MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2011
ROUND 1 ANALYTIC GEOMETRY: ANYTHING

ANSWERS

A) _______________________________

B) ____________________________ = 0

C) _______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Given: $\dfrac{(x+1)^2}{2}+\dfrac{(y-3)^2}{4}=1$

Compute the largest possible value of y.

B) Find the equation of the tangent line to the point (3, 2) on the circle $x^2+y^2-8y+3=0$.
Express your answer in simplified $\boxed{Ax+By+C=0}$ form, where A, B and C are integers and $A>0$.

C) Let S be the locus of points in the plane that are equidistant from the line $x = 2$ and the point $P(6, 3)$. Compute the coordinates (x, y) of all points of intersection between S and the line whose equation is $2x - 2y - 7 = 0$.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2011
ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) $x =$ ______________________________

B) $k =$ ______________________________

C) $x =$ ______________________________

***** NO CALCULATORS ON THIS ROUND *****

A) Find all possible values of x such that $x^2 + 2x + 1 = 10000$.

B) Determine the value of k for which $36x^2 - kx + 16 = A(Bx - C)^2$, where A, B and C are positive integers.

C) For some real constant A, $A^2 - 4x = x^2 + 4A$. Find all simplified expressions for x in terms of A.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2011
ROUND 3 TRIG: EQUATIONS WITH A REASONABLE NUMBER OF SOLUTIONS

ANSWERS

A) $k =$ ____________________

B) $x =$ ____________________

C) $x =$ ____________________

******* NO CALCULATORS ON THIS ROUND *******

A) The positive solutions of $\tan(5x) = 1$ are arranged in a list in increasing order. The fourth value in the list is $k°$. Compute k.

B) Solve for x over $0 \le x < \pi$. $\quad (\sin 2x)(\cos^4 x - \sin^4 x) = 0$

C) Solve for x over $\frac{\pi}{2} < x < \frac{5\pi}{2}$: $\quad \sin\left(2x + \frac{\pi}{4}\right) = \cos\left(\frac{7\pi}{4}\right)$

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2011
ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) ________________________________

B) __________________________ inches

C) ________________________________

***** NO CALCULATORS ON THIS ROUND *****

A) Given: $x+\frac{1}{x}=\frac{13}{6}$

Compute the numerical value of $(x+1)\left(\frac{1}{x}+1\right)$.

B) A rectangular piece of cardboard is 13 inches longer than it is wide. If squares whose perimeter is 16 inches are cut from each corner and the resulting figure is folded up to form an open-topped box, the volume will be 1200 cubic inches. Find the <u>sum</u> of the three dimensions of the box.

C) For what value(s) of m are the roots of $x^2 - mx + m + 3 = 0$ nonzero and real?

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2011
ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

ANSWERS

A) _______________ : _______________

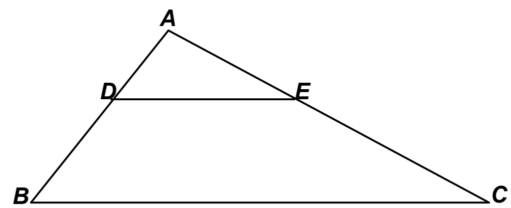
B) _______________ : _______________

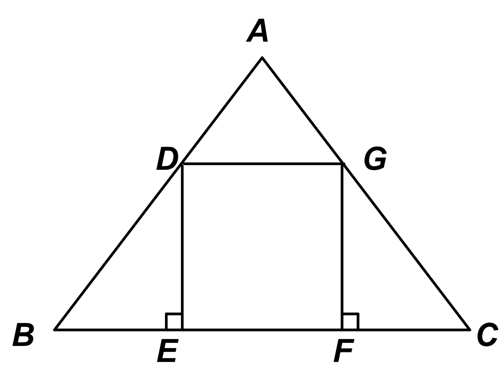
C) ______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Given: $\overline{DE} \parallel \overline{BC}, DE = 10, BC = 25$

Compute the ratio of the area of ΔADE to the area of trapezoid *DECB*.





B) ΔABC is equilateral, $\overline{DG} \parallel \overline{BC}$.

The area $BDGC$ is $\frac{15}{16}^{\text{th}}$ the area of ΔABC.

Compute the ratio of the area of ΔABC to the area of ΔBED.

C) Given: Four regular hexagons *A*, *B*, *C* and *D*

A has an area of $\frac{9\sqrt{3}}{4}$ square units.

A longer diagonal in *B* has length $4\sqrt{6}$.

Sides of regular hexagon *C* have the same length as a shorter diagonal of *A*.

Sides of regular hexagon *D* have the same length as a shorter diagonal of *B*.

Compute the <u>sum</u> of the areas of hexagons *C* and *D*.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2011
ROUND 6 ALG 1: ANYTHING

ANSWERS

A) (_______ , _______ , _______)

B) ____________________________ %

C) ____________________________

A) Find the ordered triple (x, y, t) if $x = 4t + 1$, $y = 11 - 3t$ and $x + y + 3t = 0$.

B) What is the exact percent error made when the repeating decimal $0.1\overline{6}$ is approximated as 0.17 ?

Definition: % Error = $\dfrac{|\text{Estimate} - \text{Exact}|}{\text{Exact}} \cdot 100\%$

C) A collection of coins consists exclusively of nickels, dimes and quarters.
The value of the collection is \$6.50.
The value of the dimes is 4 times the value of the nickels.
If there are 50 coins in the collection, how many quarters are there?

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2011
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ________________________________ D) $y =$ ____________________

B) ________________________________ E) ________________ sq. units

C) ________________________________ F) (__________ , __________)

***** NO CALCULATORS ON THIS ROUND *****

A) There are four squares with sides parallel to the x- and y-axes that have a common vertex at $P(3, 4)$ and whose sides have length 5. What is the sum of the distances from the four vertices of the square formed by the union of the four original squares to the line $y = x$?

B) Factor completely as the product of polynomials with integer coefficients: $x^8 + x^4 + 1$

C) Some older textbooks define the following trigonometric functions: $\begin{cases} \text{vers } A = 1 - \cos A \\ \text{covers } A = 1 - \sin A \\ \text{hav } A = \dfrac{1}{2} \text{vers } A \end{cases}$

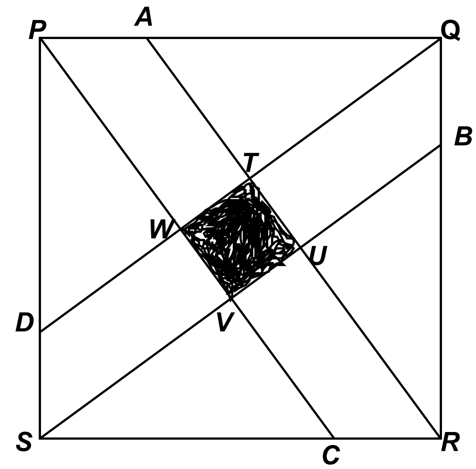
Compute all values of A, where $0 \le A < 2\pi$ for which $\text{hav}(2A) + \dfrac{\text{covers}(A)}{2} = 1$.

D) Solve for y in terms of x.

$$x^2 - xy - 6y^2 + x + 7y - 2 = 0$$

Answers must be simplified.

E) A, B, C and D are points placed so that their distances to the endpoints of their corresponding sides of square $PQRS$ are in a 3 : 1 ratio, as indicated in the diagram at the right. ($AQ > AP$). Find the area of the shaded region, if $PQ = 8$.



F) Let $A = \sqrt{x + 37}$ and $B = \sqrt{x - N}$, where x, A, B and N are positive integers. The number of possible ordered pairs (A, B) depends on the value of N. The smallest value of N for which there are exactly three possible ordered pairs (A, B) is k. Let T denote the sum of the values of x for which this happens. Compute the ordered pair (k, T).

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2011 ANSWERS

Round 1 Analytic Geometry: Anything

A) 5

B) $3x - 2y - 5 = 0$

C) (4.5, 1), (16.5, 13)
- or equivalent -

Round 2 Alg1: Factoring

A) $99, \ -101$

B) 48

C) $A - 4, \ -A$

Round 3 Trig: Equations

A) 117

B) $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$
(in any order)

C) $\pi, \frac{5\pi}{4}, 2\pi, \frac{9\pi}{4}$
(in any order)

Round 4 Alg 2: Quadratic Equations

A) $\frac{25}{6}$ (or equivalent)

B) 41

C) $m \le -2$ or $m \ge 6$ $(m \ne -3)$
- or equivalent -

Round 5 Geometry: Similarity

A) 4 : 21

B) 32 : 9

C) $\frac{459\sqrt{3}}{4}$

Round 6 Alg 1: Anything

A) (–11, 20, –3)

B) 2%

C) 14

Team Round

A) $11\sqrt{2}$

B) $\left(x^4 - x^2 + 1\right)\left(x^2 - x + 1\right)\left(x^2 + x + 1\right)$
- or equivalent -

C) $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

D) $y = \frac{x+2}{3}, \frac{1-x}{2}$
(-or equivalent simplified expressions-)

E) $\frac{64}{25}$

F) (8, 548)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2011 SOLUTION KEY

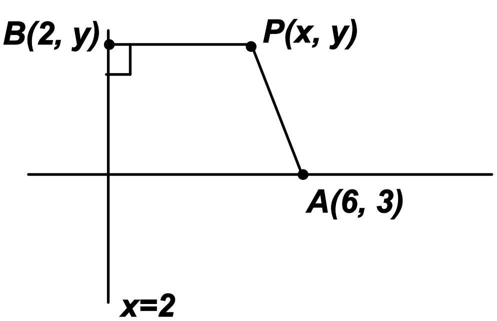
Round 1

A) The equation $\frac{(x+1)^2}{2}+\frac{(y-3)^2}{4}=1$ defines a vertical ellipse with center at $(-1, 3)$ and $a = 2$.
Thus, the maximum possible value of y is $3 + 2 = \underline{\mathbf{5}}$.

Alternate solution (I don't know about ellipses.)
Both terms being added are nonnegative and their sum is a constant. Minimizing the value of the x-expression will maximize the value of the y-expression. Let $x = -1$. Then:
$\frac{(y-3)^2}{4}=1 \rightarrow (y-3)^2 = 4 \rightarrow y - 3 = \pm 2 \rightarrow y = \underline{\mathbf{5}}$.

B) The equation of the circle $x^2+y^2-8y+3=0$ is equivalent to $x^2+(y-4)^2=13$. The slope of the line from the center (0, 4) to the point of tangency (3, 2) is $\frac{2}{-3}$ and the slope of the tangent line is $+\frac{3}{2}$. Using point-slope, the equation of the tangent line is $(y - 2) = \frac{3}{2}(x - 3)$
$\rightarrow 2y - 4 = 3x - 9 \rightarrow \underline{\mathbf{3x - 2y - 5 = 0}}$.

C) Equidistant from a point and a line describes a parabola.
Since the line $x = 2$ is vertical the equation must be of the form $(y - k)^2 = 4p(x - h)$.
$(h, k) = (6, 3)$ and since the vertex is at $(4, 3)$, $p = 2$. Thus, the equation of the parabola is $(y - 3)^2 = 8(x - 4)$. Rewriting the equation of the line, we have $x = 3.5 + y$.
Substituting, $(y-3)^2=8\left(y-\frac{1}{2}\right)=8y-4 \rightarrow y^2-14y+13=(y-1)(y-13)=0 \rightarrow y = 1, 13$
$\rightarrow$ **<u>(4.5, 1), (16.5, 13)</u>** or equivalent.



Alternative solution: (I only know the distance formula.)
$PA = PB \rightarrow x-2=\sqrt{(x-6)^2+(y-3)^2}$
$\rightarrow (x-2)^2=(x-6)^2+(y-3)^2$
$\rightarrow -4x+4=-12x+36+(y-3)^2$
$\rightarrow 8x-32=8(x-4)=(y-3)^2$ Substitute for x and solve the resulting equation for y, obtaining $(x, y) =$ **<u>(4.5, 1), (16.5, 13)</u>**.

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Round 2

A) $x^2+2x+1=10000 \rightarrow (x+1)^2=10000=10^4 \rightarrow x+1=\pm 10^2 \rightarrow x=$ **<u>99, –101</u>**.

B) Factoring out a 4, the first and third terms are perfect squares.
$36x^2 - kx + 16 = 4\left((3x)^2 + \_\_\_ + (2)^2\right)$. Since A, B and C are positive integers,
$A(Bx-C)^2 = 4(3x-2)^2$ and $k = 4(2)(3)(2) =$ **<u>48</u>**.

C) $A^2 - 4x = x^2 + 4A \rightarrow \left(x^2+4x+4\right) = A^2-4A+4 \rightarrow (x+2)^2 = (A-2)^2 \rightarrow x+2=\pm(A-2)$
$\rightarrow x =$ **<u>$A-4, -A$</u>**

Round 3

A) Tan($5x$) = 1 $\rightarrow 5x = 45° + 180n \rightarrow x = 9° + 36n$
$n = 0, 1, 2, 3, \ldots \rightarrow x = 9°, 45°, 81°, 117°, \ldots \rightarrow k =$ **<u>117</u>**

B) $(\sin 2x)\left(\cos^4 x - \sin^4 x\right) = (\sin 2x)\left(\cos^2 x - \sin^2 x\right)\left(\cos^2 x + \sin^2 x\right) = \sin 2x \cos 2x = \frac{1}{2}\sin 4x = 0$

$\rightarrow 4x = 0 + n\pi \rightarrow x =$ $\underline{\mathbf{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}}}$ (in any order)

C) $\sin\left(2x+\frac{\pi}{4}\right) = \cos\left(\frac{7\pi}{4}\right) \rightarrow \sin\left(2x+\frac{\pi}{4}\right) = +\frac{\sqrt{2}}{2} \rightarrow 2x+\frac{\pi}{4} = \begin{cases} \frac{\pi}{4}+2n\pi \\ \frac{3\pi}{4}+2n\pi \end{cases} \rightarrow x = \begin{cases} n\pi \\ \frac{\pi}{4}+n\pi \end{cases}$

$\rightarrow x = \pi, 2\pi, \frac{5\pi}{4}, \frac{9\pi}{4}$ (in any order)

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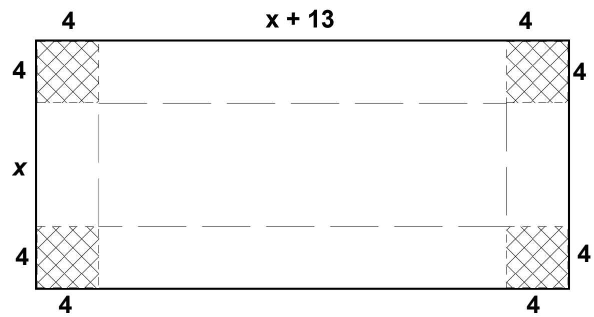
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2011 SOLUTION KEY

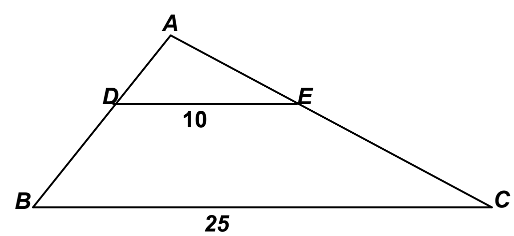
Round 4

A) $(x+1)\left(\frac{1}{x}+1\right)=1+x+\frac{1}{x}+1=x+\frac{1}{x}+2=\frac{13}{6}+2=$ $\underline{\mathbf{\frac{25}{6}}}$



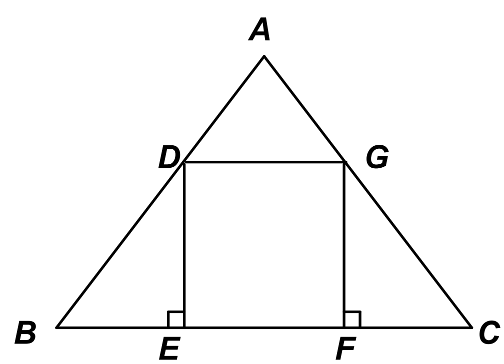
B) Let the dimensions of the cardboard be
$(x + 8)$ by $(x + 21)$
→ dimensions of the box are 4 by x by $(x + 13)$
$V = 4x(x + 13) = 1200$ → $4(x^2 + 13x) - 1200 = 0$
→ $x^2 + 13x - 300 = 0$
→ $(x + 25)(x - 12) = 0$ → $x = 12$, $\cancel{-25}$
→ dimensions: 4 x 12 x 25 → sum: **41**.

C) To have real roots the discriminant $B^2 - 4AC$ must be nonnegative.
$(-m)^2 - 4(m+3)(1) = m^2 - 4m - 12 = (m-6)(m+2) \geq 0$
The critical points on the number line are -2 and $+6$.
Testing in the 3 intervals on the number line, the product is positive when $m \leq -2$ or $m \geq 6$.
However, we also require that the roots be nonzero, that is
$\frac{m \pm \sqrt{m^2-4m-12}}{2} \neq 0$ → $m \neq \pm\sqrt{m^2-4m-12}$ → $m^2 \neq m^2 - 4m - 12$ → $4m \neq -12$ → $m \neq -3$
Therefore, the required set of m-values is: $\underline{\boldsymbol{m \leq -2 \text{ or } m \geq 6\ (m \neq 3)}}$
Alternately, $m < -3$ or $-3 < m \leq -2$ or $m \geq 6$
Also acceptable: $(-\infty,-3)\cup(-3,-2]\cup[6,\infty)$



Round 5

A) $\Delta ADE \sim \Delta ABC$ and the ratio of corresponding sides is 2 : 5.
Thus the ratio of the areas is 4 : 25.
Therefore, the ratio of the required areas is 4 : (25 − 4) = **4 : 21**.



B) Let s denote the length of the side of equilateral triangle ABC
$|ABC| = \frac{s^2\sqrt{3}}{4}$, where $|ABC|$ denotes the area of ΔABC
Area(Trap $BDGC$) = 15/16 area (ΔABC) → $AD : AB = 1 : 4$

→ $BD = \frac{3}{4}s$, $BE = \frac{3}{8}s$ and $DE = \frac{3}{8}s\sqrt{3}$
Thus, $a = |BDE| = \frac{1}{2}\left(\frac{3}{8}s\right)\left(\frac{3}{8}s\sqrt{3}\right) = \frac{9}{128}\sqrt{3}s^2$

Taking the required ratio, we have $\frac{1/4}{9/128}$ = **32 : 9**

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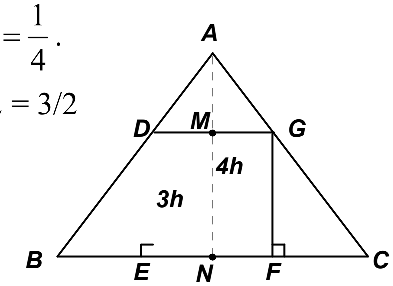
Round 5 - continued

B) Alternate solution #1

Drop a perpendicular from A to $\overline{BC}$, intersecting $\overline{DG}$ and $\overline{BC}$ at M and N respectively.

$\dfrac{\text{area}(BDGC)}{\text{area}(ABCD)} = \dfrac{15}{16} \rightarrow \dfrac{area(\Delta ADG)}{area(\Delta ABC)} = \dfrac{1}{16}$. $\Delta ADG \sim \Delta ABC \rightarrow \dfrac{AD}{AB} = \dfrac{AM}{AN} = \dfrac{1}{4}$.



Let $AD = 1$ and $AM = h$. Then: $DG = EF = 1$, $AB = BC = 4$, $BE = (4 - 1)/2 = 3/2$ and $DE = MN = 3h$.

$$\frac{area(\Delta ABC)}{area(\Delta BED)} = \frac{\frac{1}{2}BC \cdot AN}{\frac{1}{2}BE \cdot DE} = \frac{4 \cdot 4h}{\frac{3}{2} \cdot 3h} = \frac{16}{9/2} = \underline{\mathbf{32:9}}.$$

Alternate solution #2 (Norm Swanson): Let $BE = FC = 6$ and $EF = 4$. $\rightarrow \dfrac{\frac{1}{2}4h \cdot 16}{\frac{1}{2}3h \cdot 6} = \underline{\mathbf{\dfrac{32}{9}}}$

C) In a regular hexagon (with side s), the lengths of the diagonals are either $2s$ or $\sqrt{3}s$.

$\text{Area}(A) = 6\left(\dfrac{(s_A)^2\sqrt{3}}{4}\right) = \dfrac{9\sqrt{3}}{4} \rightarrow s_A = \dfrac{\sqrt{6}}{2}$; long diag$_B = 4\sqrt{6} \rightarrow s_B = 2\sqrt{6}$

Thus, $s_C = \dfrac{\sqrt{6}}{2} \cdot \sqrt{3} = \dfrac{3}{2}\sqrt{2}$ and $s_D = 2\sqrt{6} \cdot \sqrt{3} = 6\sqrt{2}$

The sum of the areas is $\dfrac{3}{2}\sqrt{3}\left(\dfrac{9}{4} \cdot 2 + 36 \cdot 2\right) = \dfrac{3}{2}\sqrt{3}\left(\dfrac{153}{2}\right) = \underline{\mathbf{\dfrac{459}{4}\sqrt{3}}}$.

Round 6

A) Substituting for x we get $4t + 1 + y + 3t = 0$ or $y + 7t = -1$. Subtracting $y + 3t = 11$, we get $4t = -12$ or $t = -3$. Substituting, we get $x = -11$ and $y = 20 \rightarrow$ **(–11, 20, –3)**

B) Convert the repeating decimal to a ratio of integers as follows:
$$\begin{array}{r} 100N = 16.\overline{6} \\ 10N = 1.\overline{6} \\ \hline 90N = 15 \end{array} \rightarrow N = \frac{15}{90} = \frac{1}{6}$$

Thus, $\dfrac{\frac{17}{100} - \frac{1}{6}}{\frac{1}{6}} = \dfrac{\frac{102-100}{600}}{\frac{1}{6}} = \dfrac{2}{600} \cdot \dfrac{6}{1} = \dfrac{2}{100} = \underline{\mathbf{2\%}}$.

C) Let n, d and q denote the number of nickels, dimes and quarters respectively. Then:

$$\begin{cases} 5n + 10d + 25q = 650 \\ 10d = 4(5n) \\ n + d + q = 50 \end{cases} \rightarrow \begin{cases} n + 2d + 5q = 130 \\ d = 2n \\ q = 50 - n - d = 50 - 3n \end{cases}$$ Then: $n + 2(2n) + 5(50 - 3n) = 130$

$\rightarrow 250 - 10n = 130 \rightarrow n = 12 \rightarrow q = \underline{\mathbf{14}}$.

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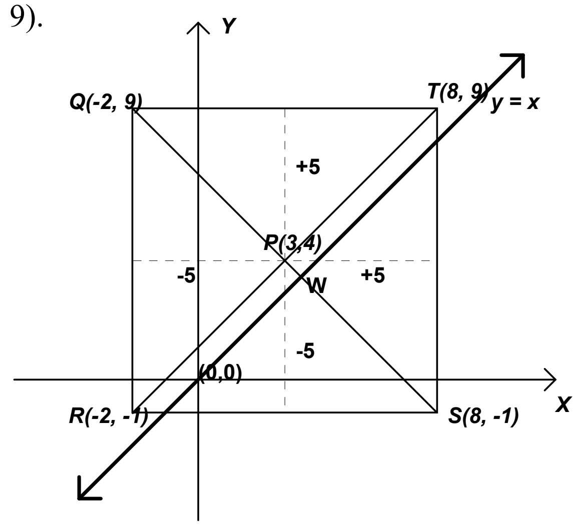
CONTEST 4 - JANUARY 2011 SOLUTION KEY

Team Round

A) The 4 vertices are $Q(-2, 9)$, $R(-2, -1)$, $S(8, -1)$ and $T(8, 9)$.
Note: The distance from the point $P(h, k)$ to the line L: $Ax + By + C = 0$ is given by the formula $\dfrac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$



The equation $y = x$ may be rewritten $x - y + 0 = 0$
Thus, from point Q, the calculations are:
$$\frac{|1(-2)+(-1)(9)+0|}{\sqrt{(1)^2+(-1)^2}} = \frac{|-2-9|}{\sqrt{2}} = \frac{11\sqrt{2}}{2}$$

$R \rightarrow \frac{1}{2}\sqrt{2}$

$S \rightarrow \frac{9}{2}\sqrt{2}$

$T \rightarrow \frac{1}{2}\sqrt{2}$

Thus, the required sum is $\frac{11}{2}\sqrt{2} + 2\left(\frac{\sqrt{2}}{2}\right) + \frac{9}{2}\sqrt{2} = \underline{\mathbf{11\sqrt{2}}}$.

Alternate solution (Norm Swanson):
The sum of the distances from $(-2, 9)$ and $(8, -1)$ to the line $y = x$ is the length of a diagonal of the square, i.e. $10\sqrt{2}$. The sum of the distances of the other vertices from $y = x$ is twice the distance from $x - y = 0$ to $x - y = -1$ which is $2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \rightarrow \underline{\mathbf{11\sqrt{2}}}$.

B) $x^8 + x^4 + 1 = (x^8 + 2x^4 + 1) - x^4 = (x^4+1)^2 - (x^2)^2 = (x^4 - x^2 + 1)(x^4 + x^2 + 1)$
$= (x^4 - x^2 + 1)\left((x^4 + 2x^2 + 1) - x^2\right) = (x^4 - x^2 + 1)\left((x^2+1)^2 - x^2\right)$
$= (x^4 - x^2 + 1)(x^2 + 1 - x)(x^2 + 1 + x) = \underline{\mathbf{(x^4 - x^2 + 1)(x^2 - x + 1)(x^2 + x + 1)}}$ or equivalent.

C) $\text{hav}(2A) = \frac{1}{2}(1 - \cos 2A) = \frac{1}{2}\left(1 - (2\cos^2 A - 1)\right) = \frac{1}{2}(2 - 2\cos^2 A) = 1 - \cos^2 A = \sin^2 A$

Thus, $\text{hav}(2A) + \frac{\text{covers}(A)}{2} = 1 \rightarrow \sin^2 A + \frac{1 - \sin A}{2} = 1 \rightarrow 2\sin^2 A - \sin A - 1 = 0$

$\rightarrow (\sin A - 1)(2\sin A + 1) = 0 \rightarrow \sin A = 1, -\frac{1}{2} \rightarrow A = \underline{\mathbf{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}}}$

Team Round – continued

D) Using the quadratic formula to solve for y in terms of x:

Re-arranging the terms of $x^2 - xy - 6y^2 + x + 7y - 2 = 0$, we have

$\rightarrow$ $6y^2 + y(x-7) + \left(2 - x - x^2\right) = 0$

(This is a quadratic equation of the form $Ay^2 + By + C = 0$.)

Thus, $y = \dfrac{(7-x) \pm \sqrt{(x-7)^2 - 4 \cdot 6 \cdot \left(2 - x - x^2\right)}}{12} = \dfrac{(7-x) \pm \sqrt{25x^2 + 10x + 1}}{12} = \dfrac{(7-x) \pm \sqrt{(5x+1)^2}}{12}$

$= \dfrac{(7-x) \pm (5x+1)}{12} = \dfrac{(8+4x)}{12}, \dfrac{6-6x}{12} = \underline{\boldsymbol{\dfrac{x+2}{3}, \dfrac{1-x}{2}}}$

Alternate Solution (Method of Indeterminant Coefficients)

Suppose $x^2 - xy - 6y^2 + x + 7y - 2 = \left(x + Ay + B\right)\left(x + Cy + D\right)$ for some constants A, B, C and D.

Multiplying out the trinomials,

$\left(x + Ay + B\right)\left(x + Cy + D\right) = x^2 + (A+C)xy + ACy^2 + (B+D)x + (AB+CD)y + BD$.

Equating the coefficients,

$$\begin{cases} (1) \quad A + C = -1 \\ (2) \quad AC = -6 \\ (3) \quad B + D = 1 \\ (4) \quad AD + BC = 7 \\ (5) \quad BD = -2 \end{cases}$$

(1), (2) $\rightarrow$ $(A, C) = (2, -3)$ or $(-3, 2)$, (3), (5) $\rightarrow$ $(B, D) = (2, -1)$ or $(-1, 2)$

Testing these possible ordered pairs in (4), the only combinations that works are $(A, B, C, D) = (-3, 2, 2, -1)$ or $(2, -1, -3, 2)$.

Thus, $\left(x + Ay + B\right)\left(x + Cy + D\right) = (x - 3y + 2)(x + 2y - 1) = 0$

The second possibility just reverses the two factors.

Setting each factor equal to zero, we have $y = \underline{\boldsymbol{\dfrac{x+2}{3}, \dfrac{1-x}{2}}}$

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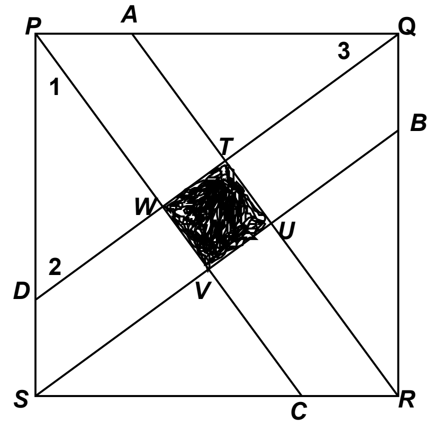
Team Round – continued



E) $PA = QB = RC = CD = 2$,
$AQ = BR = CS = DP = 6$
$\rightarrow PC = AR = DQ = SB = 10$
Also $\overline{PC} \parallel \overline{AR}$ and $\overline{DQ} \parallel \overline{SB}$.

$\Delta QPD \cong \Delta PSC$ (SAS) $\rightarrow \angle 3 \cong \angle 1$.
Since ΔQPD is a right triangle, m$\angle 2$ + m$\angle 3$ = 90°.
Substituting for m$\angle 3$, m$\angle 2$ + m$\angle 1$ = 90°.
Therefore, $\angle PWD$ and $\angle TWV$ are right angles.
Similarly, all angles with vertices at W, T, U and V are right angles.
$\Delta PWD \cong \Delta QTA \cong \Delta RUB \cong \Delta SVC$ (ASA)
$\rightarrow PW = QT = RU = SV$ and $WD = TA = UB = VC$
By subtraction of equals, $TU = UV = VW = WT$.
Thus, $TUVW$ is equiangular and equilateral and must be a square.

Continuing, Solution #1:

$\Delta VSC \sim \Delta RSB \rightarrow \dfrac{VS}{RS} = \dfrac{SC}{SB} \rightarrow \dfrac{VS}{8} = \dfrac{6}{10} \rightarrow VS = \dfrac{24}{5}$

$\Delta RUB \sim \Delta SRB \rightarrow \dfrac{BU}{BR} = \dfrac{RB}{SB} \rightarrow \dfrac{BU}{6} = \dfrac{6}{10} \rightarrow BU = \dfrac{18}{5}$

Finally, $VU = 10 - \dfrac{24}{5} - \dfrac{18}{5} = \dfrac{50-24}{5} = \dfrac{8}{5}$ and the area of square $TUVW = \underline{\mathbf{\dfrac{64}{25}}}$.

Note ΔVSC, ΔRSB and ΔURB are scaled versions of a 3 – 4 – 5 right triangle!

Continuing, Solution #2:

Drop a perpendicular from A to $\overline{PC}$, intersecting $\overline{PC}$ at point N.

$\Delta PAN \sim \Delta DQP \rightarrow \dfrac{PA}{DQ} = \dfrac{AN}{QP} \rightarrow \dfrac{2}{10} = \dfrac{AN}{8} \rightarrow AN = \dfrac{8}{5} \rightarrow TW = \dfrac{8}{5} \rightarrow$ area = $\underline{\mathbf{\dfrac{64}{25}}}$.

Note also that ΔPAN is a scaled version of a 3 – 4 – 5 right triangle!

Solution #3: (Norm Swanson)

Set up a coordinate system where $S(0, 0)$, $\overline{SP}$ lies on the Y-axis and $\overline{SR}$ along the X-axis.

Then: $\overleftrightarrow{DQ}: 3x - 4y = -8$ and $\overleftrightarrow{SB}: 3x - 4y = 0$ The distance between these parallel lines is 8/5.

Verify that the distance between $Ax + By = C$ and $Ax + By = D$ is $\dfrac{|C-D|}{\sqrt{A^2+B^2}}$.

$\overleftrightarrow{PC}: 4x + 3y = 24$ and $\overleftrightarrow{AR}: 4x + 3y = 32$ These lines are parallel and perpendicular to the above pair and also 8/5 apart. Thus, $TUVW$ is a square with area **<u>64/25</u>**. What if $PQRS$ were a rectangle, or a rhombus or a parallelogram? Can you generalize?

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E) Solution #4 (Tuan Lee)

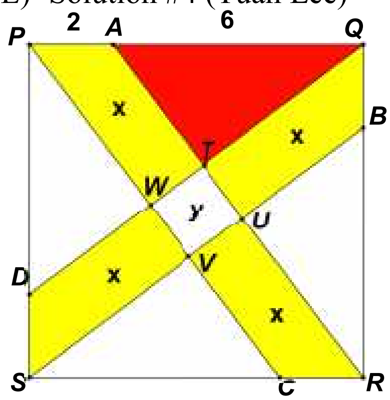

Diagram #1

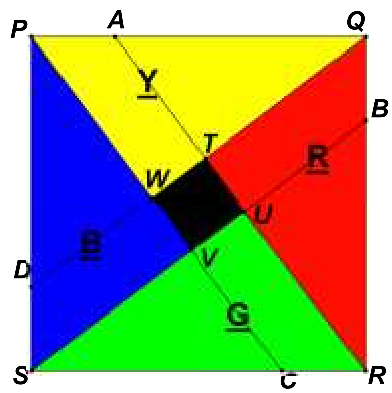

Diagram #2

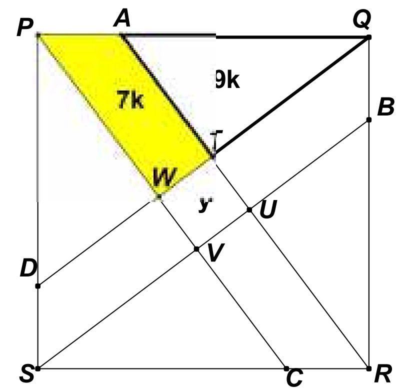

Diagram #3

From diagram #1:

Convince yourself that $ATWP$, $QBUT$, $RCVU$ and $DWVS$ have equal areas (yellow regions, labeled x).

area(ΔPQD) = 24, $PQBS$ is a right-angled trapezoid and its area is $\frac{1}{2}8(2+8)=40$

➔ area($QBSD$) = 16 ➔ equation #1: $\boxed{2x+y=16}$

Diagram #2:

Clearly, the area of the large square is equal to the sum of the areas of the five shaded regions.

We aim to show that the area of ΔPQW (yellow region, labeled Y) can be expressed entirely in terms of the area of the trapezoidal region $ATWP$ (which was called x in diagram #1)

$\Delta QAT \sim \Delta QPW$ ➔ $\frac{QT}{QW}=\frac{QA}{QP}=\frac{3}{4}$ ➔ $\frac{area(\Delta QAT)}{area(\Delta QPW)}=\frac{9}{16}$

Diagram #3:

If the area(ΔQAT) = $9k$ and the area(ΔQWP) = $16k$, then the area($ATWP$) = $7k$.

$16k=\frac{16}{7}\cdot 7k$ ➔ $area(\Delta PQW)=\frac{16}{7}area(ATWP)$

Let absolute value notation be shorthand for "the area of".

Similarly, $|QRT|=\frac{16}{7}|QBUT|$ (red-R), $|RSU|=\frac{16}{7}|RCVU|$ (green-G) and $|SPV|=\frac{16}{7}|DWVS|$ (blue-B).

$|PQRS| = |PQW| + |QRT| + |RSU| + |SPV| + |TUVW|$ = (Y + R + G + B + y)

$=\frac{16}{7}\left(|ATWP|+|QBUT|+|RCVU|+|DWVS|\right)+TUVW=\frac{16}{7}(4x)+y$

➔ Equation #2: $\boxed{64=\frac{64}{7}x+y}$. Solving, $(x, y)=\left(\frac{168}{25}, \mathbf{\frac{64}{25}}\right)$.

Team Round – continued

F) $A=\sqrt{x+37}$ and $B=\sqrt{x-N}$ ➔ $\begin{cases} x+37=A^2 \\ x-N=B^2 \end{cases} \rightarrow 37+N=A^2-B^2=(A+B)(A-B)$

Thus, we must look at the factors of (37 + N). We know that for any pair of unequal factors of (37 + N), (A + B) will equal the larger factor (call it L) and (A – B), the smaller (call it S).

Specifically, $\begin{cases} A+B=L \\ A-B=S \end{cases}$ ➔ $A=\dfrac{L+S}{2}, B=\dfrac{L-S}{2}$. To insure that A and B are integers, L and S must be of the same parity, i.e. both even or both odd. We need to find the <u>smallest</u> value of N for which there are three such factor pairs (L, S).

By trial and error,

N = 1 ➔ 38 (none – no same parity factor pairs)

N = 2 ➔ 39 (only 39·1, 13·3) N = 3 ➔ 40 (only 20·2, 10·4)

N = 4 ➔ 41 (prime) N = 5 ➔ 42 (no same parity factor pairs) N = 6 ➔ 43 (prime)

N = 7 ➔ 44 (22·2 only) N = 8 ➔ 45 (45·1, 15·3, 9·5) – Bingo!

$$\begin{cases} A+B=45 \quad 15 \quad 9 \\ A-B=\ \ 1 \quad\ \ 3 \quad 5 \end{cases} \rightarrow (A,B)=\ (23,22),(9,6),(7,2) \rightarrow x=492,\ 44,\ 12$$

Thus, (k, T) = (8, 492 + 44 + 12) = <u>**(8, 548)**</u>.

We could have made the observation that we were looking for a number with 6 odd factors. The smallest such positive number would have been $3^2 \cdot 5^1 = 45$ which would have avoided and laborious plug and check.