MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2011
ROUND 1 ALG 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS

ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) (________ , ________ , ________)

******* NO CALCULATORS ON THIS ROUND *******

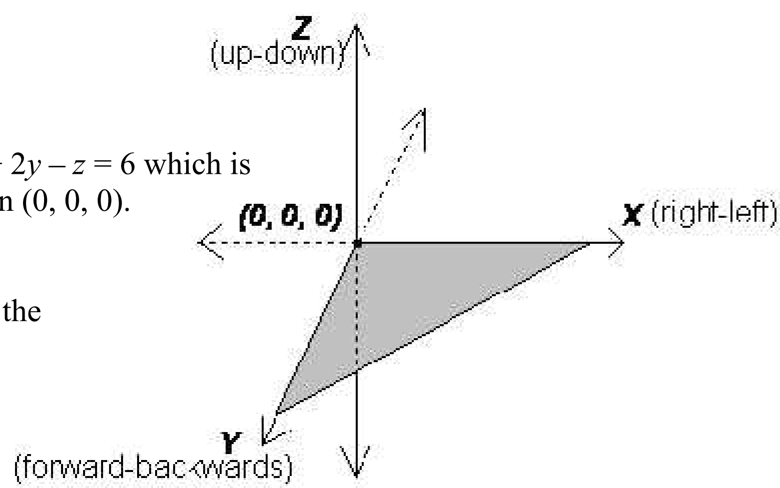
A) Find <u>all</u> ordered pairs (x, y) which satisfy the system $\begin{cases} 5x - 11y = 96 \\ x^2 - y^2 = 0 \end{cases}$.

B) Given: the matrix $\begin{bmatrix} x & 2 \\ x+1 & 10-x \end{bmatrix}$

Compute the ordered pair (x, M), where x is the value for which the determinant of this matrix attains its maximum value .

C) Let $P(a, b, c)$ be the point in the plane $3x + 2y - z = 6$ which is also in the xy-plane and <u>closest</u> to the origin (0, 0, 0). Determine the ordered triple (a, b, c).

Note: The xy-plane is the plane containing the shaded region.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2011
ROUND 2 ALG1: EXPONENTS AND RADICALS

ANSWERS

A) ________________________________

B) (_____ , _____), (_____ , _____)

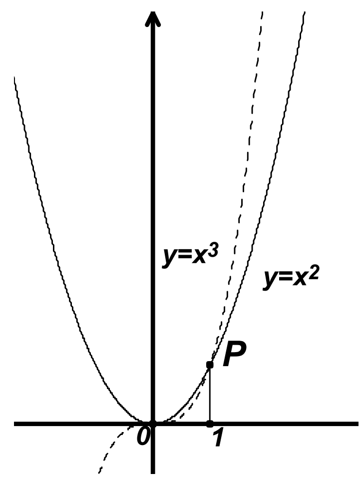
C) (_____________ , _____________)

******* NO CALCULATORS ON THIS ROUND *******

A) Clearly, the cube of a positive integer is always greater than or equal to the square of a positive integer. However, this is not true for all positive real numbers as you can see from the graph at the right. The value of x^2 exceeds the value of x^3 over the interval $0 < x < 1$, since the solid line is above the dotted line. The difference D between x^2 and x^3 is largest for one of the following values:

$$\left(x = \frac{1}{3}\right), \left(x = \frac{1}{2}\right) \text{ or } \left(x = \frac{2}{3}\right).$$

Compute the largest possible value of D.



B) Find both ordered pairs of integers (A, B) that satisfy the system $\begin{cases} (A+B)^3 = -8 \\ (A-B)^2 = 2^{2^3} \end{cases}$.

Recall: $x^{y^z} = x^{\left(y^z\right)}$

C) In simplified form, $\sqrt{37 - 20\sqrt{3}} = a + b\sqrt{3}$, where a and b are integers.
Determine the quadrant in which point $P(a, b)$ lies and compute its distance from the origin.
Express your answer as an ordered pair (quadrant, distance).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2011
ROUND 3 TRIGONOMETRY: ANYTHING

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Let P denote the radian equivalent of $\pi°$.
Let Q denote the degree equivalent of π radians.
Find the numerical product of P and Q.

B) Compute all possible values of x over $0° \le x < 360°$ for which

$$2\sin(x)+3\cot(x)=0$$

C) Find the four 4th roots of $-8+8\sqrt{3}i$. Express your answers in $a + bi$ form.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2011
ROUND 4 ALG 1: ANYTHING

ANSWERS

A) _______ , ______ , ______ , ______

B) ______________________________

C) ______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Determine the <u>four</u> integer values of x for which $Q = \dfrac{10+3x}{6-x}$ is a <u>positive</u> integer.

B) At East Statistics High School, 100 students are in the math club, 120 are in the statistics club and 130 are in the science club. Forty students belong to just the math club, 40 students belong to just the statistics club, and 40 students belong to just the science club. Exactly 12 students belong to all three clubs. If there are 1000 students in this school, how many students belong to <u>none</u> of these clubs?

C) Simplify <u>completely</u>: $\dfrac{1-4x^{-2}}{8-x^{-3}} \div \dfrac{x+2}{\dfrac{4}{x^{-2}}+\dfrac{2}{x^{-1}}+1}$

Recall: $x^{-n} = \dfrac{1}{x^n}$ and $A^3 - B^3 = (A-B)(A^2+AB+B^2)$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2011
ROUND 5 PLANE GEOMETRY: ANYTHING

ANSWERS

A) ______________________________ °

B) ______________________________

C) _______________ : _____________

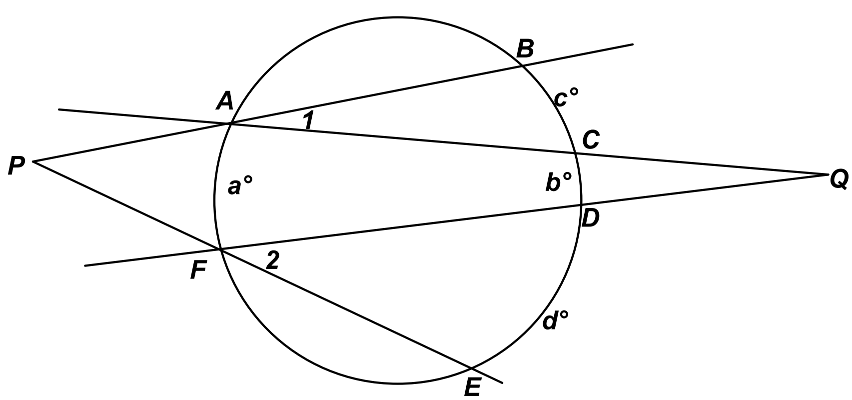
***** NO CALCULATORS ON THIS ROUND *****

A) An isosceles triangle has interior angles which measure $(3x)°$ and $(6x)°$. Compute the degree-measure of the smallest possible interior angle.

B) ΔPQR is similar to a triangle with sides 3, 4 and 5. If the shortest distance in ΔPQR from the vertex of the right angle to the hypotenuse is h units, compute the perimeter of ΔPQR (in terms of h).

C) Given: $m\angle P = 2m\angle Q$,
$AB = FE$

Express $\dfrac{m\angle 1 + m\angle 2}{m\angle P}$ as a simplified ratio of integers.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2011
ROUND 6 ALG 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) The probability of hitting the bull's-eye of a target is 0.4 . Compute the probability of <u>missing</u> the bull's-eye three times with five shots.

B) The simplified ratio of the middle terms in the expansion of $(2x-3y)^{11}$ is $a:b$.
If $\frac{y}{x}=\frac{7}{3}$, compute ab.

C) The probability that Tom will solve a particular problem is 1/2, that Dick will solve it is 2/3 and that Harry will solve it is k. If all three try to solve the problem independently, the probability that <u>at least two</u> of them will solve the problem is $\frac{3}{4}$. Compute k.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2011
ROUND 7 TEAM QUESTIONS

ANSWERS

A) (_______ , _______ , _______)   D) ____________________________

B) ____________________________   E) ____________________________

C) ____________________________   F) (____________ , ____________)

******* NO CALCULATORS ON THIS ROUND *******

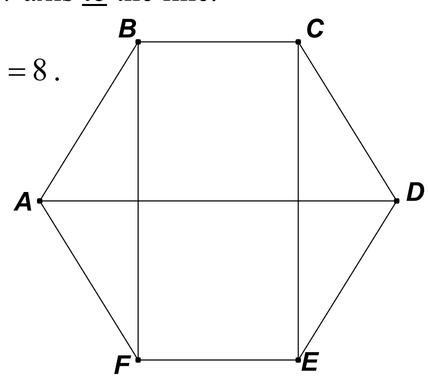
A) Suppose A and B are relatively prime integers such that $A > B > 0$ and the equation.
$A \cdot \det\left(\begin{bmatrix} x+1 & 1 \\ 1 & x+2 \end{bmatrix}\right) + B \cdot \det\left(\begin{bmatrix} x-1 & 1 \\ 1 & x-2 \end{bmatrix}\right) = 0$ has exactly one real root R.
Compute the ordered triple (A, B, R).

B) Let S be the set of positive two digit integers generated by the expression $\sqrt{1+4n}$ for integer values of n. An element of set S is randomly selected. What is the probability that the selected element is greater than 20?

C) A line with slope m passes through (3, 6) and cuts off a triangle in quadrant 1 with area 100.
Compute the smallest possible value for the angle of inclination of this line.
Give your answer (in radians) as a simplified expression in terms of a first quadrant reference value.
Use an inverse trig function which avoids the use of any radicals.
Recall: An angle of inclination is the directed angle from the positive x-axis to the line.

D) Compute the two real values of x for which $(x+1)(x+2)(x+3)(x+4) = 8$.

E) In hexagon $ABCDEF$, $AB = CD = DE = FA = 1$.
In rectangle $BCEF$, $CE = 2BC$.
If $AD = 2.2$, compute both possible lengths of $\overline{CE}$.



F) Given: $S = (A+B)^n + (C+D)^{n+1} + (E-F)^{n+2}$ for n, a positive integer > 10.
Let P denote the sum of the binomial coefficients of S divided by 16.
P can be written in the simplified form $k \cdot 2^x$, where k is a constant and x denotes an expression in terms of n. Determine the ordered pair (k, x).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2011 ANSWERS

Round 1 Alg 2: Simultaneous Equations and Determinants

A) (6, –6), (–16, –16) B) (4, 14) C) $\left(\frac{18}{13},\frac{12}{13},0\right)$

Round 2 Alg 1: Exponents and Radicals

A) $\frac{4}{27}$ B) (7, –9), (–9, 7) C) $\left(4,\sqrt{29}\right)$

Round 3 Trigonometry: Anything

A) π^2 B) 120, 240 C) $\sqrt{3}+i,-1+\sqrt{3}i,-\sqrt{3}-i,1-\sqrt{3}i$

Round 4 Alg 1: Anything

A) –1, 2, 4 and 5 B) 771 C) $\frac{x(x-2)}{2x-1}$ ($\frac{x^2-2x}{2x-1}$ or equivalent)

- in any order –

Round 5 Plane Geometry: Anything

A) 36 B) $5h$ C) 3 : 2

Round 6 Alg 2: Probability and the Binomial Theorem

A) $\frac{216}{625}$ (or 0.3456 or 34.56%) B) –14 C) $\frac{5}{6}$

Team Round

A) (5, 1, –1) D) $\frac{-5\pm\sqrt{17}}{2}$

B) $\frac{8}{9}$ E) $\frac{14}{25},\frac{6}{5}$ (or 0.56, 1.2)

C) $\pi - Arc\tan\left(18\right)$ F) $(3, n-4)$

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2011 SOLUTION KEY

Round 1

A) The 2nd condition requires $y = x$ or $y = -x$.
Substituting in the first equation , $y = x$ ➔ $-6x = 96$ ➔ **(–16, –16)**
$y = -x$ ➔ $16x = 96$ ➔ **(6, –6)**

B) $\det\left(\begin{bmatrix} x & 2 \\ x+1 & 10-x \end{bmatrix}\right) = 10x - x^2 - 2x - 2 = -x^2 + 8x - 2$

Completing the square, $-\left(x^2 - 8x + 16\right) + 16 - 2 = -(x-4)^2 + 14$ which defines a downward opening parabola with vertex at **(4, 14)**.

C) Since the point is in the xy-plane, $z = 0$.
Thus, in the xy-plane, we want the point on L: $3x + 2y = 6$ closest to (0, 0).
This point lies on the line through the origin perpendicular to L ➔ $y = \frac{2}{3}x$

Substituting, $3x + 2\left(\frac{2}{3}x\right) = 6$ ➔ $13x = 18$ ➔ $(x, y, z) = \left(\mathbf{\frac{18}{13}, \frac{12}{13}, 0}\right)$.

Round 2

A) Just need to evaluate $x^2 - x^3$ for the three given values and pick the largest difference.
Substitution is easier if the expression is factored as $x^2(1-x)$.

$\frac{1}{3}$ ➔ $\frac{1}{9}\left(1-\frac{1}{3}\right) = \frac{2}{27}$ $\quad \frac{1}{2}$ ➔ $\frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8}$ $\quad \frac{2}{3}$ ➔ $\frac{4}{9}\left(1-\frac{2}{3}\right) = \frac{4}{27}$

'Cross Multiplying', $\frac{4}{27} > \frac{1}{8}$ because $4 \cdot 8 > 27 \cdot 1$. Thus, $D = \mathbf{\frac{4}{27}}$.

B) 2^{2^3} is evaluated from right to left. $2^{2^3} = 2^8 = 256$ (not $4^3 = 64$).

$$\begin{cases} (A+B)^3 = -8 \\ (A-B)^2 = 256 \end{cases} \Rightarrow \begin{cases} A+B = -2 \\ A-B = \pm 16 \end{cases}$$

Adding, $2A = -2 \pm 16$ ➔ $A = -1 \pm 8$ ➔ 7, –9
Therefore, $(A, B) =$ **(7, –9), (–9, 7)**

C) If the given radical can be simplified, then the radicand must be expressible as a perfect square.

$37 - 20\sqrt{3} = \left(a + b\sqrt{3}\right)^2 = a^2 + 3b^2 + 2ab\sqrt{3}$ ➔ $a^2 + 3b^2 = 37$ and $ab = -10$

a and b have opposite signs. The ordered pairs (5, –2) and (–5, 2) satisfy both equations.
$a + b\sqrt{3}$ must represent a positive number, so $-5 + 2\sqrt{3}$ is rejected.
$(a, b) = (5, -2)$ ➔ quadrant = 4, distance = $\sqrt{29}$ ➔ $\left(\mathbf{4, \sqrt{29}}\right)$.

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2011 SOLUTION KEY

Round 3

A) $\pi \cdot \pi = \pi^2$ It's just that simple – no conversion necessary!
If the question had been multiply the inch equivalent of 3 feet by the foot equivalent of 3 inches, the product would be 36(1/4) = 9 = 3^2.

For the disbelievers,
In the case of degree-radian conversion, 180° = π radians (halfway around a circle)
Therefore, $\pi° = \dfrac{\pi^2}{180}$ radians and $\dfrac{\pi^2}{180} \cdot 180 = \underline{\boldsymbol{\pi^2}}$

B) $2\sin(x)+3\cot(x)=0 \rightarrow \dfrac{2\sin^2(x)+3\cos(x)}{\sin(x)}=0 \rightarrow 2\left(1-\cos^2(x)\right)+3\cos(x)=0$

$\rightarrow 2\cos^2(x)-3\cos(x)-2=(2\cos x+1)(\cos x-2)=0 \rightarrow \cos x=-\dfrac{1}{2} \rightarrow x = \underline{\mathbf{120}}, \underline{\mathbf{240}}$.

C) Let $z^4=(rcis\theta)^4=r^4cis(4\theta)=-8+8\sqrt{3}i=Acis\alpha$, where $\alpha \in Q2$

Converting to trigonometric form, $-8+8\sqrt{3}i=16cis\left(\dfrac{2\pi}{3}\right)$, because for $x+yi$

$A^2=x^2+y^2$ and $\tan\alpha=\dfrac{y}{x} \rightarrow A^2 = 64 + 64\cdot 3 = 256 \rightarrow A = 16$ and $\tan\alpha = \sqrt{3} \rightarrow \theta=\dfrac{2\pi}{3}$

Thus, $r^4cis(4\theta)=16cis\left(\dfrac{2\pi}{3}\right) \rightarrow \begin{cases} r^4=16 \\ 4\theta=\dfrac{2\pi}{3}+2n\pi \end{cases} \rightarrow \begin{cases} r=2 \\ \theta=\dfrac{\pi(1+3n)}{6} \end{cases}$, for $n=0,1,2,3$

Thus, the roots are $2cis(\theta)$, where $\theta = \dfrac{\pi}{6}, \dfrac{2\pi}{3}, \dfrac{7\pi}{6}$ and $\dfrac{5\pi}{3}$.

Converting back to rectangular form, $2cis\left(\dfrac{\pi}{6}\right)=2\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)=2\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i\right)=\sqrt{3}+i$ etc.

Thus, the four roots are: $\underline{\boldsymbol{\sqrt{3}+i, -1+\sqrt{3}i, -\sqrt{3}-i, 1-\sqrt{3}i}}$

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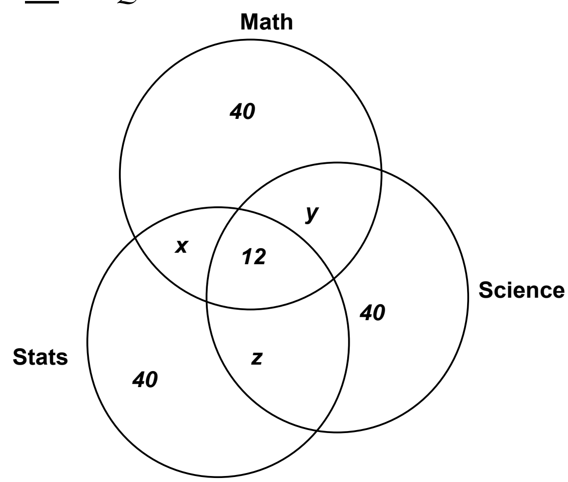
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2011 SOLUTION KEY

Round 4

A) Given: $Q = \dfrac{10+3x}{6-x}$ The value of x must be less than 6 for the value of the fractional expression to be positive. Testing $x = \underline{\mathbf{5}} \rightarrow Q = 25$, $x = \underline{\mathbf{4}} \rightarrow Q = 11$, $x = \underline{\mathbf{2}} \rightarrow Q = 4$
$x = 1, 3$ fail! Where's the 4$^{\text{th}}$ value?!
There was no requirement that x had to be positive. $x = \underline{\mathbf{-1}} \rightarrow Q = +1$



B) Using the Venn diagram to the right
$100 = x + z + 40 + 12$
$120 = x + y + 12 + 40$
$130 = y + z + 40 + 12$
Then:
$48 = x + z$
$68 = x + y$
$78 = y + z$

Solving, $x = 19$, $y = 49$ and $z = 29$
Thus, the number of students belonging to no club is:
$1000 - (40 + 40 + 40 + 12 + 19 + 49 + 29) = 1000 - 229 = \underline{\mathbf{771}}$

C) $$\frac{1-4x^{-2}}{8-x^{-3}} \div \frac{x+2}{\frac{4}{x^{-2}}+\frac{2}{x^{-1}}+1} = \frac{\left(x^3-4x\right)\left(4x^2+2x+1\right)}{\left(8x^3-1\right)(x+2)} = \frac{x\left(x^2-4\right)\left(4x^2+2x+1\right)}{(2x-1)\left(4x^2+2x+1\right)(x+2)} = \underline{\frac{\mathbf{x(x-2)}}{\mathbf{2x-1}}}$$

Note: $8x^3-1$ is factored as $A^3-B^3=\left(A-B\right)\left(A^2+AB-B^2\right)$, where $A = 2x$ and $B = 1$.

Round 5

A) The vertex angle measures either $(3x)°$ or $(6x)°$.
In the first case, $15x = 180 \rightarrow x = 12$ and $\angle$s measure 36, 72 and 72
In the second case, $12x = 180 \rightarrow x = 15$ and $\angle$s measure 90, 45 and 45
Thus, the smallest possible interior angle measures $\underline{\mathbf{36}}$°.

B) The shortest distance is measured along the altitude. Let the length of the altitude in ΔPQR be h units. The sides of ΔPQR have lengths $3x$, $4x$ and $5x$, where x represents the proportionality constant. The area of ΔPQR is $\frac{1}{2}(3x)(4x)$ or $\frac{1}{2}h(5x)$. Equating, $h = \frac{12}{5}x$ or $x = \frac{5h}{12}$.

Perimeter $= 12x = 12\left(\frac{5h}{12}\right) = \underline{\mathbf{5h}}$.

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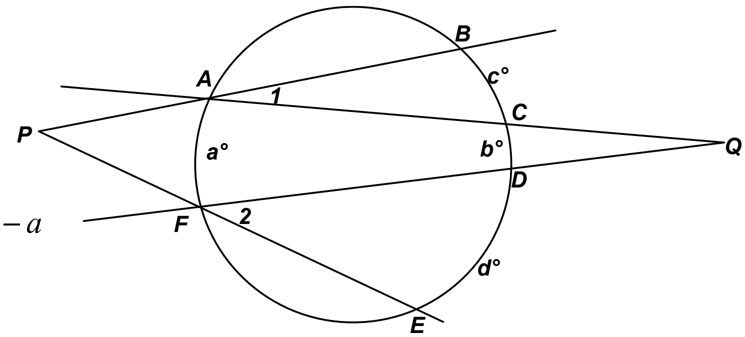
Round 5 – continued



C) Let m$\angle P = 2x$, m$\angle Q = x$ and the degree measures of the minor arcs as indicated in the diagram

$$m\angle P = \frac{1}{2}(c+d+b-a) \rightarrow 4x = c+d+b-a$$

$$m\angle Q = x = \frac{1}{2}(a-b) \rightarrow 2x = a-b$$

Adding, $c+d=6x$.

$$m\angle 1 + m\angle 2 = \frac{1}{2}c + \frac{1}{2}d = 3x$$

Thus, without finding specific values for our variables and only using the first piece of given information, the required ratio $\frac{m\angle 1 + m\angle 2}{m\angle P}$ is $\frac{3x}{2x} = \frac{3}{2}$ ➔ **3 : 2**.

Round 6

A) MMMHH or any rearrangement of 3 Misses and 2 Hits

$$\binom{5}{3}(0.4)^2(0.6)^3 = \frac{10(4)^2(6)^3}{10^5} = \frac{2^4\cdot 2^3\cdot 3^3}{2^4\cdot 5^4} = \frac{2^3\cdot 3^3}{5^4} = \mathbf{\frac{216}{625}}$$ (or **0.3456** or **34.56%**)

B) Since there are 12 terms in the expansion, the middle terms are the 6th and 7th terms.

The required ratio is $\dfrac{\binom{11}{5}(2x)^6(-3y)^5}{\binom{11}{6}(2x)^5(-3y)^6}$ or the reciprocal. Since the combinatorial terms are equal, we have $\frac{2x}{-3y} = \frac{2}{-3}\cdot\frac{3}{7} = \frac{2}{-7}$ and the required product is **–14**.

C) Given: P(T) = $\frac{1}{2}$, P(D) = $\frac{2}{3}$ and P(H) = k. Let ~ denote not.

P(at least 2) = P(exactly 2) + P(exactly 3)

= P(T)·P(D)·P(~H) + P(T)·P(~D)·P(H) + P(~T)·P(D)·P(H) + P(T)·P(D) ·P(H)

$$= \tfrac{1}{2}(\tfrac{2}{3})(1-k) + \tfrac{1}{2}(\tfrac{1}{3})(k) + \tfrac{1}{2}(\tfrac{2}{3})(k) + \tfrac{1}{2}(\tfrac{2}{3})(k) = \frac{3}{4}$$

$$= \frac{1-k}{3} + \frac{k}{6} + 2\left(\frac{k}{3}\right) = \frac{3}{4}$$ ➔ $4(1-k) + 2k + 8k = 9$ ➔ $6k = 5$ ➔ $k = \mathbf{\frac{5}{6}}$

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MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2011 SOLUTION KEY

Team Round

A) $A \cdot \det\left(\begin{bmatrix} x+1 & 1 \\ 1 & x+2 \end{bmatrix}\right) + B \cdot \det\left(\begin{bmatrix} x-1 & 1 \\ 1 & x-2 \end{bmatrix}\right) = 0 \rightarrow A\left(x^2+3x+1\right) + B\left(x^2-3x+1\right) = 0$

$\rightarrow (A+B)x^2 + 3(A-B)x + (A+B) = 0 \rightarrow x = \dfrac{-3(A-B) \pm \sqrt{9(A-B)^2 - 4(A+B)^2}}{2(A+B)}$.

If there is only one real root then, $9(A-B)^2 - 4(A+B)^2 = 0$ and $x = \dfrac{-3(A-B)}{2(A+B)}$

The radicand as a difference of perfect squares is
$\left(3(A-B)+2(A+B)\right)\left(3(A-B)-2(A+B)\right) = (5A-B)(A-5B) = 0$
$A > B > 0 \rightarrow A = 5B$.
A and B relatively prime $\rightarrow$ $(A, B) = (5, 1)$ $\rightarrow x = \dfrac{-3(5-1)}{2(5+1)} = \underline{\mathbf{-1}}$ $\rightarrow (A, B, R) =$ $\underline{\mathbf{(5, 1, -1)}}$.

B) $\sqrt{1+4n}$ generates integers for $n = 0, 2, 6, 12, \ldots$
Note these values of n are of the form $k(k-1)$ for $k = 1, 2, 3, \ldots$

$\sqrt{1+4n} = \sqrt{1+4k(k-1)} = \sqrt{4k^2-4k+1} = \sqrt{(2k-1)^2} = 2k-1$ for $k = 1, 2, 3, \ldots$.
$10 < 2k - 1 < 100 \rightarrow 6 \leq k \leq 50$
$k = 6 \rightarrow n = 30 \rightarrow \sqrt{121} = 11$ (or $2\cdot 6 - 1$)
$k = 7 \rightarrow n = 42 \rightarrow \sqrt{169} = 13$ (or $2\cdot 7 - 1$)
…
$k = 50 \rightarrow n = 2450 \rightarrow \sqrt{9801} = 99$ (or $2\cdot 50 - 1$)
Thus, S contains the 45 odd integers between 11 and 99 inclusive.
The condition "less than 20" is satisfied by only 11,13,15,17 and 19.
Thus, the required probability is $\dfrac{45-5}{45} = \underline{\dfrac{\mathbf{8}}{\mathbf{9}}}$.

Alternate solution [Michael Zanger-Tishler (BB & N)]
The square of any integer is congruent to either 0 or 1 mod 4, specifically, the squares of even integers are congruent to 0 mod 4 and the squares of odd integers are congruent to 1 mod 4. [Note: Being "congruent to 1 mod 4" is a fancy way of saying "leaves a remainder of 1 when divided by 4".]
So the question then becomes "what percent of two digit odd numbers are greater than 20?"
which is $\dfrac{\{21, 23, \ldots, 99\}}{\{11, 13, \ldots, 99\}} \rightarrow \dfrac{40}{45} = \underline{\dfrac{\mathbf{8}}{\mathbf{9}}}$.

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Team Round - continued

C) (3, 6) / $y = mx + b$ → $b = 6 - 3m$

The y-intercept is at $(0, b) = (0, 6 - 3m)$.

The x-intercept as at $\left(-\frac{b}{m}, 0\right) = \left(-\frac{6-3m}{m}, 0\right)$

The area of the first quadrant triangle is given by $\frac{1}{2}(6-3m)\cdot\left(-\frac{6-3m}{m}\right) = 100$

→ $(6-3m)^2 = -200m$ → $9m^2 + 164m + 36 = (9m+2)(m+18) = 0$ → $m = -2/9, -18$

If θ denotes the angle of inclination, then $m = \tan\theta$

Negative slopes correspond to <u>obtuse</u> angles of inclination.

$\theta = \pi - Arc\tan(18)$ or $\pi - Arc\tan\left(\frac{2}{9}\right)$ Since $y = \text{Arctan}(x)$ is a strictly increasing function, $Arc\tan(18) > Arc\tan\left(\frac{2}{9}\right)$ and the smaller of these is $\underline{\boldsymbol{\pi - Arc\tan(18)}}$

D) Regroup the terms on the left side as follows:

$\left((x+2)(x+3)\right)\left((x+1)(x+4)\right) = \left(x^2+5x+6\right)\left(x^2+5x+4\right) = 8$

If $A = (x^2 + 5x)$, then the equation becomes $(A+6)(A+4) = 8$.

→ $A^2 + 10A + 16 = (A+2)(A+8) = 0$ → $A = -2, -8$

Substituting and applying the quadratic formula,

$x^2 + 5x + 2 = 0$ → $x = \underline{\boldsymbol{\frac{-5 \pm \sqrt{17}}{2}}}$ and $x^2 + 5x + 8 = 0$ → $b^2 - 4ac = 25 - 32 < 0$ (rejected)

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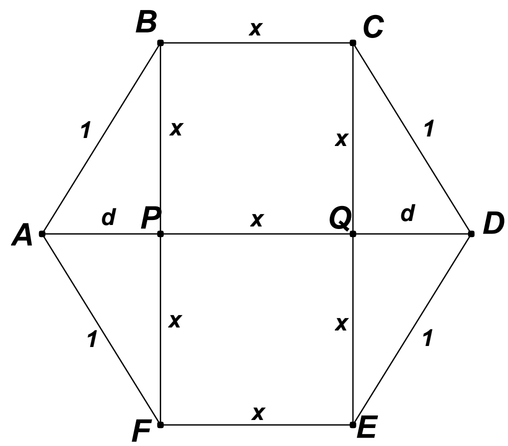
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2011 SOLUTION KEY

Team Round - continued

E) The measurements are indicated in the diagram at the right:



$d=\sqrt{1-x^2}$ → $AD = x+2\sqrt{1-x^2}=2.2=\dfrac{11}{5}$

$10\sqrt{1-x^2}=11-5x$ → $100(1-x^2) = 121 - 110x + 25x^2$

→ $125x^2 - 110x + 21 = (25x - 7)(5x - 3) = 0$ → $x = \dfrac{7}{25},\dfrac{3}{5}$

→ $CE = \underline{\mathbf{\dfrac{14}{25},\dfrac{6}{5}}}$ (or **<u>0.56</u>**, **<u>1.2</u>**)

Double check that both answers are OK.

Alternate solution (Norm Swanson)

$\left(2\sqrt{1-x^2}\right)^2=(2.2-x)^2$ → $5x^2-4.4x+0.84=0$

Suppose the roots are multiplied by 10. Since the linear coefficient determines the sum of the roots and the constant term determines the product of the roots, this changes the linear coefficient by 10 and the constant term by 100.

→ $5x^2-4.4(10)x+0.84(100)=5x^2-44x+84=(5x-14)(x-6)=0$

Thus, the roots of the original quadratic are $\dfrac{14}{50},\dfrac{6}{10}$ → $\dfrac{7}{25},\dfrac{3}{5}$ → $CE = \underline{\mathbf{\dfrac{14}{25},\dfrac{6}{5}}}$.

F) Consider a specific case first, let $n = 2$.

$(A+B)^2+(C+D)^3+(E-F)^4=$

$\left(A^2+2AB+B^2\right)+\left(C^3+3C^2D+3CD^2+D^3\right)+\left(E^4-4E^3F+6E^2F^2-4EF^3+F^4\right)$ ****

Examining the coefficients only, (1 + 2 + 1) + (1 + 3 + 3 + 1) + (1 – 4 + 6 – 4 + 1) = 2 + 8 + 0 = 10

We notice:

- The coefficients are terms in Pascal's triangle.
- The first two sums are powers of 2.
- In the third sum, the signs of the coefficients alternate and the resulting sum is 0.

Were these coincidences?

If we want only the coefficients in **** above, we could let $A = B = C = D = E = F = 1$.

This forces all the literal parts to evaluate to 1 and we are left with the coefficients.

Therefore, the sum of the coefficients in S divided by 16 is

$$\frac{(1+1)^n+(1+1)^{n+1}+(1-1)^{n+2}}{16}=\frac{2^n+2^{n+1}+0}{16}=\left(\frac{2^n}{2^4}\right)(1+2)=3\left(2^{n-4}\right)$$

→ (k, x) = **<u>(3, *n* – 4)</u>**