MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 – DECEMBER 2010
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

ANSWERS

A) ________________________________

B) ________________________________

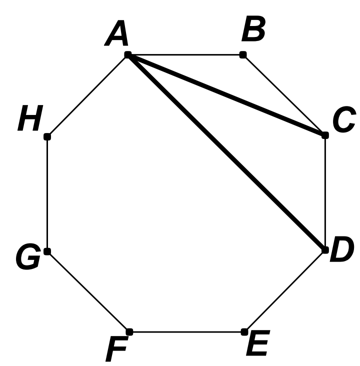
C) ________________________________

****** NO CALCULATORS ON THIS ROUND ******

A) Given: ΔABC with $AC = 8$, $BC = 15$ and $m\angle C = 60°$
Compute AB.

B) The sides of a right triangle have lengths $k + 1$, $4k + 1$ and $4k$.
Compute all possible sums of the lengths of the two legs.

C) In <u>regular</u> octagon $ABCDEFGH$, $AC^2 = AD$.
Compute AB.



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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010
ROUND 2 ARITHMETIC/NUMBER THEORY

ANSWERS

A) ________________________________

B) ______________________________ (4)

C) $P =$ ____________________________

****** NO CALCULATORS ON THIS ROUND ******

A) Find the largest prime factor of 7659.

B) Change $2144_{(7)}$ to an equivalent number is base 4.

C) 42875 is written in the form $(a+b)^n \cdot (a-b)^n$, where a, b and n are positive integers and $n > 1$. Compute the largest possible value of the product abn.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

****** NO CALCULATORS ON THIS ROUND ******

A) Points A and B are each equidistant from $P(1, -1)$ and $Q(3, -6)$.
If A and B lie on the x-axis and y-axis respectively, determine the slope of $\overline{AB}$.

B) Let P and Q denote the points of intersection between $(x-4)^2+(y+2)^2=20$ and $2x^2+2y^2-9x-13y=0$. Compute PQ.

C) A circle whose center is located in quadrant 1 is tangent to both coordinate axes and passes through the point $A(1, 8)$. The point P on the circle closest to the origin O lies on the line $y = x$. Compute OP.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

ANSWERS

A) $n =$ ______________________________

B) $x =$ ______________________________

C) $x =$ ______________________________

****** NO CALCULATORS ON THIS ROUND ******

A) Compute n, if the coordinates of A, B and C are log 60, log n and log 90 respectively, and B is the midpoint of $\overline{AC}$.

.

B) Solve for x.

$$5^{2\log_5 x} - 12\left(4^{\log_2 \sqrt{x}}\right) - 27^{\log_3 4} = 0$$

C) Solve for all real numbers x satisfying the equation

$$100^x - 3 \cdot 2^{x+1} \cdot 5^x + 5 = 0$$

If necessary, an answer may be left as a simplified log expression.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

ANSWERS

A) y = ___________________________

B) Ben: $_________ Joe: $_________

C) (____________ , ___________)

****** NO CALCULATORS ON THIS ROUND ******

A) Given: y varies directly as x and z.
If $y = 5$, when $(x, z) = (3, 4)$, then compute y when $(x, z) = (36, 134)$.

B) Two brothers, Ben and Joe, bought a single family home for $180,000.
Ben invested $3,600 of his own money in repairs.
Joe invested $2,000 of his own money in repairs.
The house was sold for $227,000 and $12,000 covered all expenses (closing costs, real estate commissions, etc.). If the brothers split the profit based on the contributions of their own money towards repairs, how much should each brother receive?

C) Given: A and B are integers,
$60 < A < 70$, but the units' digit is illegible.
$20 < B < 30$, but the units' digit is illegible.

Two students computed the ratio of $\frac{A}{B}$ incorrectly.
The first student reversed the digits of A, but not B.
The second student reversed the digits of B, but not A.
Amazingly, both students get an answer of $\frac{3}{2}$. Compute the ordered pair (A, B)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

ANSWERS

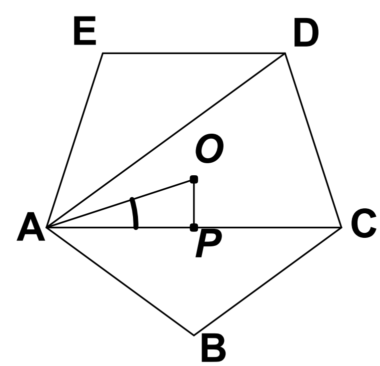
A) ______________________

B) ______________________ °

C) ______________________

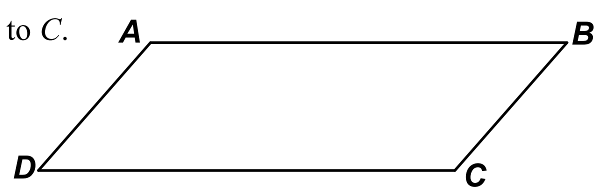
****** NO CALCULATORS ON THIS ROUND ******

A) O is the center of regular pentagon $ABCDE$ and $\overline{OP} \perp \overline{AC}$. A regular polygon has an exterior angle with the same measure as the marked angle? How many sides does this polygon have?



B) In isosceles trapezoid $ABCD$, where $\overline{AB} \parallel \overline{CD}$, the diagonals intersect at point E.
If $m\angle DEC = 3 \cdot m\angle BAE$ and $m\angle DAE : m\angle ADE = 5 : 4$, compute $m\angle BCE$.

C) $ABCD$ is a parallelogram.
Let M and N lie on $\overline{AB}$. M is closer to A and N is closer to B.
$AM : MB = 2 : 7$, $AN : NB = 5 : 4$
Let P and Q lie on $\overline{CD}$. P is closer to D and Q is closer to C.
$DP : PC = 1 : 5$, $DQ : QC = 5 : 7$
Compute $MB : PC$.



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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) (_______ , _______ , _______)

C) ____________________________ F) (____________ , ___________)

****** NO CALCULATORS ON THIS ROUND ******

A) In ΔABC, $AB = 10$, $AC = 12$ and $m\angle B = 2m\angle A$.
If A is the smallest angle in ΔABC, compute BC.

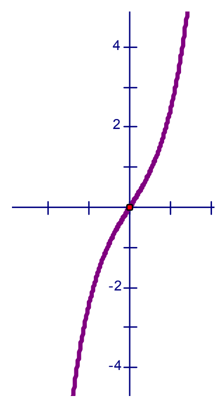
B) Using the letters, arrange the following in order from largest to smallest.

$A = (16{,}874{,}535)^{5000} g(16{,}874{,}537)^{5000}$

$B = (16{,}874{,}533)^{5000} g(16{,}874{,}539)^{5000}$

$C = (16{,}874{,}536)^{5000} g(16{,}874{,}536)^{5000}$

$D = (16{,}874{,}534)^{5000} g(16{,}874{,}538)^{5000}$

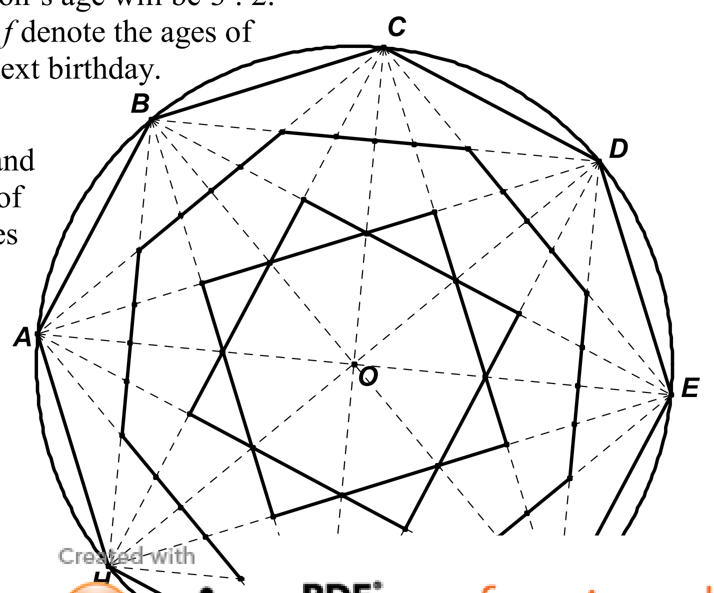


C) Three vertices of <u>rectangle</u> $PQRS$ are $P(-8, -1)$, $Q(k, k)$ and $R(14, 2)$.
Determine <u>all</u> possible coordinates (x, y) of vertex S.

D) The graph of $y = \dfrac{5^x - 5^{-x}}{2}$ is shown at the right.
Solve for x in terms of y.

E) My son, his daughter (my granddaughter) and I have the same birthday! I was 21 when my son was born. On our next birthday, our combined ages will total 100 years, and when my granddaughter turns 25, the ratio of my age to my son's age will be 3 : 2.
Compute the ordered triple (g, s, f), where g, s and f denote the ages of granddaughter, son and father respectively on our next birthday.

F) Eight points lie on a circle and form the vertices of an octagon. Let M and m denote the maximum and minimum number of <u>interior</u> points of intersection of the diagonals respectively, i.e. <u>excluding</u> the vertices of the octagon. Determine the ordered pair (M, m).
The diagram at the right shows a <u>regular</u> octagon $ABCDEFGH$ with all its diagonals.



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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010 ANSWERS

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

A) 13 B) 31 C) $\frac{\sqrt{2}}{2}$

Round 2 Arithmetic/Elementary Number Theory

A) 37 B) $23333_{(4)}$ C) 918

Round 3 Coordinate Geometry of Lines and Circles

A) $\frac{2}{5}$ (or 0.4) B) $2\sqrt{10}$ C) $5\left(\sqrt{2}-1\right)$, $13\left(\sqrt{2}-1\right)$

or equivalent (Both answers required)

Round 4 Alg 2: Log and Exponential Functions

A) $30\sqrt{6}$ B) 16 C) $0, \log_{10} 5$

[$\log 5$ is also acceptable]

Round 5 Alg 1: Ratio, Proportion or Variation

A) 2010 B) Ben: \$22,500
Joe: \$12,500 C) (63, 24)

Round 6 Plane Geometry: Polygons (no areas)

A) 20 B) 48° C) 14 : 15

Team Round

A) 8

B) CADB

C) (12, 7), (−3.5, −8.5)

D) $\log_5\left(y+\sqrt{y^2+1}\right)$

[$\log_5\left(y-\sqrt{y^2+1}\right)<0$ is extraneous]

E) (15, 32, 53)

F) (70, 49)

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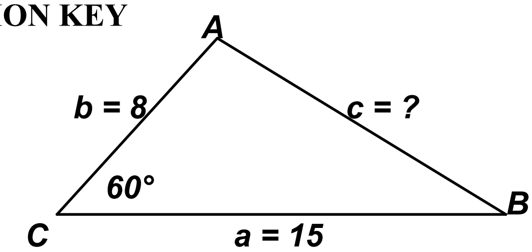
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010 SOLUTION KEY

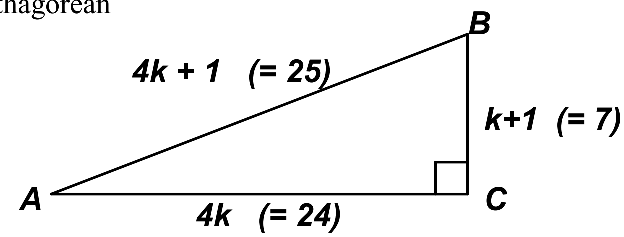


Round 1

A) Using the laws of Cosines, $c^2 = a^2 + b^2 - 2ab\cos C$

$\rightarrow$ $c^2 = 8^2 + 15^2 - 2(8)(15)\cos 60°$

$= 4 + 225 - 120 = 169$ $\rightarrow$ $c =$ **<u>13</u>**



B) Assuming $4k + 1$ is the hypotenuse, and applying the Pythagorean theorem, $(k+1)^2 + (4k)^2 = (4k+1)^2$.

Expanding and canceling, $k^2 + 2k = 8k$

$\rightarrow$ $k^2 - 6k = k(k-6) = 0$ $\rightarrow$ $k = 6$.

$\rightarrow$ $7 - 24 - 25$ right triangle

$\rightarrow$ a sum of 31.

But was $4k + 1$ necessarily the hypotenuse???

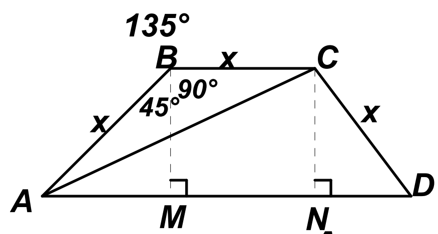
For $k > 0$ (to insure that $4k$ is positive), $4k + 1 > 4k$.

Also $k > 0$ $\rightarrow$ $3k > 0$

$\rightarrow$ $3k + 1 > 1$ (adding 1 to both sides of the inequality)

$\rightarrow$ $4k + 1 > k + 1$ (adding k to both sides of the inequality)

$\rightarrow$ $4k + 1$ is the longest side and must be the hypotenuse. Thus, **<u>31</u>** is the only possible sum.



C) $AM = \dfrac{x}{\sqrt{2}}$ $\rightarrow$ $AD = \dfrac{x\sqrt{2}}{2} + x + \dfrac{x\sqrt{2}}{2} = x\left(\sqrt{2}+1\right)$

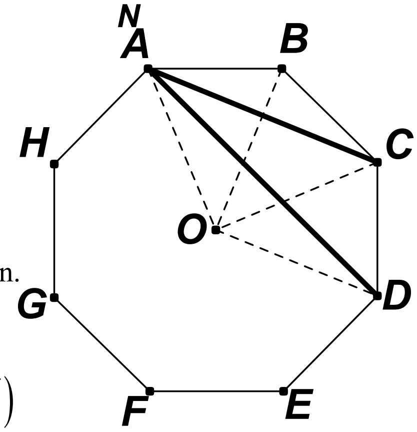
Using the law of cosines on ΔABC,

$\rightarrow$ $AC^2 = x^2\left(2+\sqrt{2}\right)$.

$AC^2 = AD$ $\rightarrow$ $x^2\left(2+\sqrt{2}\right) = x\left(\sqrt{2}+1\right)$

Dividing by x ($\neq 0$), $x\left(2+\sqrt{2}\right) = \sqrt{2}+1$

$\rightarrow$ $AB = x = \dfrac{\sqrt{2}+1}{2+\sqrt{2}} \cdot \dfrac{2-\sqrt{2}}{2-\sqrt{2}} = \dfrac{2\sqrt{2}-2+2-\sqrt{2}}{2} = \underline{\dfrac{\sqrt{2}}{2}}$



Alternate Solution #1 (Tuan Le)

Let O be the center of the circle of radius R circumscribed about the regular octagon.

Now, $m\angle AOB = 45°$ $\rightarrow$ $m\angle AOC = 90°$ $\rightarrow$ ΔAOC is an isosceles right triangle with legs of length R; hence, $AC = R\sqrt{2}$ or $AC^2 = 2R^2$.

Applying the Law of Cosines to ΔAOD, $AD^2 = 2R^2 - 2R^2\cos 135° = R^2\left(2+\sqrt{2}\right)$

The given $AC^2 = AD$ implies $2R^2 = R\sqrt{2+\sqrt{2}}$ $\rightarrow$ $R = \dfrac{\sqrt{2+\sqrt{2}}}{2}$ $\rightarrow$ $R^2 = \dfrac{2+\sqrt{2}}{4}$

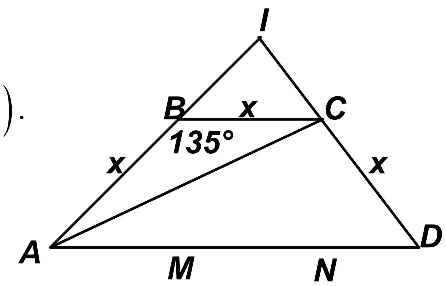
Applying the Law of Cosines to ΔAOB, $AB^2 = 2R^2 - 2R^2\cos 45° = R^2\left(2-\sqrt{2}\right)$

Substituting, $AB^2 = \dfrac{2+\sqrt{2}}{4}\cdot\left(2-\sqrt{2}\right) = \dfrac{4-2}{4} = \dfrac{1}{2}$ $\rightarrow$ $AB = \underline{\dfrac{\sqrt{2}}{2}}$

Round 1 – continued

C) Alternate solution #2 (Norm Swanson)

Using the law of cosines on ΔABC, $AC^2 = AD = x^2\left(2+\sqrt{2}\right)$.



Extend $\overline{AB}$ and $\overline{DC}$ to meet at point I.
ΔIBC and ΔIAD are 45-45-90 right triangles with $BI = \dfrac{x}{\sqrt{2}}$. By similar triangles, $\dfrac{AI}{BI} = \dfrac{AD}{BC}$ or

$$\frac{x+\frac{x}{\sqrt{2}}}{\frac{x}{\sqrt{2}}} = \frac{x^2\left(2+\sqrt{2}\right)}{x} \rightarrow \sqrt{2}+1 = x\left(2+\sqrt{2}\right) = x\sqrt{2}\left(\sqrt{2}+1\right)$$

$$\rightarrow 1 = x\sqrt{2} \rightarrow x = \underline{\boldsymbol{\frac{\sqrt{2}}{2}}}$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010 SOLUTION KEY

Round 2

A) Since the digit sum of 7659, namely 27, is divisible by 9, 9 must be a factor of 7659. Thus, 7659 = 9(851) and we need to determine whether 851 is prime or factors as a product of smaller primes. If any prime factor exists, there must be one smaller than $\sqrt{851}$. Since $30^2 = 900 > 851$, we restrict our search to primes less than or equal to 29.
2 and 5 obviously fail. By long division, 7, 11, 13, 17 and 19 fail, but 851 = 23(37).
Thus, the largest prime factor of 7659 is **<u>37</u>**.

B) Converting from base 7: $2144_{(7)} = 2\left(7^3\right)+1\left(7^2\right)+4\left(7^1\right)+4\left(7^0\right) = 686+49+28+4 = 767_{(10)}$

The **<u>shortcut</u>** looks like synthetic substitution:

$$\begin{array}{r|rrrr} & 2 & 1 & 4 & 4 \\ & & 14 & 105 & 763 \\ \hline 7 & 2 & 15 & 109 & \mathbf{767} \end{array}$$

Converting to base 4: The digit values in base 4 are $\underset{1024}{\underline{4^5}}\ \underset{256}{\underline{4^4}}\ \underset{64}{\underline{4^3}}\ \underset{16}{\underline{4^2}}\ \underset{4}{\underline{4^1}}\ \underset{1}{\underline{4^0}}$

Since 767 < 1024, only the five rightmost positions will be filled with a digit.

$\dfrac{767}{256} \to \mathbf{2}\ r255$ or 767 – 2(256) = 767 – 512 = 255. The leftmost digit must be 2.

Continuing, $\dfrac{255}{64} \to \mathbf{3}\ r63\quad \dfrac{63}{16} \to \mathbf{3}\ r15\quad \dfrac{15}{4} \to \mathbf{3}\ r\,\mathbf{3}$, we see the remaining digits are 3333.

The **<u>shortcut</u>** looks like long division, recording quotients and *remainders*.
Eventually, the divisor (in this case 4) will be larger than the dividend, the quotient will be zero and we will have the last remainder.
The remainders are the digits and they are read from the bottom up.

$$\begin{array}{l} 4\underline{|767} \to r_1 = 3 \\ 4\underline{|191} \to r_2 = 3 \\ 4\underline{|47} \to r_3 = 3 \\ 4\underline{|11} \to r_4 = 3 \\ 4\underline{|\ 2} \to r_5 = 2 \\ \quad 0 \end{array}$$

Reading up, we have **<u>23333$_{(4)}$</u>**.

C) $42875 = 25(1715) = 125(343) = 5^3 \cdot 7^3$ or $35^3 \cdot 1^3$
Thus, $n = 3$ and $(a + b, a - b) = (7, 5)$ or $(35, 1)$
(7, 5) ➔ $(a, b) = (6, 1)$
(35, 1) ➔ $(a, b) = (18, 17)$
The maximum value of abn = (18)(17)(3) = 18(51) = **<u>918</u>**

$$\begin{array}{l} 5\underline{|42875} \\ 5\underline{|8574} \\ 5\underline{|1715} \\ 7\underline{|343} \quad \Rightarrow 5^3 \cdot 7^3 \\ 7\underline{|49} \\ \quad 7 \end{array}$$

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010 SOLUTION KEY

Round 3

A) Points A and B lie on the perpendicular bisector of $\overline{PQ}$. The slope of $\overline{PQ}$ is $\dfrac{-6-(-1)}{3-1}=\dfrac{-5}{2}$

Thus, the slope of $\overline{AB}$ must be $\underline{\mathbf{+\frac{2}{5}}}$ (the negative reciprocal).

No need to find the coordinates of *A* and *B*!!! But if you insist:

$A(a, 0)$ and $PA = QA$ ➔ $(a-1)^2+1=(a-3)^2+36$ ➔ $-2a + 2 = -6a + 45$ ➔ $a = \dfrac{43}{4} = 10.75$

$B(0, b)$ and $PB = QB$ ➔ $1 + (b + 1)^2 = 9 + (b + 6)^2$ ➔ $2b + 2 = 12b + 45$ ➔ $b = -4.3$

The slope of $\overline{AB}$ equals $\dfrac{0-b}{a-0}=\dfrac{-b}{a}=\dfrac{4.3}{10.75}=\dfrac{430}{1075}=\dfrac{5(2)(43)}{5(5)(43)}=\underline{\mathbf{\dfrac{2}{5}}}$

B) $P(0, 0)$ is clearly one of the points of intersection.

$(x-4)^2+(y+2)^2=20 \Leftrightarrow x^2+y^2-8x+4y=0$

$2(x^2+y^2-8x+4y=0)-(2x^2+2y^2-9x-13y=0) \Leftrightarrow 7x+21y=0 \Leftrightarrow x=-3y$

Substituting, $2(9y^2)+2y^2-9(-3y)-13y=0$ ➔ $20y^2-40y=20y(y-2)=0$

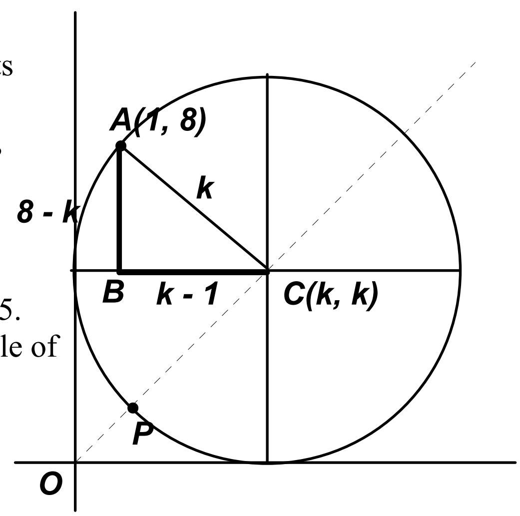
➔ $y = 2, x = 6$. Thus, $\overline{PQ}$ connects (0, 0) and (6, 2) and $PQ = \sqrt{6^2+2^2}=\sqrt{40}=\underline{\mathbf{2\sqrt{10}}}$.

C) Consider the diagram at the right. Since the circle is tangent to both axes, its center must be at (k, k) and its radius must be k units. In ΔABC, $AB = |8 - k|$ and $BC = |k - 1|$. Applying the Pythagorean theorem,

$(8-k)^2+(k-1)^2=k^2$ ➔ $k^2 - 18k + 65 = 0$

➔ $(k - 5)(k - 13) = 0$ ➔ $k = 5, 13$.

Both k- values are possible. The diagram at the right shows the relative position of P for a circle of radius 5. You are encouraged to re-draw the diagram for a circle of radius 13. The required point is P and



$OP = OC - PC = k\sqrt{2}-k=k\left(\sqrt{2}-1\right)$

Thus, $OP = \underline{\mathbf{5\left(\sqrt{2}-1\right),\ 13\left(\sqrt{2}-1\right)}}$

Both answers are required.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010 SOLUTION KEY

Round 4

A) B is the midpoint of $\overline{AC}$ ➔ $\log n = \dfrac{\log 60 + \log 90}{2} = \dfrac{1}{2}\log 5400 = \log\sqrt{5400}$

Therefore, $n = \sqrt{5400} = \sqrt{100\cdot 9\cdot 6} = \underline{\mathbf{30\sqrt{6}}}$

B) $5^{2\log_5 x} - 12\left(4^{\log_2\sqrt{x}}\right) - 27^{\log_3 4} = 0$ ➔ $5^{\log_5 x^2} - 12\left(4^{\log_4 x}\right) - 3^{\log_3\left(4^3\right)} = 0$

➔ $x^2 - 12x - 64 = (x - 16)(x + 4) = 0$ ➔ $x = \underline{\mathbf{16}}$ ($x = -4$ is extraneous.)

C) Let $a = 10^x$. Then $100^x - 3\cdot 2^{x+1}\cdot 5^x + 5 = 0$ ➔ $a^2 - 6a + 5 = (a - 5)(a - 1) = 0$
Thus, $10^x = 1$ ➔ $x = \underline{\mathbf{0}}$ or $10^x = 5$ ➔ $x = \underline{\mathbf{\log_{10} 5}}$ (or simply $\underline{\mathbf{\log 5}}$)

Round 5

A) y varies directly as x and z ➔ $y = kxz$, for some constant k.
Substituting, $5 = k(3)(4)$ ➔ $k = 5/12$.

Therefore, $y = \dfrac{5}{12}\cdot 36\cdot 134 = 15(134) = \underline{\mathbf{2010}}$

B) The profit from the sale of the house was \$227000 – (\$180000 + \$12000) = \$35000

A total of \$5600 worth of repairs were done and Ben contributed $\dfrac{3600}{5600} = \dfrac{9}{14}^{\text{th}}$ of the money,

and Joe contributed $\dfrac{5}{14}^{\text{th}}$. $14k = 35000$ ➔ $k = 12500$ ➔ **Ben: <u>\$22,500</u> Joe: <u>\$12,500</u>**

C) Let $A = 60 + x$ and $B = 20 + y$.

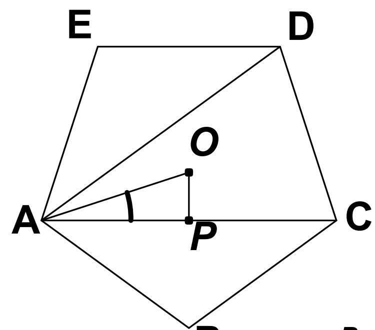
According to the first student, $\dfrac{10x+6}{20+y} = \dfrac{3}{2}$. According to the second student, $\dfrac{60+x}{10y+2} = \dfrac{3}{2}$.

Cross multiplying, $20x + 12 = 3y + 60$ ➔ $20x - 3y = 48$ and
$120 + 2x = 30y + 6$ ➔ $2x - 30y = -114$

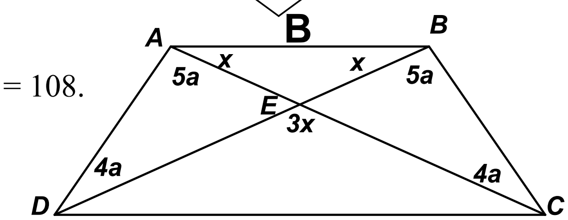
$\begin{cases} -200x + 30y = -480 \\ 2x - 30y = -114 \end{cases}$ ➔ $-198x = -594$ ➔ $x = 3, y = 4$ ➔ $(A, B) = \underline{\mathbf{(63, 24)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010 SOLUTION KEY

Round 6

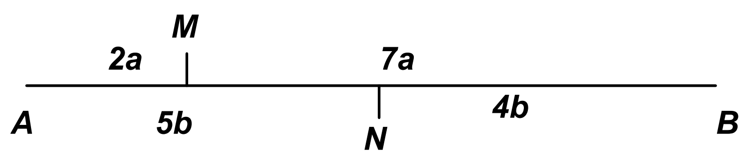


A) m$\angle E = 108°$ ➔ m$\angle EAD = 36°$ (Base angle of isosceles ΔEAD)
Likewise, m$\angle BAC = 36°$ ➔ m$\angle DAC = 36°$
m$\angle OAP = \frac{1}{2}$ m$\angle DAC = 18°$
$\frac{360}{n} = 18$ ➔ $n = \underline{\mathbf{20}}$.

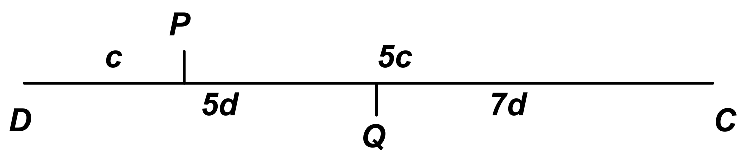


B) In ΔAEB, we have $x + x + 3x = 180$ ➔ $x = 36$ ➔ m$\angle DEC = 108$.
Since $\angle DEC$ is an exterior angle of ΔAED, $9a = 108$
➔ $a = 12$ ➔ m$\angle ADE$ = m$\angle BCE$ = $\underline{\mathbf{48}}°$.

C) The fact that $\frac{AM}{MB} = \frac{2}{7}$ and $\frac{AN}{NB} = \frac{5}{4}$ is shown in the diagram at the right.



Clearly, $9a = 9b$ and $a = b$.
The fact that $\frac{DP}{PC} = \frac{1}{5}$ and $\frac{DQ}{QC} = \frac{5}{7}$ is shown in the diagram at the right.
Clearly, $6c = 12d$ and $c = 2d$.

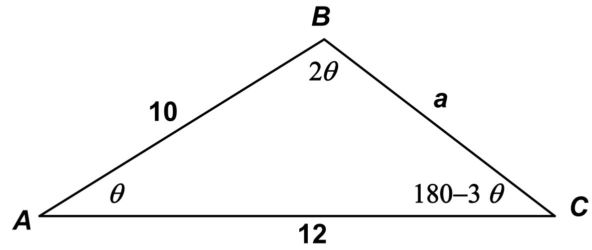


As opposite sides of a parallelogram, $AB = CD$ ➔ $9a = 12d$ ➔ $\frac{a}{d} = \frac{4}{3}$

Thus, the required ratio is $\frac{MB}{PC} = \frac{7a}{5c} = \frac{7a}{10d} = \frac{7}{10} \cdot \frac{4}{3} = \underline{\mathbf{\frac{14}{15}}}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2010 SOLUTION KEY

Team Round



A) Let $m\angle A = \theta$. Then $m\angle B = 2\theta$ and $m\angle C = 180 - 3\theta$.

Using the law of Cosines,
$a^2 = 10^2 + 12^2 - 2(10)(12)\cos\theta = 244 - 240\cos\theta$ (#1).
Using the law of Sines,

$$\frac{\sin\theta}{a} = \frac{\sin 2\theta}{12} = \frac{\sin(180-3\theta)}{10} \rightarrow \frac{\sin\theta}{a} = \frac{2\sin\theta\cos\theta}{12} = \frac{\sin(3\theta)}{10} = \frac{\sin\theta\left(3-4\sin^2\theta\right)}{10}$$

Since $\sin\theta \neq 0$, we have $\frac{1}{a} = \frac{\cos\theta}{6} = \frac{3-4\sin^2\theta}{10}$ (#2).

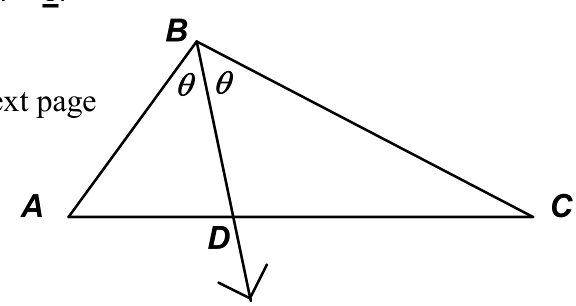
Method #1: Substituting $\cos\theta = \frac{6}{a}$, we have $a^2 = 244 - 240\left(\frac{6}{a}\right) \rightarrow a^3 - 244a + 1440 = 0$

As the smallest side, $a < 10$. Since $a = 7 \rightarrow +75$ and $a = 9 \rightarrow -27$, we try $a =$ **<u>8</u>** and hit paydirt!

Method #2: Using the last two ratios in #2, $5\cos\theta = 3\left(3-4\sin^2\theta\right) = 3\left(4\cos^2\theta - 1\right) \rightarrow$

$12\cos^2\theta - 5\cos\theta - 3 = (4\cos\theta - 3)(3\cos\theta + 1) = 0 \rightarrow \cos\theta = +\frac{3}{4}$

($\cos\theta = -\frac{1}{3}$ would imply θ was obtuse which is impossible for the smallest angle in ΔABC.)
Substituting in #1, $a^2 = 244 - 240(3/4) = 244 - 180 = 64 \rightarrow a =$ **<u>8</u>**.

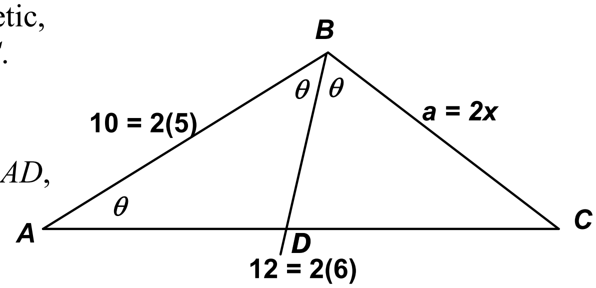


Alternate Solution (Norm Swanson)
Requisite Notions (using diagram at right) - <u>proved</u> on the next page

Angle Bisector Theorem #1: $\frac{AD}{CD} = \frac{AB}{CB}$

Angle Bisector Theorem #2: $BD^2 = (AB)(BC) - (AD)(DC)$

Draw the angle bisector of $\angle B$, intersecting $\overline{AC}$ at point D.
ΔBAD is isosceles with $AD = BD$. To simplify the arithmetic,
let $AB = 5$, $AC = 6$ and then <u>double</u> our answer for $x = BC$.



Using angle bisector theorem #1, $AD = BD = \frac{30}{x+5}$.

Using angle bisector theorem #2, substituting for BD and AD,

$$\frac{900}{(x+5)^2} = 5x - \frac{180x}{(x+5)^2}$$

$\rightarrow x(x+5)^2 - 36x = 180 \rightarrow x^3 + 10x^2 - 11x = 180$
The left side of this equation factors to $x(x-1)(x+11)$; the right side factors as $2^2 \cdot 3^2 \cdot 5$.
By inspection $4(3)15) = 180$, so $x = 4$ and $a = BC = 8$.
Also the cubic $x^3 + 10x^2 - 11x = 180$ factors as $(x-4)(x+5)(x+9)$, so again $x = 4$.

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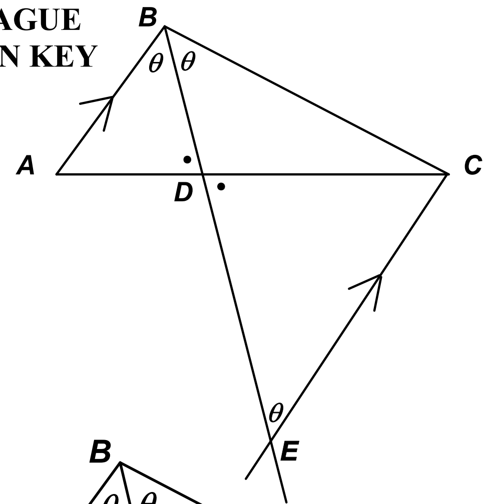
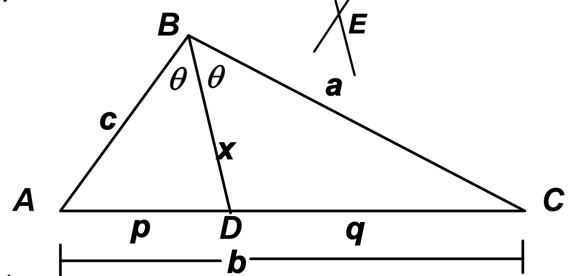


Proof of Angle Bisector Theorem #1:

Draw a line through C parallel to $\overline{AB}$, intersecting $\overrightarrow{BD}$ in E.
Note: As alternate interior angles of parallels, $\angle ABD \cong \angle CED$
ΔBCE is isosceles, with $BC = CE$.
$\Delta ABD \sim \Delta CED$. As corresponding sides of similar triangles,

$\dfrac{AD}{CD} = \dfrac{AB}{CE}$. Substituting BC for CE gives us the required result.



Proof of Angle Bisector Theorem #2:
In the diagram at the right, let $BD = x$, $AD = p$ and $CD = q$.
Using Angle Bisector theorem #1, note that
$\dfrac{p}{q} = \dfrac{p}{b-p} = \dfrac{a}{c} \rightarrow p = \dfrac{bc}{a+c}$. Similarly, show that $q = \dfrac{ab}{a+c}$.
Now a double application of the Law of Cosines, some substitution and rather impressive simplification!

ΔBAD: $x^2 = c^2 + p^2 - 2pc\cos A$ (***) ΔABC: $a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

Now substituting for p and cos$A$ in (***), $x^2 = c^2 + \left(\dfrac{bc}{a+c}\right)^2 - \dfrac{2bc^2}{a+c}\cdot\dfrac{b^2+c^2-a^2}{2bc}$

$$= c^2 + \frac{b^2c^2}{(a+c)^2} + \frac{a^2-b^2-c^2}{a+c} = \frac{c^2(a+c)^2 + b^2c^2 + (a+c)(a^2-b^2-c^2)}{(a+c)^2}$$

$$= \frac{a^2c^2 + 2ac^3 + c^4 + b^2c^2 + a^3c - ac^3 - ab^2c + a^2c^2 - c^4 - b^2c^2}{(a+c)^2} = \frac{2a^2c^2 + ac^3 + a^3c - ab^2c}{(a+c)^2}$$

$$= \frac{ac(2ac + c^2 + a^2 - b^2)}{(a+c)^2} = \frac{ac((a+c)^2 - b^2)}{(a+c)^2} = ac - \frac{ab}{a+c}\cdot\frac{bc}{a+c} = ac - pq$$

or $BD^2 = (AB)(BC) - (AD)(DC)$, as required. A truly remarkable result - worth remembering for future contests!

For those familiar with Stewart's Theorem, ($c^2q + a^2p = x^2b + bpq$ in terms of the diagram above), the algebraic manipulations are greatly simplified. Since $p = \dfrac{bc}{a+c}$ and $q = \dfrac{ab}{a+c}$ from Angle Bisector Theorem #1, we have

$$\frac{abc^2}{a+c} + \frac{a^2bc}{a+c} = x^2b + bpq \rightarrow \frac{ac^2}{a+c} + \frac{a^2c}{a+c} = \frac{ac\cancel{(c+a)}}{\cancel{a+c}} = x^2 + pq \rightarrow x^2 = ac - pq.$$

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B) You need only consider the rightmost three digits in each case.
536(536) > 535(537) > 534(538) > 533(539)
We know this because 535(537) = (536 – 1)(536 + 1) = $536^2 - 1$,
534(538) = (536 – 2)(536 + 2) = $536^2 - 4$ and 533(539) = (536 – 3)(536 + 3) = $536^2 - 9$
Thus, without multiplying out any of these products, the order from largest to smallest is
<u>**CADB**</u>.

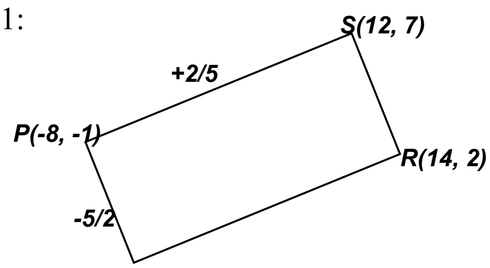
C) In rectangle *PQRS*, $\overline{PQ} \perp \overline{QR}$. Thus, the slopes of these segments are negative reciprocals of each other and it follows that product of the slopes will be −1.

$\left(\dfrac{k+1}{k+8}\right)\left(\dfrac{k-2}{k-14}\right) = -1 \rightarrow (k+1)(k-2) = -(k+8)(k-14) \rightarrow k^2 - k - 2 = -k^2 + 6k + 112$

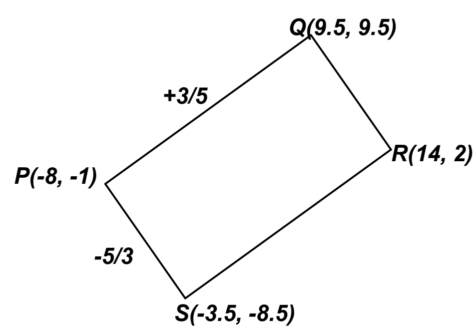
$\rightarrow 2k^2 - 7k - 114 = 0 \rightarrow (2k - 19)(k+6) = 0 \rightarrow k = 19/2$ or -6

$Q(-6, -6)$ to $P(-8, -1)$ = 2 left 5 up
Starting at $R(14, 2)$ and moving 2 left and 5 up, we arrive at S<u>**(12, 7)**</u>
$Q(19/2, 19/2)$ to $P(-8, 1)$ = 17.5 left 10.5 down
Starting at $R(14, 2)$ and moving 17.5 left and 10.5 down, we arrive at S<u>**(−3.5, −8.5)**</u>

Case 1:



Case 2:



D) $2y = 5^x - 5^{-x} \rightarrow 2y = 5^x - \dfrac{1}{5^x} \rightarrow 2y = \dfrac{\left(5^x\right)^2 - 1}{5^x} = \dfrac{5^{2x} - 1}{5^x} \rightarrow 5^{2x} - (2y)5^x - 1 = 0$

Applying the quadratic formula (treating 5^x as the variable and 1, $(-2y)$ and -1 as the coefficients)

$5^x = \dfrac{2y \pm \sqrt{4y^2 + 4}}{2} \rightarrow 5^x = y + \sqrt{y^2 + 1}$ $\quad$ ($y - \sqrt{y^2+1} < 0$ and is extraneous)

$x = \underline{\mathbf{\log_5\left(y + \sqrt{y^2 + 1}\right)}}$

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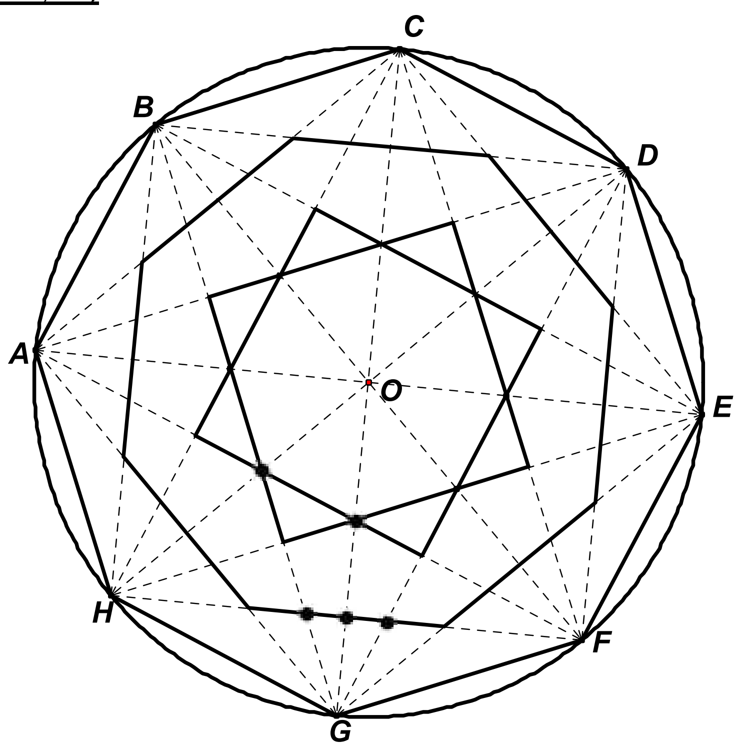
E) When my son is x years old, I will be $(x + 21)$ years old.
In a certain number of years, say n years, my age and my son's age are in a 3 : 2 ratio.
$\frac{x+21+n}{x+n}=\frac{3}{2}$ ➔ $x + n = 42$. If $n = 25$, then $x = 17$ and, on that birthday, our ages are $g = 25$, $s = 42$ and $f = 63$. On that birthday, my son is 17 years older than his daughter and I am 38 years older than my granddaughter. This condition is invariant, as long as we are all alive.
k years after my granddaughter is born our ages total 100 ➔ $k + (17 + k) + (38 + k) = 100$ ➔ $k = 15$
Thus, (g, s, f) = **(15, 32, 53)**.



F) The minimum number of points of intersection will occur in a regular octagon. Points A through H are excluded. The inner octagon contains 3 points of intersection on each side, plus the 8 vertices, for a total of 32. The two intersecting squares contain 16 additional points of intersection, two on each side and the 8 vertices. Adding the center point, we have a minimum, $m = 49$.

To maximize the number of points of intersection, we must examine the points where more than two lines intersect, i.e. the 8 points where the two squares intersect as well as the center point. At each of the 8 points there are three intersecting lines which could have determined 3 points, instead of a single point of concurrency. This would add an additional $24 - 8 = 16$ points.

At the center point there are four intersecting lines which could have determined $\binom{4}{2} = 6$ points, instead of a single point of concurrency. This would add $6 - 1 = 5$ additional points of intersection. Thus, the maximum $M = 49 + 16 + 5 = 70$. (M, m) = **(70, 49)**

The original problem 2C did not specify that n must be greater than 1.
Therefore, the answer was 459, 566,406 – the product of two 5-digit numbers
This was an unintended exercise in number crunching without a calculator.
Everyone was given credit for the problem.

$a + b = 42875$ and $a - b = 1$ → $(a, b) = (21438, 21437)$ → $ab =$ **459,566,406**.
With the added condition n > 1, the original answer/solution is correct.

Problem 4B:
The solution rejected $x = -4$ since substitution in the original equation required taking the base 2 logarithm of a complex number ($2i$) which is not defined in algebra 2.
Thus, simplifying the equation to $x^2 - 12x + 64 = 0$ invokes the rule
$a^{\log_a x} = x$. In algebra 2, there is a restriction that $x > 0$.
Appeal submitted by coach of student who was taking a Complex Variables course that both answers should be accepted is denied. Actual written appeal of student never sent to me.