MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2011
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Given: $\begin{cases} f(x) = 4x + 5 \\ g(x) = 6x^2 + x - 4 \end{cases}$

Compute $f^{-1}(3) + g(f(-1))$.

B) Given: $f(x) = 4x - 1$ and $g(t) = 3 - 2t$

Determine all values of a for which $f\left(g^{-1}(2a)\right) = g\left(f^{-1}(2-a)\right)$.

C) A, B, C and D are the four rational zeros of the function defined by $f(x) = 3x^4 - 8x^3 - 11x^2 + 28x - 12$.

Compute $(1+A)(1+B)(1+C)(1+D)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2011
ROUND 2 ARITHMETIC / NUMBER THEORY

ANSWERS

A) (____________ , ____________)

B) ______________________________

C) ______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) The number 312 is a multiple of 13.
The number 688 is a multiple of 43.
Their sum is 1000.
Find the other pair of positive integers (a, b), where $a < b$, $a + b = 1000$, if one of these numbers must be a multiple of 13 and the other a multiple of 43.

B) Given: a and b are base 10 digits.
Determine the ordered pair (a, b) such that $N = 33650ab97$ is divisible by 99.

Note: N is a 9-digit integer.

C) When $129600_{(10)}$ is converted to base 12, its rightmost digits are k consecutive zeros.
Determine the value of k.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2011
ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

***** NO CALCULATORS ON THIS ROUND *****

A) Determine the two smallest positive values of x (in degrees) for which $\sin x = \cos 110°$.

B) If A is the smallest positive angle (in radians) for which $\sin 3A = -\frac{1}{2}$, compute $\cos 12A \cos 6A$.

C) Compute $\sin\left(Arc\tan\left(\frac{-2}{3\sqrt{5}}\right)\right) + \tan\left(Arc\cos\left(-\frac{35}{37}\right)\right)$.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2011
ROUND 4 ALG 1: WORD PROBLEMS

ANSWERS

A) _______________________________ kg

B) x = ______________________________

C) ____________________________ days

***** NO CALCULATORS ON THIS ROUND *****

A) At a candy store, a mixture of red and black licorice is $\frac{1}{4}$ black licorice. If 4 kg of black licorice is added to this mixture, the new mixture is then only $\frac{1}{3}$ black. Compute the total weight (in kg) of the original mixture.

B) Given: y varies inversely as the square root of x. When $y = 4$, $x = 25$. Find x, when $y = 100$.

C) Nine house painters working at a constant rate can complete a job in 7 days. Compute how many days it would take a new group of ten painters (working at the same rate as the previous painters) to complete the job, if four of these ten painters were not on the job until the 4^{th} day?

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ROUND 5 PLANE GEOMETRY: CIRCLES

ANSWERS

A) ______________________ sq. units

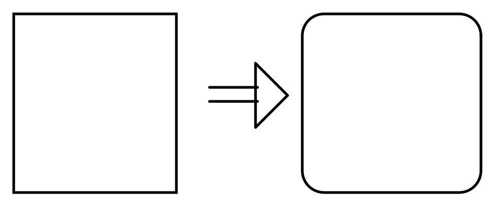
B) ______________________________

C) _____________________________ °

******* NO CALCULATORS ON THIS ROUND *******

A) A quarter is placed on top of a table. Then k quarters are placed around the given quarter so that each is tangent to the given quarter and to two others. Compute the minimum area of a single coin that will cover all of these $(k + 1)$ quarters. Assume the diameter of a quarter is exactly 1 inch.

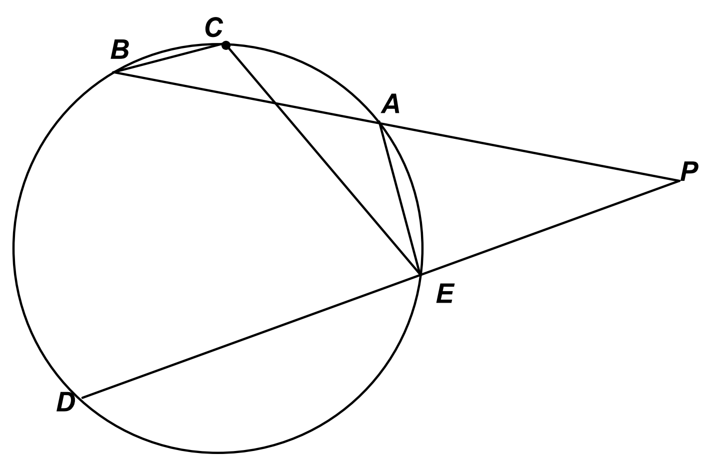
B) A square is replaced by a square with rounded corners, thereby losing 1/10 of its area. If x and r denote the edge of the square and the radius of the rounded corner respectively , then compute $\frac{x^2}{r^2}$.



C) Given: $m\angle BCE = 140°$, $m\angle P = (5x+3)°$

$m\left(\overset{\frown}{BD}\right) = (15x+8)°, m\left(\overset{\frown}{AE}\right) = (6x-6)°$

Compute $m\angle AED$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2011
ROUND 6 ALG 2: SEQUENCES AND SERIES

ANSWERS

A) $d =$ ____________________________

B) $t_{10} =$ ____________________________

C) ____________________________

******* NO CALCULATORS ON THIS ROUND *******

A) What is the common difference d in an arithmetic sequence, where the first term is 2, the last term is 29 and the sum of the terms is 155 ?

B) $x, y, -2x$ are the first three terms in an arithmetic progression.
$3x, -3y, x-1$ are the first three terms in a geometric progression.
If $xy \neq 0$, compute the 10^{th} term in the geometric progression.

C) Given a sequence generated by $a_4 = 11,\ a_6 = 64$ and $a_{n+2} = 2a_{n+1} + a_n$ for integers $n \geq 1$.
Compute $a_3 + a_7$.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2011
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________ D) ____________________

B) ____________________ E) ____________________

C) ____________________ F) ____________________

******* NO CALCULATORS ON THIS ROUND *******

A) Given: $f(x)=\dfrac{3x+1}{2(x-1)}$, $g(t)=\dfrac{1}{3t-2}$

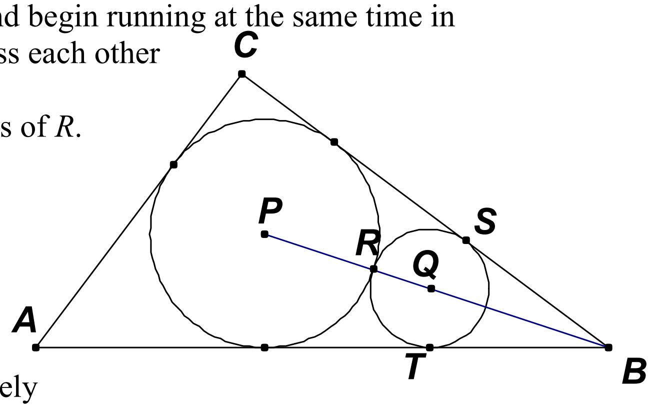
Determine all values of m for which $f^{-1}(m) \le g^{-1}(m)$.

B) The sum of the digits of a two-digit positive <u>even</u> integer N is 9.
Determine the <u>sum</u> of all possible values of N which have 12 divisors.

C) Given: $a > 0$, $b > 0$ and $Tan^{-1}\left(\dfrac{a}{b}\right)+Sin^{-1}\left(\dfrac{a}{b}\right)=90°$. Compute $\dfrac{a^2}{b^2}$.

D) Faster runner #1 (running at R feet/sec) completes A laps around a quarter-mile track in the time that runner #2 completes B laps, where A and B are relatively prime positive integers. If the two runners start at the same point on the track and begin running at the same time in <u>opposite</u> directions at the rates specified above, they pass each other for the first time in 45 seconds. If A and B are integers, where $0 < A - B < 3$, compute <u>all</u> possible integer values of R.

Note: 1 mile = 5280 feet



E) $AC = 6$, $BC = 8$ and $AB = 10$
Circle P is inscribed in ΔABC.
Circle Q is tangent to $\overline{BC}$ and $\overline{AB}$ at S and T respectively and to circle P at R.
Compute BS.

F) The sum of the first three terms in an infinite geometric progression of rational numbers is 1792. The sum of the first 11 terms is 2047. If the 56th term in an arithmetic progression is equal to the sum of the terms in this infinite geometric progression and the first term a and the common difference d are positive integers (with $a < 50$), compute the 55th term of this arithmetic progression.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2011 ANSWERS

Round 1 Alg 2: Algebraic Functions

A) 2.5 B) $\frac{7}{9}$ C) $-\frac{40}{3}$

Round 2 Arithmetic/ Number Theory

A) (129, 871) B) (1, 2) C) 3

Round 3 Trig Identities and/or Inverse Functions

A) 200, 340 B) $-\frac{1}{4}$ C) $-\frac{22}{35}$

Round 4 Alg 1: Word Problems

A) 32 kg B) $\frac{1}{25}$ C) 7.5 days

Round 5 Geometry: Circles (Exact equivalents in terms of π are acceptable.)

A) $\frac{9\pi}{4}$ B) $10(4-\pi)$ C) 83

Round 6 Alg 2: Sequences and Series

A) 3 B) $\frac{3}{128}$ C) 159

Team Round

A) $-3 \le m \le -\frac{1}{2}$ or $0 < m < \frac{3}{2}$

B) 162

C) $\frac{\sqrt{5}-1}{2}$

D) 15, 16, 22

E) $\frac{2}{3}\left(11-2\sqrt{10}\right)$

F) 2011

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2011 SOLUTION KEY

Round 1

A) Knowing that $f^{-1}(a)=b \Leftrightarrow f(b)=a$, we need not bother finding $f^{-1}(x)$.

Solving $4x + 5 = 3 \rightarrow x = -\frac{1}{2}$. Thus, $f\left(-\frac{1}{2}\right)=3 \Leftrightarrow f^{-1}(3)=-\frac{1}{2}$.

$f(-1) = 1$. Thus, $f^{-1}(3)+g(f(-1)) = -\frac{1}{2}+g(1)= -\frac{1}{2}+6+1-4=$ **<u>2.5</u>** $\left(2\frac{1}{2} \text{ or } \frac{5}{2}\right)$

B) $f(x) = 4x - 1 \rightarrow f^{-1}(x)=\frac{x+1}{4}$. $g(t) = 3 - 2t \rightarrow g^{-1}(t)=\frac{t-3}{-2}=\frac{3-t}{2}$

$f\left(g^{-1}(2a)\right)=f\left(\frac{3-2a}{2}\right)=4\left(\frac{3-2a}{2}\right)-1=6-4a-1=5-4a$

$g\left(f^{-1}(2-a)\right)=g\left(\frac{(2-a)+1}{4}\right)=g\left(\frac{3-a}{4}\right)=3-2\left(\frac{3-a}{4}\right)=\frac{6}{2}-\frac{3-a}{2}=\frac{3+a}{2}$

Equating, $5-4a=\frac{3+a}{2} \rightarrow 10 - 8a = 3 + a \rightarrow a = \underline{\mathbf{\frac{7}{9}}}$

C) Solution #1: (Brute Force - Find the 4 roots, plug and chug.)
The possible integer roots are factors of 12, the constant term. Testing by synthetic substitution:

	3	−8	−11	28	−12
1	3	−5	−16	12	0
−2	3	−11	6	0	

, we discover two integer roots, 1 and −2 and the remaining roots can be determined by factoring the quotient $3x^2 - 11x + 6 = (3x - 2)(x - 3) = 0 \rightarrow 3, \frac{2}{3}$.

Let $(A, B, C, D) = \left(1, -2, 3, \frac{2}{3}\right)$.

$(1+A)(1+B)(1+C)(1+D) = 2\cdot -1\cdot 4\cdot\frac{5}{3}=\underline{\mathbf{-\frac{40}{3}}}$

This method depends on being able to *<u>factor the given expression</u>*. This is <u>not</u> always possible.
Ex: Try $f(x) = 2x^4 - 3x^3 + 5x^2 - 7x + 11$.
This polynomial does not factor over the integers. With a graphing calculator you could approximate the four zeros, plug values into the expression and <u>approximate</u> the product. However, the computations would be extremely messy. How can this be avoided??

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Round 1 - continued

C) – continued

Solution #2 shows how we can compute the required product **without specifically knowing** A, B, C and D. Expanding the product, $(1+A)(1+B)(1+C)(1+D)=$
$1+(A+B+C+D)+(AB+AC+AD+BC+BD+CD)+(ABC+ABD+ACD+BCD)+ABCD$

Normalizing a polynomial function (making its lead coefficient 1) does not change its zeros. Normalizing $f(x)$ whose zeros are A, B, C and D, we have $F(x) = (x-A)(x-B)(x-C)(x-D)$
Expanding, we have
$x^4-(A+B+C+D)x^3+(AB+AC+AD+BC+BD+CD)x^2-(ABC+ABD+ACD+BCD)x+ABCD$

Lo and behold the coefficients match the expressions we need to evaluate!
After normalizing, each of these sums can be determined by inspection.

$$\underline{\mathbf{3}}x^4 \ \underline{-\mathbf{8}}x^3 \ \underline{-\mathbf{11}}x^2+\underline{28}x\ \underline{-\mathbf{12}}=0$$

$(A+B+C+D)=8/3$ (the opposite of the cubic coefficient divided by the lead coefficient)
$(AB+AC+AD+BC+BD+CD)=-11/3$ (the quadratic coefficient divided by …)
$(ABC+ABD+ACD+BCD)=-28/3$ (the opposite of the linear coefficient divided by …)
$ABCD=-12/3=-4$ (the constant term divided by …)
Thus, the required product is $1+\left(\dfrac{8-11-28-12}{3}\right)=1-\dfrac{43}{3}=\underline{-\dfrac{\mathbf{40}}{\mathbf{3}}}$.
[Using this relationship between the coefficients and the zeros, you can verify that for the unfactorable polynomial, the computation is simply $(3+5+7+11)/2=\underline{\mathbf{13}}$.

Alternative Solution #3 A REAL GEM! (Norm Swanson): By synthetic substitution,

$$\begin{array}{r|rrrrr} -1 & 3 & -8 & -11 & 28 & -12 \\ \hline & 3 & -11 & 0 & 28 & \boxed{-40} \end{array}$$

. Divide by 3 and we have our answer. Why does this work?

Since A is a zero, $3A^4-8A^3-11A^2+28A-12=0$ and similarly for B, C and D.
Consider $g(x) = 3(x-1)^4-8(x-1)^3-11(x-1)^2+28(x-1)-12$.
Since $g(1+A) = 3A^4-8A^3-11A^2+28A-12=0$, $1+A$ is a zero of g and so is $1+B$, etc.

In the expansion of $g(x)$, we only need to know the constant term which is determined by letting $x = 0$ or evaluating the original polynomial expression for $x=-1$.

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Round 2

A) 13 x 43 = 559 is a multiple of both 13 and 43.
Add 559 to 312 to produce another multiple of 13, namely 871.
Subtract 559 from 688 to produce another multiple of 43, namely 129.
The sum is unaffected by this transformation, so (a, b) = **(129, 871)**.

Algebraic solution (much more of a pain! – but it does limit the number of possible answers)

$13x + 43y = 43 \rightarrow 13x = 1000 - 43y \rightarrow x = \dfrac{1000-43y}{13} = 76 - 3y + \boxed{\dfrac{4(3-y)}{13}}$

The boxed fractional expression must evaluate to an integer. Clearly, this is true for $y = 3$.
Multiples of 13 are 13 apart, so y = 3, 16, 29, …
$y = 3 \rightarrow x = 67 \rightarrow 13x = 871,\ 43y = 129$
$y = 16 \rightarrow x = 24 \rightarrow 13x = 312,\ 43y = 688$
$y = 29 \rightarrow x = -19$ (rejected, since both x and y must be positive)
Thus, the solution found by the arithmetic approach is unique.

B) $N = 33650ab97$ must be divisible by 9 and 11.
$\div 9 \rightarrow (3 + 3 + 6 + 5 + 0 + a + b + 9 + 7) = 33 + a + b$ is a multiple of 9
$\rightarrow a + b = 3$ or 12 (since $0 \leq a + b \leq 18$)
$\div 11 \rightarrow (b + 7 + 0 + 6 + 3) - (a + 9 + 5 + 3) = b - a - 1$ is a multiple of 11
Case 1: $a + b = 3 \rightarrow b = 3 - a$
Substituting, $b - a - 1 = 2 - 2a = 2(1 - a)$
$a = 1 \rightarrow b = 2$ OK, $a = 2 \rightarrow -2$ fails, $a = 3 \rightarrow -4$ fails, … $a = 9 \rightarrow -16$ fails (all values are even)

Case 2: $a + b = 12 \rightarrow b - a - 1 = 11 - 2a$
Only $a = 0$ produces a multiple of 11, but $a = 0 \rightarrow b = 12$ which is not an allowable digit

Therefore, the solution **(1, 2)** is unique.

C) Factoring 129600 as a product of primes, we get $2^6 \cdot 3^4 \cdot 5^2$.
In base 10, an integer ending in zero is divisible by 10 ($= 2 \cdot 5$).
Looking at the factorization ($2^6 \cdot 3^4 \cdot 5^2$), we see by inspection that the product contains 10^2, i.e., it must end in exactly two zeros.
In this case, we are limited by the number of factors of 5.
Ignore the 3s. Using all the factors of 5 and two of the factors of 2, we get two factors of 10.

In base 12, an integer ending in zero is divisible by 12 ($= 2^2 \cdot 3$).
We do not need to convert 129600 to base 12. Simply examine the factorization above and determine the maximum number of 12s that can be produced.
We are limited by the number of factors of 2. Ignore the 5s and consider only the 2s and 3s.
Since $2^6 \cdot 3^3 = (2^2 \cdot 3)^3 = 12^3$, 129600 will end in **3** zeros.

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Round 3

A) $\sin x = \cos 110°$ ➔ $\sin x = -\cos 70° = -\sin 20°$
Thus, x belongs to the 20° family and must have a negative sine value; therefore, it must be located in quadrant 3 or 4 ➔ **<u>200, 340</u>**.

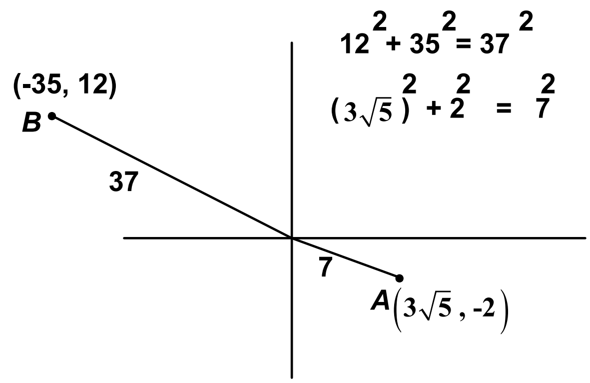
B) $\sin 3A = -\frac{1}{2}$ ➔ $3A = \frac{7\pi}{6} + 2k\pi$ ➔ $12A = \frac{14\pi}{3} + 8k\pi$ and $6A = \frac{7\pi}{3} + 4k\pi$, where k is an integer.
Since $\cos(x + 2k\pi) = \cos(x)$, we get:

$$\cos\left(\frac{14\pi}{3}\right)\cos\left(\frac{7\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \underline{\mathbf{-\frac{1}{4}}}$$

C) $Arc\tan\left(\frac{-2}{3\sqrt{5}}\right)$ denotes a value in quadrant 4.

$Arc\cos\left(-\frac{35}{37}\right)$ denotes a value in quadrant 2.

$$-\frac{2}{7} + \left(-\frac{12}{35}\right) = \frac{-10-12}{35} = \underline{\mathbf{-\frac{22}{35}}}$$



Round 4

A) Let X denote the weight of the original mixture.
Black: $\frac{X}{4} + 4 = \frac{1}{3}(X+4)$ ➔ $3X + 48 = 4X + 16$ ➔ $X =$ **<u>32</u>** kg

B) Inverse variation ➔ $y = \frac{k}{\sqrt{x}}$ Substituting $(x, y) = (25, 4)$ ➔ $k = 20$.

Thus, $100 = \frac{20}{\sqrt{x}}$ ➔ $\sqrt{x} = \frac{1}{5}$ ➔ $x = \underline{\mathbf{\frac{1}{25}}}$

C) The painting of the house takes 63 painter-days. Therefore, in one day, the initial 6 painters can complete $\frac{6}{63}$ th of the job, whereas when all 10 painters are on the job they complete $\frac{10}{63}$ th of the job. Let x denote the number of days the crew of 10 painters work. Then:
$\left(\frac{10}{63}\right)x + \left(\frac{6}{63}\right)3 = 1$ ➔ $10x + 18 = 63$ ➔ $x = 4.5$ and the total time required is
$3 + 4.5 =$ **<u>7.5</u>** days

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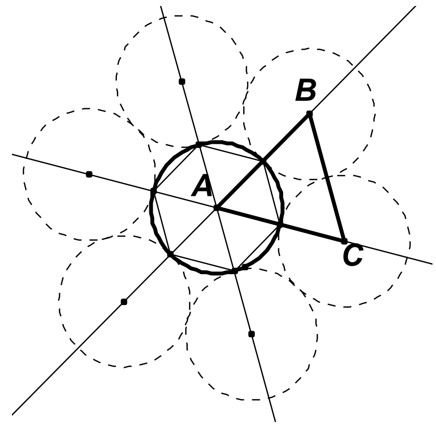
Round 5 (Exact equivalents in terms of π are acceptable.)

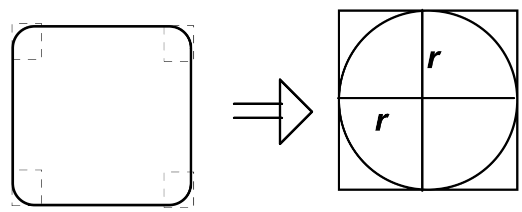
A) Clearly, $k = 6$.
Place a quarter tangent to circle of the innermost quarter at each vertex of a regular hexagon inscribed inside this circle. Every adjacent pair is tangent to each other as well as the innermost circle, since the lines connecting the centers (ex. *ABC*) form an equilateral triangle.



From the diagram, the radius of the covering coin is $\frac{3}{2}$.

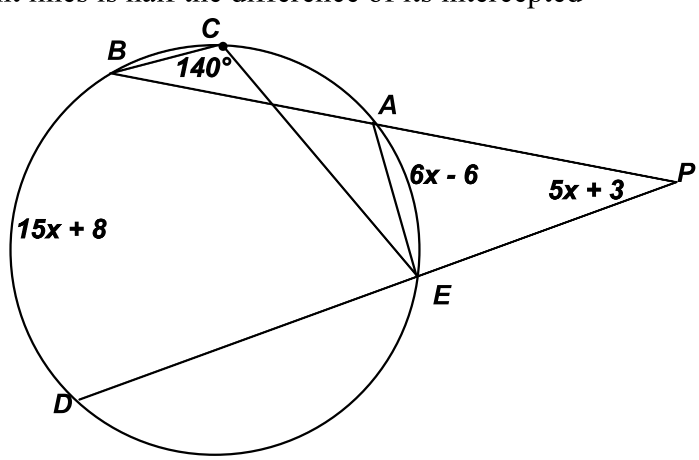
Therefore, the required area is $\underline{\boldsymbol{\frac{9\pi}{4}}}$.

B) Let x denote the side of the original square. Telescoping the four corners, the area lost equals the area of the region inside a square with edge $2r$ and outside a circle of radius r, i.e. $4r^2 - \pi r^2$



➔ $r^2(4-\pi) = 0.1x^2$ ➔ $\frac{x^2}{r^2} = \frac{(4-\pi)}{0.1}$ ➔ $\underline{\mathbf{10(4-\pi)}}$

C) The measure of an angle formed by two secant lines is half the difference of its intercepted arcs. Thus, $5x+3 = \frac{1}{2}(15x+8-6x+6)$



➔ $10x + 6 = 9x + 14$ ➔ $x = 8$
➔ $\overset{\frown}{BD} = 128°$, $\overset{\frown}{AE} = 42°$,
$\overset{\frown}{DE} = 280 - 128 = 152°$,
$\overset{\frown}{BCA} = 38°$
Finally, as an inscribed angle,
$m\angle AED = \frac{1}{2}(128+38) = \underline{\mathbf{83}}°$

Alternate solution (Tuan Lee):
As above $x = 8$ ➔ $m\angle P = 43°$.
Since $\angle BCE$ and $\angle BAE$ are both inscribed angles intercepting the same arc $\overset{\frown}{BDE}$, each measures 140°. As an exterior angle of ΔAPE, $m\angle BAE = m\angle BPD + m\angle AEP$
$= m\angle BPD + 180 - m\angle AED$ and we have: $m\angle AED = 180 - 140 + 43 = \underline{\mathbf{83}}°$.

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Round 6

A) $\frac{n}{2}(2+29) = 155$ ➔ $n/2 = 5$ ➔ $n = 10$. Thus, $t_{10} = 2 + 9d = 29$ ➔ $d = \underline{\mathbf{3}}$

B) (1) $-2x - y = y - x$ ➔ $x = -2y$

(2) $\frac{x-1}{-3y} = \frac{-3y}{3x}$

Substituting, $\frac{-2y-1}{-3y} = \frac{-3y}{-6y} = \frac{1}{2}$ ➔ $-3y = -4y - 2$ ➔ $(x, y) = (4, -2)$

Thus, the GP is 12, 6, 3, … ➔ $t_{10} = \underline{12\left(\frac{1}{2}\right)^9 = \mathbf{\frac{3}{128}}}$

C) $a_{n+2} = 2a_{n+1} + a_n$ ➔ $\begin{cases} a_6 = 2a_5 + a_4 \\ a_7 = 2a_6 + a_5 \end{cases}$ ➔ $\begin{cases} 64 = 2a_5 + 11 \\ a_7 = 128 + a_5 \end{cases}$ ➔ $a_5 = \frac{53}{2}$ and $a_7 = 128 + \frac{53}{2}$

Also $a_{n+2} = 2a_{n+1} + a_n$ ➔ $a_n = a_{n+2} - 2a_{n+1}$

If $n = 3$, we have $a_3 = a_5 - 2a_4 = \frac{53}{2} - 2(11) = \frac{9}{2}$

Thus, $a_3 + a_7 = 128 + \frac{62}{2} = \underline{\mathbf{159}}$

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Team Round

A) Let $y = f(x) = \dfrac{3x+1}{2(x-1)}$. Interchanging variables: $x = \dfrac{3y+1}{2(y-1)}$

Solving for y: $2xy - 2x = 3y + 1 \rightarrow 2xy - 3y = y(2x - 3) = 2x + 1 \rightarrow y = f^{-1}(x) = \dfrac{2x+1}{2x-3}$

Let $y = g(t) = \dfrac{1}{3t-2}$ Interchanging variables: $t = \dfrac{1}{3y-2}$

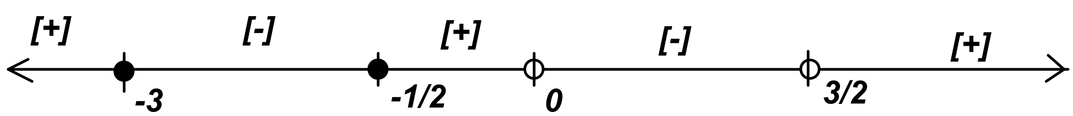
Solving for y: $3ty - 2t = 1 \rightarrow 3ty = 2t + 1 \rightarrow y = g^{-1}(t) = \dfrac{2t+1}{3t}$

Thus, we require that $\dfrac{2m+1}{2m-3} \le \dfrac{2m+1}{3m} \rightarrow \dfrac{2m+1}{2m-3} - \dfrac{2m+1}{3m} \le 0 \rightarrow (2m+1)\left(\dfrac{1}{2m-3} - \dfrac{1}{3m}\right) \le 0$

$\rightarrow (2m+1)\left(\dfrac{3m-(2m-3)}{(2m-3)(3m)}\right) \le 0 \rightarrow \dfrac{(2m+1)(m+3)}{(2m-3)(3m)} \le 0 \quad (m \ne 0, 3/2)$

The critical values are: $-3, -\dfrac{1}{2}, 0, \dfrac{3}{2}$

At the extreme left on the number line all four factors are negative, producing a positive quotient and as we move to the right, the sign of the quotient alternates as we pass each critical point. This is summarized in the following diagram:



Thus, the inequality is satisfied if and only if $\underline{\mathbf{-3 \le m \le -\frac{1}{2}} \text{ or } \mathbf{0 < m < \frac{3}{2}}}$.

B) Nine two-digit integers can be formed, but only 5 of them are even, namely 18, 36, 54, 72 and 90. Examining the factorization of each of these

$$18 = 2^1 \cdot 3^2, 36 = 2^2 \cdot 3^2, 54 = 2^1 \cdot 3^3, 72 = 2^3 \cdot 3^2,\ 90 = 2^1 \cdot 3^2 \cdot 5^1,$$

we can determine the number of factors by adding 1 to each exponent and then taking the product of all these sums.

18: 2(3) = 6 36: 3(3) = 9 54: 2(4) = 8 **72**: 4(3) = 12 **90**: 2(3)(2) = 12 $\rightarrow$ **<u>162</u>**.

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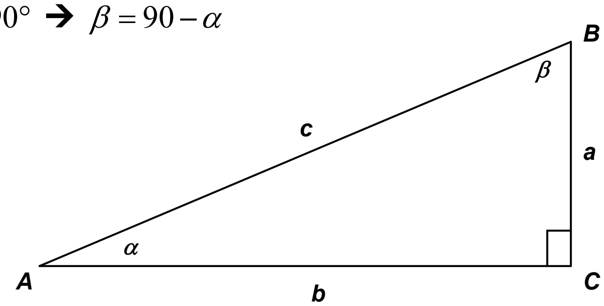
C) Let $\alpha = Tan^{-1}\left(\frac{a}{b}\right)$ and $\beta = Sin^{-1}\left(\frac{a}{b}\right)$. Then: $\alpha + \beta = 90°$ → $\beta = 90 - \alpha$



$a, b > 0$ → $0 < \alpha, \beta \le 90$

$\sin\beta = \frac{a}{b} = \sin(90-\alpha) = \cos\alpha = \frac{b}{c} = \frac{b}{\sqrt{a^2+b^2}}$

→ $b^2 = a\sqrt{a^2+b^2}$ → $b^4 = a^2(a^2+b^2)$

→ $b^4 - a^2b^2 - a^4 = 0$

→ $b^2 = \frac{a^2 \pm \sqrt{a^4+4a^4}}{2} = a^2\left(\frac{1\pm\sqrt{5}}{2}\right)$ →

$\frac{b^2}{a^2} = \frac{1+\sqrt{5}}{2}$ (~~$\frac{1-\sqrt{5}}{2}$~~ < 0 is rejected.)

Inverting, $\frac{a^2}{b^2} = \frac{2}{1+\sqrt{5}} = \boldsymbol{\frac{\sqrt{5}-1}{2}}$

Note: Using a calculator, $\frac{1+\sqrt{5}}{2} \approx 1.6180339887...$ and $\frac{\sqrt{5}-1}{2} \approx 0.6180339887...$.

The first constant is called ϕ, the golden ratio and the second is $\phi - 1$.

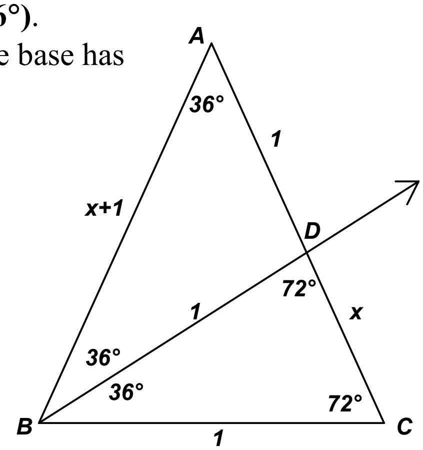
Check: $\frac{a}{b} \approx 0.7861513778$ → $\alpha \approx 38.17270763°$, $\beta \approx 51.82729238°$ and $\alpha + \beta \approx 90.00000001 \to 90°$.

<u>**An aside**</u>:

Actually, did you know that besides the 30°, 45° and 60° families of angles, it is also possible to compute an <u>exact</u> value for the trig functions of 36° ? In fact, cos(36°) = $\phi/2 = \frac{1+\sqrt{5}}{4}$

Here's how you can determine a closed (exact) expressions for cos(36°).

Start with an isosceles triangle ABC whose vertex angle is 36° and whose base has length 1. Bisect a base angle. Let $CD = x$ and mark the remaining sides accordingly. Then: $\Delta BAC : \Delta CBD$ → $\frac{BA}{CB} = \frac{BC}{CD}$ → $\frac{x+1}{1} = \frac{1}{x}$



Cross multiplying and using the quadratic formula, $x = \frac{\sqrt{5}-1}{2}$.

Using the law of cosines on ΔCBD, $x^2 = 1^2 + 1^2 - 2\cdot1\cdot1\cdot\cos36°$

Substituting for x and solving for cos36°, we have

$\cos36° = 1 - \frac{x^2}{2} = 1 - \frac{6-2\sqrt{5}}{8} = \frac{1+\sqrt{5}}{4} = \frac{\phi}{2}$.

Q.E.D

Euclid ended many of his proofs with these 3 letters, an abbreviation for the Latin phrase "quod erat demonstratum" (which was to be proven).

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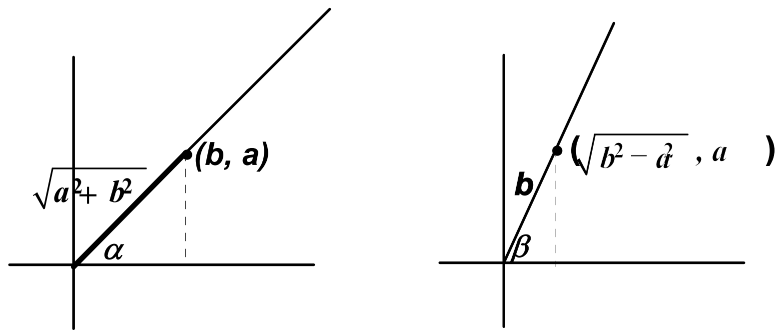
C) - continued

Alternate solution #1: Let $\alpha = Arc\tan\left(\frac{a}{b}\right)$ and $\beta = Arc\sin\left(\frac{a}{b}\right)$



Taking the cosine of both sides, $\cos\left(Arc\tan\left(\frac{a}{b}\right)+Arc\sin\left(\frac{a}{b}\right)\right)=\cos 90°$

➔ $\cos\left(Arc\tan\left(\frac{a}{b}\right)\right)\cos\left(Arc\sin\left(\frac{a}{b}\right)\right)-\sin\left(Arc\tan\left(\frac{a}{b}\right)\right)\sin\left(Arc\sin\left(\frac{a}{b}\right)\right)=0$

➔ $\frac{b}{c}\cdot\frac{\sqrt{b^2-a^2}}{b}-\frac{a}{c}\cdot\frac{a}{b}=0$, where c replaces $\sqrt{a^2+b^2}$

➔ $b\sqrt{b^2-a^2}=a^2$

Squaring both sides, $b^2\left(b^2-a^2\right)=a^4$ ➔ $b^4-a^2b^2-a^4=0$ and then proceed as above.

Alternate solution #2 (Norm Swanson):

$\cos\left(Arc\tan\left(\frac{a}{b}\right)+\arcsin\left(\frac{a}{b}\right)\right)=\boxed{\left(\frac{b}{c}\right)\left(\frac{\sqrt{b^2-a^2}}{b}\right)-\left(\frac{a}{c}\right)\left(\frac{a}{b}\right)=0}$ ***, where $c=\sqrt{a^2+b^2}$

Multiplying through by $c \neq 0$, eliminates c and we have $\sqrt{b^2-a^2}=\frac{a^2}{b}$.

Dividing by a ($\sqrt{a^2}$ on the left side), we have $\sqrt{\frac{b^2}{a^2}-1}=\frac{a}{b}$ ➔ $\frac{b^2}{a^2}-1=\frac{a^2}{b^2}$

or letting $x=\frac{b^2}{a^2}$, $x-1=\frac{1}{x}$ and the result follows.

Even easier: Let $b = 1$. Then *** immediately simplifies to $\left(\frac{1}{c}\right)\sqrt{1-a^2}=\frac{a^2}{c}$

$c \neq 0$ ➔ $\sqrt{1-a^2}=a^2$ ➔ $a^4+a^2-1=0$ ➔ $a^2=\frac{-1+\sqrt{5}}{2}$ (since $a > 0$).

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D) Let x be the rate of runner #2. We have: $\frac{B}{x}=\frac{A}{R}$ ➔ $x=\frac{B}{A}\cdot R$.

Since the two runners pass each other in 45 seconds when they run in opposite direction, they have completed 1 lap, i.e. covered a distance of 1320 feet in 45 seconds. Thus,

$$R\cdot 45+\frac{B}{A}\cdot R\cdot 45=\frac{1}{4}\cdot 5280=1320 \text{ ➔ } R\left(1+\frac{B}{A}\right)=R\left(\frac{A+B}{A}\right)=\frac{1320}{45}=\frac{88}{3}$$

➔ $R=\frac{88A}{3(A+B)}$ ➔ A must be a multiple of 3

The factors of 88 are: 1, 2, 4, 8, 11, 22, 44 and 88.
Under the given restrictions,

- $A > B$,
- the sum $A + B$ can't be 1 or 2 and
- the difference $A - B$ must be 1 or 2

$A+B$	$(A,B)=$
4:	(3, 1) ➔ R = 22 ft/sec
8:	~~(5,3)~~ - 5 is not a multiple of 3
11:	(6, 5) ➔ R = 16 ft/sec
22:	~~(12,10)~~ - not relatively prime
44:	(23, 21) – 23 is not a multiple of 3
88:	(45, 33) ➔ R = 15 ft/sec

Thus, R = **<u>15, 16 or 22</u>**.

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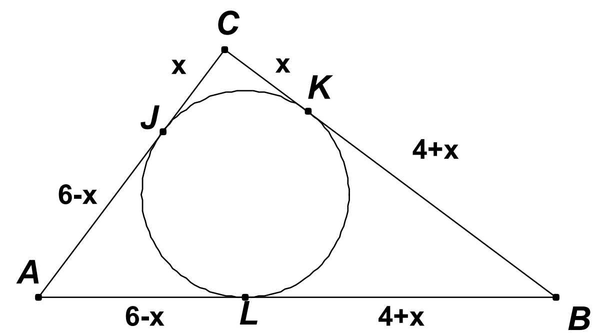
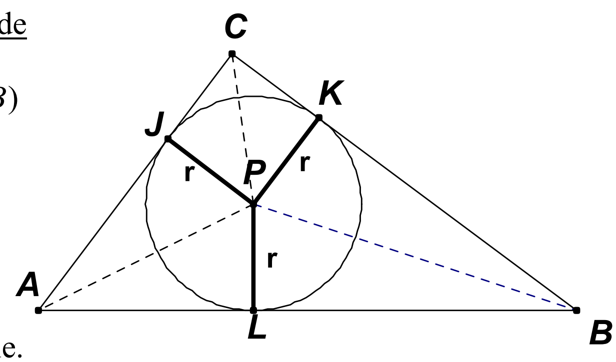
E) Tangents from an external point to a circle are congruent.
Let $CJ = CK = x$.
$AC = 6 \rightarrow AJ = AL = 6 - x$ and
$AB = 10 \rightarrow BL = BK = 4 + x$.
$BC = x + (4 + x) = 8 \rightarrow x = 2$



Since ΔABC is a right triangle, its area is $\frac{1}{2}\cdot 6\cdot 8 = 24$.

Notice that the radius of the inscribed circle is an <u>altitude</u> in triangles APC, BPC and APB.



$\text{area}(\Delta ABC) = \text{area}(\Delta APC) + \text{area}(\Delta BPC) + \text{area}(\Delta APB)$

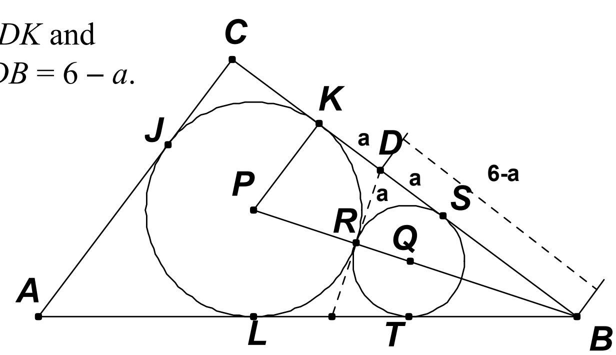
$\rightarrow 24 = \frac{1}{2}\cdot 6\cdot r + \frac{1}{2}\cdot 8\cdot r + \frac{1}{2}\cdot 10\cdot r = 12r$

$\rightarrow 24 = r\frac{(6+8+10)}{2} = 12r \rightarrow r = 2$

Note: The line above illustrates an important relationship between <u>any</u> triangle and its inscribed circle. Namely, the area of a triangle equals the product of its <u>s</u>emi-perimeter and the <u>r</u>adius of its inscribed circle. [$A(\Delta) = rs$] Semi-perimeter means half the perimeter.

Applying the Pythagorean Theorem to ΔPKB, $PB = 2\sqrt{10}$.

Draw a line perpendicular to $\overline{PB}$ at R. Note that $DR = DK$ and $DR = DS$. They are all marked a in the diagram. Now $DB = 6 - a$.



In right ΔDRB, $a^2 + \left(2\sqrt{10} - 2\right)^2 = (6 - a)^2$

$\rightarrow 44 - 8\sqrt{10} = 36 - 12a \rightarrow 12a = 8\left(\sqrt{10} - 1\right)$

$\rightarrow a = \frac{2}{3}\left(\sqrt{10} - 1\right)$

Thus, $BS = BK - 2a = 6 - 2a = 6 - \frac{4}{3}\left(\sqrt{10} - 1\right) =$

<u>$\mathbf{\frac{2}{3}\left(11 - 2\sqrt{10}\right)}$</u> or (any exact equivalent)

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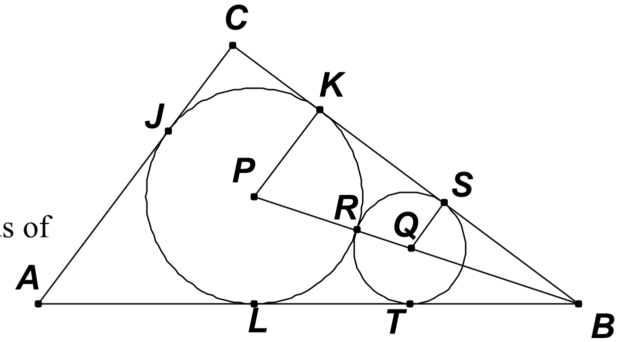


E) - continued
Solution #2 (Tuan Lee)

After showing that $CK = 2$, $BK = 6$ and the radius of the larger circle (PK) is 2, apply the Pythagorean Theorem to ΔPKB, getting $PB = 2\sqrt{10}$

➔ $BR = 2\left(\sqrt{10}-1\right)$

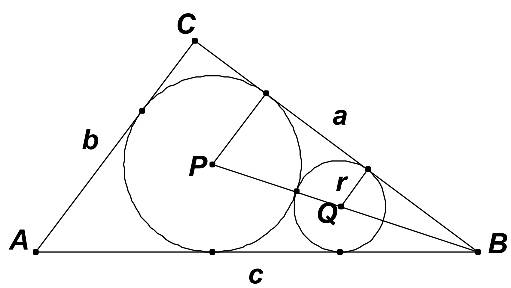
$QR = QS$ ➔ $BR = BQ + QS = 2\left(\sqrt{10}-1\right)$ (Eqtn #1)

Now $\Delta BSQ \sim \Delta BKP$ ➔ $\frac{BQ}{QS} = \frac{BP}{PK} = \frac{2\sqrt{10}}{2} = \sqrt{10}$ ➔ $BQ = \sqrt{10}\,QS$ (Eqtn #2).

Substituting for BQ in eqtn #1, $QS\,(\sqrt{10} + 1) = 2\left(\sqrt{10}-1\right)$ ➔ $QS = \frac{2}{9}\left(11-2\sqrt{10}\right)$

Using the same pair of similar triangles, $\frac{QS}{PK} = \frac{BS}{BK}$ ➔ $\frac{\frac{2}{9}\left(11-2\sqrt{10}\right)}{2} = \frac{BS}{6}$ ➔ $BS = \underline{\mathbf{\frac{2}{3}\left(11-2\sqrt{10}\right)}}$.

Conjecture: (Norm Swanson)



For any right triangle with hypotenuse c and legs a and b (a, b and c integers) and two circles externally tangent to each other and internally tangent to the three sides of the right triangle, as shown in the diagram above, the radius of the <u>larger</u> circle is $\frac{ab}{a+b+c}$ or equivalently $\frac{a+b-c}{2}$ and the radius of the <u>smaller</u> circle is $\frac{(a+b-c)\left((a+c)^2+2b^2-2b\sqrt{(a+c)^2+b^2}\right)}{2(a+c)^2}$.

Will you accept the challenge of proving (or disproving) these conjectures?

Insight gives us conjectures.
Proof gives us theorems (generalizations).

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Proof of the conjectures

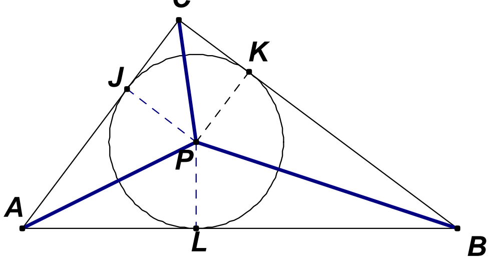

Diagram #1

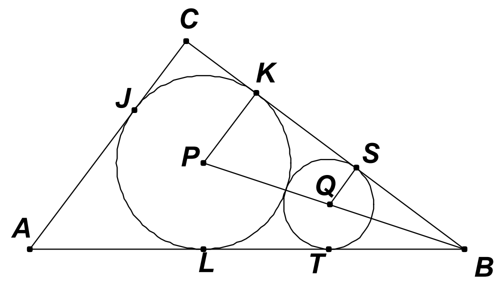

Diagram #2

Conjecture #1 (Diagram #1):
Let P denote the center of the larger circle with radii R in ΔABC with sides $BC = a$, $AC = b$ and $AB = c$.
The area of ΔABC equals the sum of the areas of Δs BPC, APC and APB.

Using $A(\Delta) = \frac{1}{2}bh$, $A(\Delta ABC) = \frac{1}{2}Ra + \frac{1}{2}Rb + \frac{1}{2}Rc = \left(\frac{a+b+c}{2}\right)R$.

Since ABC is a right triangle with hypotenuse $AB = c$ and legs $BC = a$ and $AC = b$,

we have $\frac{1}{2}ab = \left(\frac{a+b+c}{2}\right)R \rightarrow R = \boxed{\frac{ab}{a+b+c}}$.

The equivalent formula $\frac{a+b-c}{2}$ can be verified by showing the cross products are equal.

$\frac{ab}{a+b+c} = \frac{a+b-c}{2} \rightarrow (a+b+c)(a+b-c) = ((a+b)+c)((a+b)-c) = (a+b)^2 - c^2 = a^2 + 2ab + b^2 - c^2$

ΔABC is a right triangle $\rightarrow a^2 + b^2 = c^2$. Regrouping, $(a^2 + b^2 - c^2) + 2ab = 0 + 2ab = 2ab$.
Alternately, note that $CKPJ$ is a square, $R = CK$ and use argument similar to that used below to find BK.
Q.E.D

Conjecture #2 (Diagram #2):
As tangents to circle P from external points A, B and C, $AJ = AL$, $BK = BL$ and $CK = CJ$.
The perimeter of ΔABC may be expressed as $2AJ + 2CJ + 2BK = 2AC + 2BK$.

Thus, $a + b + c = 2b + 2BK$ or $BK = \frac{a+c-b}{2}$. Similarly, $AJ = \frac{b+c-a}{2}$ and $CK = \frac{a+b-c}{2}$

Now, since P and Q both lie on the bisector of $\angle ABC$, B, Q and P must be collinear.
In right triangle BPK, $PB^2 = PK^2 + BK^2$ or

$$PB^2 = R^2 + \left(\frac{a+c-b}{2}\right)^2 = \left(\frac{a+b-c}{2}\right)^2 + \left(\frac{a+c-b}{2}\right)^2 = \frac{a^2+b^2+c^2-2bc}{2} = \frac{2c^2-2bc}{2} = c(c-b)$$

Since $\Delta BQS \sim \Delta BPK$, $\frac{QB}{PB} = \frac{PB-(R+r)}{PB} = \frac{SQ}{KP} = \frac{r}{R} \rightarrow 1 - \frac{R+r}{PB} = \frac{r}{R} \rightarrow R(PB) - R(R+r) = rPB$

$\rightarrow rPB + rR = R(PB) - R^2 \rightarrow r(PB+R) = R(PB-R)$

$\rightarrow r = R\left(\frac{PB-R}{PB+R}\right)$

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Substituting $\sqrt{c(c-b)}$ for PB in $R\left(\frac{PB-R}{PB+R}\right)$ is tedious, so we revert to using the first expression for R.

$$PB^2 = R^2 + \left(\frac{a+c-b}{2}\right)^2 = \left(\frac{ab}{a+b+c}\right)^2 + \left(\frac{a+c-b}{2}\right)^2 = \frac{4(ab)^2 + \left((a+c-b)(a+c+b)\right)^2}{4(a+b+c)^2}$$

$$= \frac{4(ab)^2 + \left((a+c)^2 - b^2\right)^2}{4(a+b+c)^2} = \frac{4(ab)^2 + \left(a^2+c^2-b^2+2ac\right)^2}{4(a+b+c)^2}$$

But since $a^2 + b^2 = c^2$, this simplifies to

$$\frac{4(ab)^2 + \left(2a^2+2ac\right)^2}{4(a+b+c)^2} = \frac{(ab)^2 + a^2(a+c)^2}{(a+b+c)^2} = \frac{a^2\left(b^2 + (a+c)^2\right)}{(a+b+c)^2}$$

Thus, $PB = \frac{a}{a+b+c}\sqrt{(a+c)^2+b^2} = \frac{ab}{b(a+b+c)}\sqrt{(a+c)^2+b^2} = \boxed{\frac{R}{b}\sqrt{(a+c)^2+b^2}}$.

Now substitute for PB:

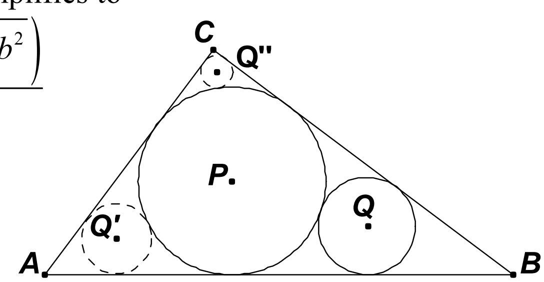
$$r = R\left(\frac{PB-R}{PB+R}\right) = R\left(\frac{\frac{R}{b}\sqrt{(a+c)^2+b^2} - R}{\frac{R}{b}\sqrt{(a+c)^2+b^2} + R}\right) = R\left(\frac{\cancel{\frac{R}{b}}\left(\sqrt{(a+c)^2+b^2} - b\right)}{\cancel{\frac{R}{b}}\left(\sqrt{(a+c)^2+b^2} + b\right)}\right)$$

Rationalizing the denominator, $R\left(\frac{\left(\sqrt{(a+c)^2+b^2} - b\right)}{\left(\sqrt{(a+c)^2+b^2} + b\right)}\right)\cdot\frac{\left(\sqrt{(a+c)^2+b^2} - b\right)}{\left(\sqrt{(a+c)^2+b^2} - b\right)} = R\frac{\left(\sqrt{(a+c)^2+b^2} - b\right)^2}{(a+c)^2+b^2-b^2}$

$$= R\frac{(a+c)^2 + 2b^2 - 2b\sqrt{(a+c)^2+b^2}}{(a+c)^2}$$

Now, using the second expression for R, the expression for r simplifies to

$$\frac{(a+b-c)\left((a+c)^2 + 2b^2 - 2b\sqrt{(a+c)^2+b^2}\right)}{2(a+c)^2}$$



Q.E.D.

You are invited to verify that

1) for a circle with center at Q' a similar formula for the radius can be derived, namely:

$$\frac{(a+b-c)\left((b+c)^2 + 2a^2 - 2a\sqrt{(b+c)^2+a^2}\right)}{2(b+c)^2}$$

2) for the circle with center at Q'', the radius is given by $\left(\frac{a+b-c}{2}\right)\left(3-2\sqrt{2}\right)$.

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F) Suppose the first term of the GP is a and the common multiplier is r.

$S_3 = \frac{a(1-r^3)}{1-r} = 1792$ and $S_{11} = \frac{a(1-r^{11})}{1-r} = 2047$

Dividing, $\frac{S_{11}}{S_3} = \frac{(1-r^{11})}{(1-r^3)} = \frac{2047}{1792}$

If the sum of the terms in the infinite geometric progression converges to a finite sum, $|r| < 1$.
Noting that 2047 is 1 less than a power of 2 and that all the terms are rational numbers,

I try $r = \frac{1}{2}$. $\left[\left(\frac{1-\frac{1}{2048}}{1-\frac{1}{8}}\right)\frac{2048}{2048} = \frac{2048-1}{2048-156} = \frac{2047}{1792}\right]$ Bingo!

Substituting, $\frac{a\left(1-\left(\frac{1}{2}\right)^3\right)}{1-\frac{1}{2}} = 1792$ ➔ $\frac{7}{4}a = 1792$ ➔ $a = 4(256) = 1024$.

The sum of the infinite G.P. is $\frac{a}{1-r} = \frac{1024}{1-\frac{1}{2}} = 2048$.

Now, for the arithmetic progression, $t_{56} = a + 55d = 2048$

➔ $d = \frac{2048-a}{55} = 37 - \boxed{\frac{13-a}{55}}$

For the boxed expression to be an integer, $a = 13 + 55k$, for integer values of k.
$a < 50$ ➔ $a = 13$, $d = 37$ ➔ $t_{55} = 13 + 54(37) =$ **2011**.