know that $EF = \frac{1}{2}$ and $FB = \frac{\sqrt{3}}{2}$.

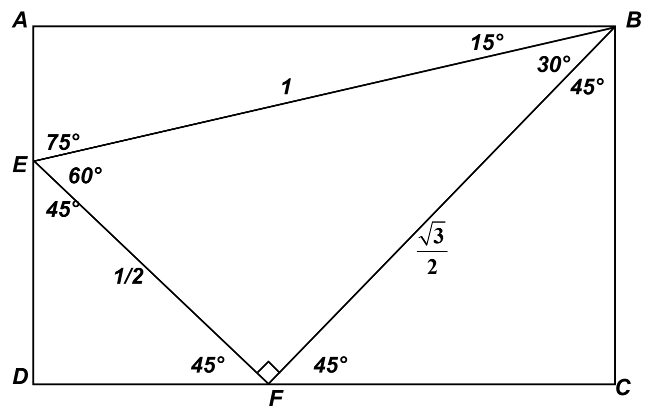
Both ΔEDF and ΔBCF are isosceles right triangles whose sides are in a 1 : 1 : $\sqrt{2}$ ratio.

$$FC = BC = AD = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{6}}{4}$$

$$DE = DF = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$AB = DF + FC = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$AE = AD - DE = \frac{\sqrt{6}-\sqrt{2}}{4}$$



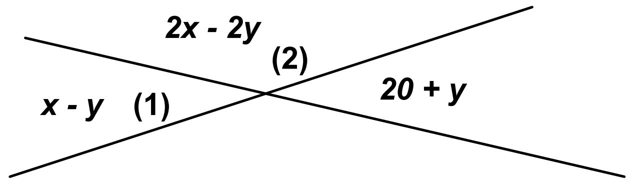
Thus, $\cos(\angle BED) = \cos(105°) = -\cos(75°) = -\frac{AE}{BE} = \boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$

This approach is an alternative to using the usual expansion of cos(A + B) which would not be known by all students in this round. Thanks again to MaryBeth McGinn for this gem. The question, diagram and solution are adapted directly from the 2008 MML Contest #2 notes and the study guide MML Contest 2 A-Questions + Solutions (05-09).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2010 SOLUTION KEY

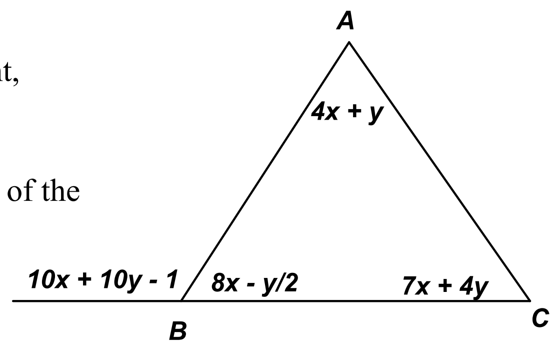
Round 6

A) $2x - 2y = 2(x - y)$
$\angle$s 1 and 2 are supplementary
$(x - y) + 2(x - y) = 3(x - y) = 180 \rightarrow x - y = 60$
Therefore, without having to solve for x and y,
the larger of the vertical angles is **<u>120</u>**°.

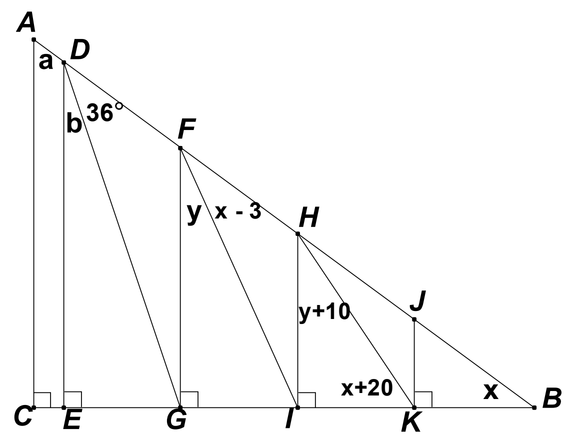


Alternate Solution:
Vertical angles $\rightarrow x - y = 20 + y \rightarrow x = 20 + 2y$
Linear Pair ($\angle$s 1 and 2) $\rightarrow 3x - 3y = 180 \rightarrow x - y = 60$
Substituting, $20 + 2y - y = 60 \rightarrow y = 40, x = 100 \rightarrow$ larger vertical angles **<u>120</u>**°.

B) Solve for (x, y) given $AB = AC$.
Since base angles of an isosceles triangle are congruent,
$7x + 4y = 8x - \frac{y}{2} \rightarrow x = \frac{9}{2}y$
Since the exterior angle of any triangle equals the sum of the measures of the two remote interior angles,
$10x + 10y - 1 = (4x + y) + (7x + 4y) \rightarrow x = 5y - 1$
Substituting, $5y - 1 = \frac{9}{2}y \rightarrow y = 2 \rightarrow (x, y) =$ **<u>(9, 2)</u>**.



C) In ΔHIK, $x + y = 60$.
Applying the fact that the measure of exterior angle equals the sum of the measures of the two remote interior angles to ΔHKB forces $m\angle KHB = 20$.
$\angle GFH$ and $\angle IHJ$, as corresponding angles of parallel lines, forces $x + (y - 3) = (y + 10) + 20$
$\rightarrow x = 33 \rightarrow \begin{cases} a = 57 \\ y = 33 \end{cases}$ and $m\angle DFG = 123$.
As alternate interior angles of parallels,
$m\angle DGF = m\angle EDG = b$.
Therefore, in ΔDFG, $b = 180 - (36 + 123) = 21$
$\rightarrow a + b =$ **<u>78</u>**.



Alternate Solution (Tuan Le)
As an exterior angle, $m\angle HKI = m\angle HKI + m\angle HKI$
$\rightarrow m\angle BHK = 20°$
In ΔHIB, $x + y + 30 = 90 \rightarrow x + y = 60$.
Since $\overline{HI} \parallel \overline{FG}$, $m\angle BFG = m\angle BHI \rightarrow$
$x + y - 3 = y + 30 \rightarrow x = 33,\ y = 27$ and $a = 57$.
$\overline{DE} \parallel \overline{AC} \rightarrow a = b + 36 \rightarrow b = 21$. Thus, $a + b = 51 + 27 =$ **<u>78</u>**.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2010 SOLUTION KEY

Team Round

A) Given : $\begin{cases} \dfrac{1}{z} = \overline{z} \\ a+b=1.4 \\ a>b \end{cases}$ and $z = a + bi$. Substitute $b = \left(\dfrac{7}{5} - a\right)$ in $\dfrac{1}{a+bi} = a - bi \Leftrightarrow a^2 + b^2 = 1$

$a^2 + \left(\dfrac{7}{5} - a\right)^2 = 1 \rightarrow 2a^2 + \dfrac{49}{25} - \dfrac{14a}{5} = 1 \rightarrow 50a^2 - 70a + 24 = 0$

$\rightarrow 25a^2 - 35a + 12 = (5a-3)(5a-4) = 0$

$a > b \rightarrow (a, b) = \underline{\left(\mathbf{\dfrac{4}{5}, \dfrac{3}{5}}\right)}$.

Alternate solution (Norm Swanson)

$z \cdot \overline{z} = 1 \rightarrow z$ lies on the unit circle with center at (0, 0). So, let $a = \cos(t)$ and $b = \sin(t)$ and consequently, $a^2 + b^2 = 1$.

Squaring the second equation, $a^2 + 2ab + b^2 = \left(\dfrac{7}{5}\right)^2 \rightarrow 2ab = \dfrac{49}{25} - 1 = \dfrac{24}{25} \rightarrow ab = \dfrac{12}{25}$.

We want two numbers whose sum is 7/5 and whose product is 12/25.
Clearly, 3/5 and 4/5 satisfy the requirement.

$a > b \rightarrow (a, b) = \underline{\left(\mathbf{\dfrac{4}{5}, \dfrac{3}{5}}\right)}$.

B) Rearranging terms, $1 - \dfrac{1}{x^2} = \dfrac{1}{2x} - \dfrac{1}{2x^3} \rightarrow \dfrac{x^2-1}{x^2} = \dfrac{x^2-1}{2x^3}$

Since $x \neq 0$, this simplifies to $\dfrac{x^2-1}{1} = \dfrac{x^2-1}{2x}$

For $x = \pm 1$, both terms are 0; otherwise, equating the denominators, $2x = 1 \rightarrow x = 1/2$
Thus, $(x^2 + 1)^2 = 2^2$ or $(5/4)^2 \rightarrow (x^2 + 1)^2 =$ **<u>4 or 25/16</u>**

Alternate Solution: ($x \neq 0$) Multiplying through by $2x^3$, $1 - 2x - x^2 + 2x^3 = 0$
$\rightarrow (1 - 2x) - x^2(1 - 2x) = 0 \rightarrow (1 - 2x)(1 - x^2) = 0 \rightarrow x = \pm 1$ or ½ $\rightarrow (x^2 + 1)^2 =$ **<u>4 or 25/16</u>**

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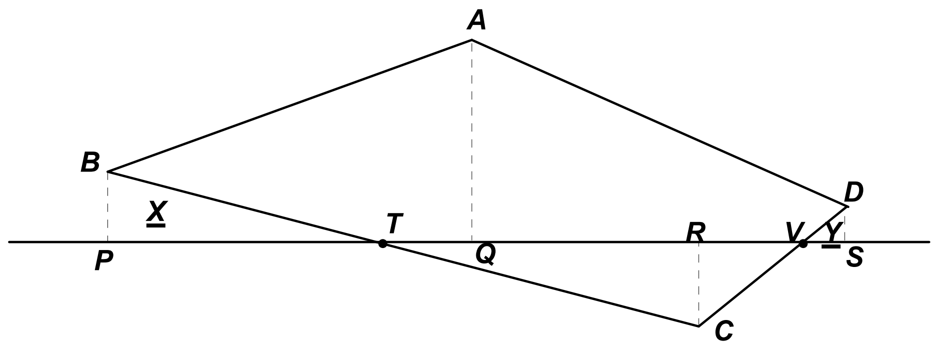
Team Round

C) Compared to $ABCD$, the trapezoids $ABPQ$ and $ADSQ$ combined include some regions that should be excluded and exclude some regions that should be included. ΔBPT and ΔDSV should be excluded, while ΔCRT and ΔCRV should be included.



Since $\Delta BPT \sim \Delta CRT$ and $BP : CR = 4 : 5$, Area(ΔBPT) : Area(ΔCRT) = 16 : 25.
Since $\Delta DSV \sim \Delta CRV$ and $DS : CR = 2 : 5$, Area(ΔDSV) : Area(ΔCRV) = 4 : 25.
Let (X, Y) denote the areas of ΔBPT and ΔDSV respectively. Then

$$\text{Area}(ABCD) = \text{Area}(ABPQ) + \text{Area}(ADSQ) - \underline{X} - \underline{Y} + \frac{25}{16}\underline{X} + \frac{25}{4}\underline{Y}$$

$$= \text{Area}(ABPQ) + \text{Area}(ADSQ) + \frac{9}{16}\underline{X} + \frac{21}{4}\underline{Y} \rightarrow (a, b) = \underline{\left(-\frac{\mathbf{9}}{\mathbf{16}}, -\frac{\mathbf{21}}{\mathbf{4}}\right)}.$$

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CONTEST 2 - NOVEMBER 2010 SOLUTION KEY

Team Round - continued

D) $x^{14k} - x^{8k} - x^{6k} + 1 = \left(x^{6k}-1\right)\left(x^{8k}-1\right) = \left(x^{3k}+1\right)\left(x^{3k}-1\right)\left(x^{4k}+1\right)\left(x^{4k}-1\right)$

$= \left(x^{k}+1\right)\left(x^{2k}-x^{k}+1\right)\left(x^{k}-1\right)\left(x^{2k}+x^{k}+1\right)\left(x^{4k}+1\right)\left(x^{2k}+1\right)\left(x^{k}+1\right)\left(x^{k}-1\right)$

Thus, the sum of the factors is $x^{4k}+3x^{2k}+4x^{k}+4$ ➔ **<u>(1, 3, 4, 4)</u>**.

E) $\sin 54° = \dfrac{\sqrt{5}+1}{4}$ ➔ $\cos 36° = \dfrac{\sqrt{5}+1}{4}$. Utilizing basic identities,

$\sin 2\theta \sin\theta = 2\sin^2\theta\cos\theta = 2\cos\theta\left(1-\cos^2\theta\right)$ (***)

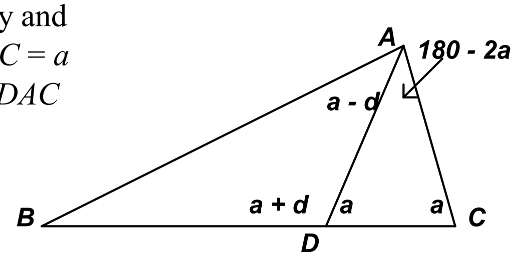
➔ $\sin 144° \sin 72° = \sin(180°-36°)\sin 72° = \sin 36° \sin\left(2(36)°\right)$

Let θ = 36. Then (***) ➔ $\sin 144° \sin 72° = 2\cos 36°(1-\cos^2 36°)$

$= 2\left(\dfrac{\sqrt{5}+1}{4}\right)\left(1-\left(\dfrac{\sqrt{5}+1}{4}\right)^2\right) = \left(\dfrac{\sqrt{5}+1}{2}\right)\left(\dfrac{16-6-2\sqrt{5}}{16}\right) = \left(\dfrac{\sqrt{5}+1}{2}\right)\left(\dfrac{5-\sqrt{5}}{8}\right) = \dfrac{4\sqrt{5}}{16} = \dfrac{\sqrt{5}}{4}$

➔ (A, B) = **<u>(5, 4)</u>**.

F) Let m$\angle BAD$, m$\angle ADC$, m$\angle ADB$ = $a - d$, a and $a + d$ respectively and $d^2 = a + 60$. Since $BA = BC$, m$\angle C$ = m$\angle BAC$, so $a - d$ + m$\angle DAC = a$ ➔ m$\angle DAC$ = d. But notice also that in ΔDAC, the vertex angle DAC has measure $180 - 2a$.



Equating and solving for a, $d = 180 - 2a$ ➔ $a = \dfrac{180-d}{2}$.

Thus, $d^2 = a + 60$ becomes $d^2 = \dfrac{180-d}{2} + 60$

➔ $2d^2 = 180 - d + 120$ ➔ $2d^2 + d - 300 = (2d+25)(d-12) = 0$ ➔ d = 12 only, a = 84.
($d = -12.5$ ➔ a = 96.25 which is impossible for the base angle in an isosceles triangle.)
Finally, m$\angle BAD$ = 84 – 12 = **<u>72</u>**.