MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 – OCTOBER 2011
ROUND 1 VOLUME & SURFACES

ANSWERS

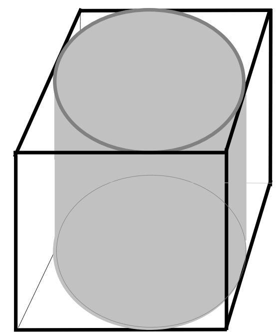
A) ________________ : _______________

B) ___________________________________

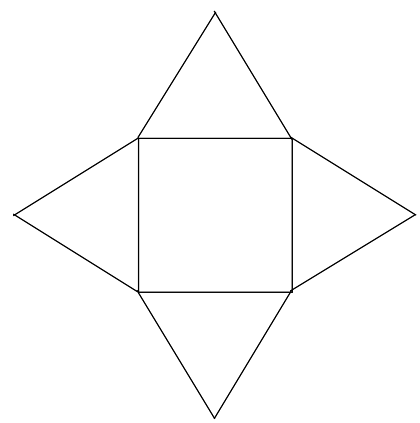
C) ___________________________________

******* NO CALCULATORS ON THIS ROUND ******

A) A cylinder is <u>inscribed</u> in a cube such that the bases lie in opposite faces of the cube. Compute the ratio of the volume of the cylinder to the volume of the cube.



B) Each segment in the template to the right has length 6. The template is comprised of a square and 4 equilateral triangles. If folded along the sides of the square, a pyramid with a square base is formed. Compute its volume.



C) The diagonal of a rectangular solid is $4\sqrt{10}$ units. The length of the solid is $\sqrt{3}$ times as long as its width. The height of the solid is 2 less than its width. Compute the volume of the solid.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011
ROUND 2 PYTHAGOREAN RELATIONS IN RECTILINEAR FIGURES

ANSWERS

A) ______________________________

B) ______________________________

C) (________, ________, ________)

******* NO CALCULATORS ON THIS ROUND ******

A) The hypotenuse and a leg of right triangle ΔABC has lengths $13\sqrt{2}$ and $6\sqrt{3}$ respectively. To the nearest integer, how long is the other leg?

B) Obtuse ΔABC has sides of length 25, 45 and 53. If the length of the shortest side is increased by the positive integer N (but still remains the shortest side), ΔABC becomes a right triangle. Compute the value of N.

C) We know right triangles exist in which the hypotenuse is 1 unit longer than a leg, e.g. 3 – 4 – 5. Compute the sides (a, b, c), where a, b and c are integers, c denotes the hypotenuse and $b > a$, of the smallest such triangle whose perimeter exceeds 100.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) (____________, ____________)

B) ______________________________

C) (____________, ____________)

******* NO CALCULATORS ON THIS ROUND ******

A) The lines $y = 4x + 1$ and $y = mx + b$ are perpendicular to each other.
The second line passes through the point $\left(2, \frac{3}{4}\right)$. Find the ordered pair (m, b).
<u>Note</u>: Perpendicular lines have negative reciprocal slopes.

B) Comparing their ages next year, John's age will be twice Andy's age. Comparing their ages two years ago, John's age was three times Andy's age. Find the <u>sum</u> of their current ages.

C) A rectangle has length 5. If the length is increased by 10 and the width is increased by 10%, the perimeter is increased by 110%. Compute the ordered pair (A, B), where A denotes the perimeter of the original rectangle and B denotes the perimeter of the new (larger) rectangle.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

******* NO CALCULATORS ON THIS ROUND ******

A) Gottfried is waiting in line to buy tickets to Calculus: The Musical. In front of him in line are 5/6 of the total number of people in line. Behind him are 1/8 of the people.
Compute the least number of people in line waiting for tickets.

B) A basketball team has won 46 out of the 60 games played so far. The team has 22 more games on the schedule. Find the minimum number of additional wins the team will need so their season record will exceed 0.800 for the first time in club history.

C) Let t be an integer.

The equations $\begin{cases} x = 1 + \dfrac{4t-3}{9} \\ y = 3 - \dfrac{13-5t}{8} \end{cases}$ generate (x, y) coordinates of points that lie on a straight line.

For example, if $t = -15$, $x = 1 + (-63/9) = -6$ and $y = 3 - (88/8) = -8$.
So this straight line passes through the point $P(-6, -8)$.
Compute the next largest value of t for which x and y are both integers.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

ANSWERS

A) ______________________________

B) ______________________________

C) ________________________ sq. units

******* NO CALCULATORS ON THIS ROUND ******

A) Specify all intervals over which the inequality $x^2 + 10x \leq 24$ is satisfied.

B) Solve: $|3x - 11| < 2x + 1$

C) Find the area of the region bounded by $\begin{cases} y + |x| < 8 \\ |x+1| < 7 \\ y \geq 0 \end{cases}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011
ROUND 6 ALG 1: EVALUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

******* NO CALCULATORS ON THIS ROUND ******

A) Compute $(394387)^2 - (394381)^2$.

B) Given: $x + y = 5$ and $2x - 3y = 8$. Compute the numerical value of $7x - 8y$.

C) Compute: $1-\cfrac{1}{2-\cfrac{1}{3-\cfrac{1}{4}}} + \left(0.2\overline{3}+0.0\overline{4}\right)$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) ____________________________

C) $_______. _________________ F) ____________________________

******* NO CALCULATORS ON THIS ROUND ******

A) Given: A box (i.e. a rectangular solid) with faces having areas of 180 square units, 240 square units and 144 square units. Compute the length of a diagonal of the box.

B) Given: ΔABC, with a right angle at A such that

$AB = 1$, $BC = \sqrt{10}$

Let M be the point on $\overline{BC}$ such that $\frac{AM}{AC} = \frac{1}{3}$ and $MC < AC$. Compute MC.

C) On February 3, 1991, the postcard rate was increased to 19¢ and a first-class letter (1 oz. or less) to 29¢. A postal clerk sold 40 stamps (19¢ and 29¢ only) for $9.20 .
The current rates for postcards and first-class mail are 28¢ and 44¢ respectively.
Using current rates, how much (in dollars and cents) would it cost to mail the same number of postcards and first-class letters?

D) Let $N = \frac{10x}{x+10}$ for integer values of x.
Compute the sum of all possible positive integer values of N.

E) The inequality $|2x + 1| < x - c$, where c is an integer, is satisfied by exactly 17 integer values of x.
Determine the largest possible value of c.

F) What value is printed by the following "program"?

$T = 0$
$m = 1$
Repeat
  $p = 4m + 1$
  $q = 4m + 3$
  If both p and q are prime, increase the value of T by $(p + q)$.
  Increase the value of m by 1.
Until $q > 100$
Print T

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011 ANSWERS

Round 1 Geometry Volumes and Surfaces

A) $\frac{\pi}{4}$ B) $36\sqrt{2}$ C) $144\sqrt{3}$

Round 2 Pythagorean Relations

A) 15 B) 3 C) (11, 60, 61)

Round 3 Linear Equations

A) $\left(-\frac{1}{4},\frac{5}{4}\right)$ B) 16 C) (19, 39.9)

Round 4 Fraction & Mixed numbers

A) 24 B) 20 C) 57

Round 5 Absolute value & Inequalities

A) $-12 \le x \le 2$ or $[-12, 2]$ B) $2 < x < 12$ C) 62

Round 6 Evaluations

A) 4732608 B) 29 C) $\frac{2}{3}$

Team Round

A) $10\sqrt{6}$ D) 335

B) $\frac{4\sqrt{10}}{5}$ E) -14

C) \$13.76 F) 396

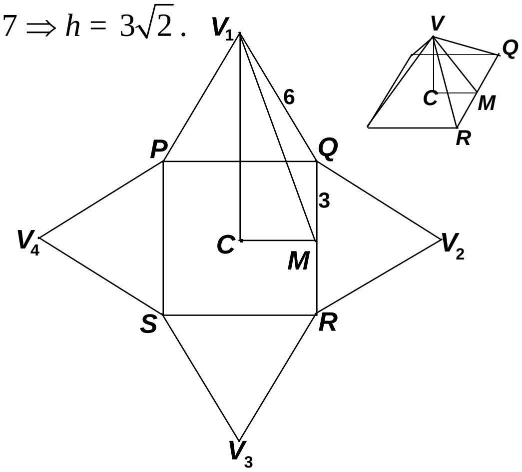
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011 SOLUTION KEY

Round 1

A) Let s denote the length of a side of the cube. Then the height of the cylinder is also s and the radius of the base is $\frac{s}{2}$. The required ratio is $\dfrac{\pi\left(\frac{s}{2}\right)^2\cdot s}{s^3}=\underline{\boldsymbol{\frac{\pi}{4}}}$.

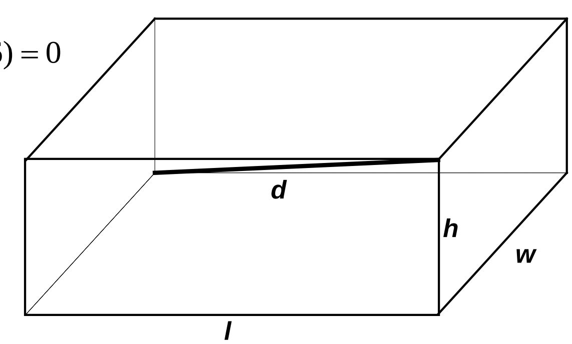
B) If h denotes the height of the pyramid, the volume of the pyramid is $\frac{1}{3}Bh$ or $12h$.

Let V denote the vertex of the pyramid, C the center of the square and M the midpoint of a side of the square.
Consider right ΔVCM, with hypotenuse VM. $VC = h$, $CM = 3$ and VM is an altitude of the equilateral triangle VQR.
$VM^2 + 3^2 = 6^2 \Rightarrow VM^2 = 27$ and, therefore, $h^2 + 9 = 27 \Rightarrow h = 3\sqrt{2}$.
Thus, the required volume is $12\left(3\sqrt{2}\right)=\underline{\mathbf{36\sqrt{2}}}$.



C) $\begin{cases} d = 4\sqrt{10} \\ l = \sqrt{3}w \\ h = w-2 \\ d^2 = l^2 + w^2 + h^2 \end{cases} \Rightarrow \left(4\sqrt{10}\right)^2 = 160 = w^2 + (w-2)^2 + \left(w\sqrt{3}\right)^2$

$160 = 5w^2 - 4w + 4 \Rightarrow 5w^2 - 4w - 156 = (5w+26)(w-6) = 0$
$\Rightarrow w = 6$
$\Rightarrow (l, w, h) = \left(6\sqrt{3}, 6, 4\right) \Rightarrow V = \underline{\mathbf{144\sqrt{3}}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011 SOLUTION KEY

Round 2

A) $x^2+\left(6\sqrt{3}\right)^2=\left(13\sqrt{2}\right)^2 \Rightarrow x^2=169(2)-36(3)=338-108=230$.
Since $15^2 = 225$ and $16^2 = 256$, we see that 230 is closer to 15^2 than 16^2.
Thus, to the nearest integer, x = **<u>15</u>**.

B) $(25+N)^2+45^2=53^2 \Rightarrow (25+N)^2=53^2-45^2=(53+45)(53-45)=98(8)=49(16)=28^2$.
Thus, $25 + N = 28 \Rightarrow N =$ **<u>3</u>**.

<u>FYI</u> - Here's how we know that the original ΔABC was obtuse:
Let $AB = 53$, then C is the largest angle. Using the Law of Cosines,

$$c^2=a^2+b^2-2ab\cos C \Rightarrow \cos C=\frac{a^2+b^2-c^2}{2ab} \Rightarrow \cos C=\frac{25^2+45^2-53^2}{2(25)(45)}$$

Since $25^2+45^2-53^2=625+2025-2809<0$, $\cos C<0$ and ΔABC must be obtuse.

C) Let $(a, b, c) = (a, n - 1, n)$. Then $a^2 = n^2 - (n - 1)^2 = 2n - 1$.
Consider these perfect squares $\{9, 25, 49, ...\}$.
Since a^2 must be odd, even perfect squares are not considered.
$a = 1$ is rejected , since $a = 1 \Rightarrow n = 1$ and this leaves a leg of length 0.

($a^2 = 2n - 1$)

a	***a²***	***N***	***a***	***b***	***c***	***Per***
3	9	5	3	4	5	12
5	25	13	5	12	13	30
7	49	25	7	24	25	56
9	81	41	9	40	41	90
11	121	61	11	60	61	**<u>132</u>**

Thus, (a, b, c) = **<u>(11, 60, 61)</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 3

A) Since the two lines are perpendicular, $m = -\frac{1}{4}$.

Since the second line passes through $(x, y) = \left(2, \frac{3}{4}\right)$, the coordinates must satisfy the equation. Substituting, $\frac{3}{4} = -\frac{1}{4} \cdot 2 + b \Rightarrow b = \frac{5}{4} \Rightarrow \underline{\left(-\frac{\mathbf{1}}{\mathbf{4}}, \frac{\mathbf{5}}{\mathbf{4}}\right)}$.

B) $\begin{cases} x+1 = 2(y+1) \\ x-2 = 3(y-2) \end{cases} \Rightarrow x = 2y + 1$

Substituting, $(2y - 1) - 2 = 3(y - 2) \Rightarrow y = 5$, $x = 11 \Rightarrow$ sum = **<u>16</u>**.

C) A 100% increase doubles the original amount; 110% adds an additional 10% or 1/10.
Therefore, a 110% increase is equivalent to 2.1 times as large.
$2.1(2 \cdot 5 + 2W) = 2(5+10) + 2(1.1W)$
$\Rightarrow 21 + 4.2W = 30 + 2.2W \Rightarrow 2W = 9 \Rightarrow W = 4.5$
The original rectangle is 5 x 4.5 with a perimeter of 19.
The new rectangle is 15 x 4.95 with a perimeter of 39.9 .
Thus, (A, B) = **<u>(19, 39.9)</u>** .

Did you know that 5 out of every 4 people profess to have difficulty with %?

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Round 4

A) $\frac{1}{8}+\frac{5}{6}=\frac{3+20}{24}=\frac{23}{24}$ Thus, Gottfried (or for those who aren't on a first name basis – Mr. Leibnitz) represents $\frac{1}{24}$ th of the people in line and the minimum number of people is **24**.

FYI: An alternate solution (He's a real intellectual and some would say egotistical and long-winded.) Courtesy of Mr. Leibnitz himself with thanks to his great grandson for translating from the original German – they both love algebra and all forms of higher math.)
Suppose there were x people behind me and y people in front of me.

It follows that $\frac{1}{8}\left(x+y+\overset{\int ME}{\boxed{1}}\right)+\overset{\int ME}{\boxed{1}}+\frac{5}{6}\left(x+y+\overset{\int ME}{\boxed{1}}\right)=x+y+\overset{\int ME}{\boxed{1}}$. Multiplying through by 24,

$3(x + y + 1) + 24 + 20(x + y + 1) = 24(x + y + 1)$. Combining terms, $(x + y + 1) =$ **24**.

BUT $x + y + 1 = 24 \Leftrightarrow x + y = 23$, so why aren't there more possibilities?
We wanted the length of the line, not the number of people in front of me and behind me.
$x = 3$ and $y = 20$ is the only possible x, y-combination. [3/24 = 1/8, 20/24 = 5/6]
For example, 4 + 19 = 23 (This would be fine with ME, I'd be closer to the ticket window.), but this would have 4/24 = 1/6 of the people behind me and 19/24 of the people in front of me, clearly violating the initial conditions and similarly for any other x, y-values.
Q.E.D – That's all I've got to say about that, but what about the law of diminishing returns?

The sequence of individual terms of the harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}$ approach 0,

but I think the series itself diverges. What do you think?

B) $\frac{W}{W+L}=\frac{46+x}{82}=\frac{4}{5} \Rightarrow 328 = 230 + 5x \Rightarrow x = 98/5 = 19.6 \Rightarrow$ **20** games
Check: 65/82 = 0.793   66/82 = 0.805

C) With respect to the x-coordinate, $(4t – 3)$ must be a multiple of 9 for x to be an integer. Multiples of 9 are 9 apart, so we look at t-values which are 9 apart, starting at –15, –6, 3, 12, 21, ….
With respect to the y-coordinate, $(13 – 5t)$ must be a multiple of 8 for y to be an integer. Multiples of 8 are 8 apart, so we look at t-values which are 8 apart, starting at –15, –7, 1, 9, 17, ….
What is the next number that these lists will have in common?
We could continue the lists until the common number appeared or simply note that the least common multiple of 8 and 9 is 72. –15 + 72 = **57**.
It's left to you to check that for $x = 57$, both x and y are integers.

Alternative Solutions
Solve for y in terms of x: $(45x – 32y = –14)$ A (reduced) slope of 45/32 $\Rightarrow x = –6 + 32 = 26$ and substituting for x, $26 = 1 + (4t – 3)/9 \Rightarrow t = (25\cdot 9 + 3)/4 = 228/4 =$ **57**.

The calculus student would have found the slope this way: $m=\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{5}{8}\div\frac{4}{9}=\frac{45}{32}$
and proceeded as above. The derivative – a powerful tool indeed!

MASSACHUSETTS MATHEMATICS LEAGUE
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Round 5

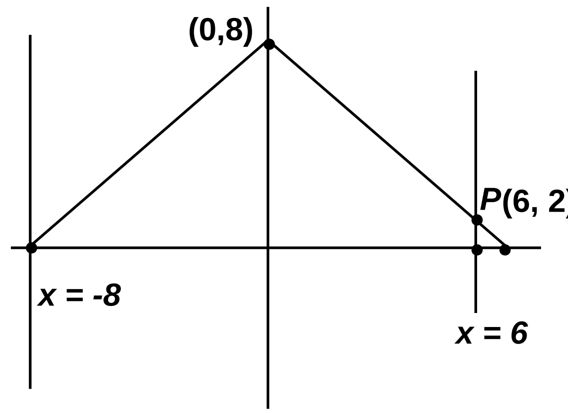
A) $x^2 + 10x \leq 24 \Leftrightarrow x^2 + 10x - 24 \leq 0 \Leftrightarrow (x + 12)(x - 2) \leq 0$. The critical values are -12, $+2$, which divide the number line into three regions. We require a negative or zero product. Testing a value in each region, only the region between -12 and 2 inclusive satisfies the required condition $\Rightarrow$ $\underline{\mathbf{-12 \leq x \leq 2}}$.

B) $-2x - 1 < 3x - 11 < 2x + 1 \Rightarrow -2x - 1 < 3x - 11$ <u>and</u> $3x - 11 < 2x + 1$
$\Rightarrow 10 < 5x$ and $x < 12 \Rightarrow$ $\underline{\mathbf{2 < x < 12}}$.

C) $y + |x| > 8 \Rightarrow y < 8 - |x| \Rightarrow \begin{cases} y = x + 8 \\ y = -x + 8 \end{cases}$

$|x + 1| < 7 \Rightarrow -7 < x + 1 < 7 \Rightarrow -8 < x < 6$

$y \geq 0$ restricts us to above the y-axis.
Examining the graphs of the related equations, we see the required region is the interior of a polygon, comprised of a triangle in quadrant 2 and a trapezoid in quadrant 1 between the y-axis and the vertical line $x = 6$.



Since $y = -x + 8$ and $x = 6$ intersect at (6, 2), we have the necessary dimensions to find the area of each region.

$$\frac{1}{2} \cdot 8 \cdot 8 + \frac{1}{2} \cdot 6 \cdot (2 + 8) = 32 + 30 = \underline{\mathbf{62}}$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011 SOLUTION KEY

Round 6

A) As a difference of perfect square,
$(394387)^2 - (394381)^2 = (394387 + 394381)(394387 - 394381) = (788768)(6) =$ **<u>4732608</u>**.

B) Avoiding solving for x and y, let's try combining the two equations and getting the required expression.
$A(x + y = 5) + B(2x - 3y = 8) \Rightarrow (A + 2B)x + (A - 3B)y = 5A + 8B$,
If $A + 2B = 7$ and $A - 3B = -8$, so we get the required expression $7x - 8y$.
Solving for A and B, $(A, B) = (1, 3)$
The required numerical value is $5(1) + 8(3) =$ **<u>29</u>**.

C) $$1-\cfrac{1}{2-\cfrac{1}{3-\cfrac{1}{4}}}=1-\cfrac{1}{2-\cfrac{1}{\frac{11}{4}}}=1-\cfrac{1}{2-\frac{4}{11}}=1-\cfrac{1}{\frac{18}{11}}=1-\frac{11}{18}=\frac{7}{18}$$
$$\left(0.2\overline{3}++0.0\overline{4}\right)=0.2\overline{7}$$
Converting the repeating decimal,
Let $N = 0.2\overline{7}$. Then: $\begin{cases}100N = 27.\overline{7}\\10N = 2.\overline{7}\end{cases} \Rightarrow 90N = 25 \Rightarrow N = \frac{5}{18}$.

Thus, $\frac{7}{18}+\frac{5}{18}=\underline{\boldsymbol{\frac{2}{3}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011 SOLUTION KEY

Team Round

A) If the dimensions of the solid are a , b and c, then $\begin{cases} (1)\ ab = 180 \\ (2)\ bc = 240 \\ (3)\ ac = 144 \end{cases}$.

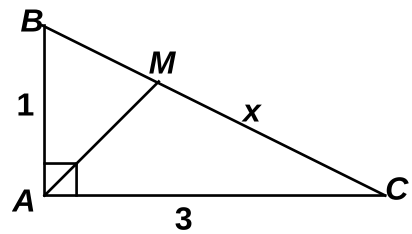
Divide (1) by (2), multiply the left hand side by c/c and substitute for ac using (3):

$\frac{a}{c} = \frac{180}{240} \Rightarrow \frac{ac}{c^2} = \frac{3}{4} \Rightarrow \frac{144}{c^2} = \frac{3}{4} \Rightarrow c^2 = 4(48) \Rightarrow c = 8\sqrt{3} \Rightarrow a = 6\sqrt{3}$ and $b = 10\sqrt{3}$

Find the edge as above and use the relationship $d^2 = L^2 + W^2 + H^2$

$\Rightarrow d^2 = \left(6\sqrt{3}\right)^2 + \left(8\sqrt{3}\right)^2 + \left(10\sqrt{3}\right)^2 = 3(36+64+100) = 600 \Rightarrow d = \underline{\mathbf{10\sqrt{6}}}$.

B) Let $MC = x$. Applying the Pythagorean Theorem to ΔABC, $AC = 3 \Rightarrow AM = 1$.



$\cos(\measuredangle ACB) = \frac{3}{\sqrt{10}}$. Now use the Law of Cosines on ΔAMC

$1^2 = 3^2 + x^2 - 2\cdot 3\cdot x\cdot \frac{3}{\sqrt{10}} \Rightarrow x^2 - \frac{18}{\sqrt{10}}x + 8 = 0 \Rightarrow$

$\sqrt{10}x^2 - 18x + 8\sqrt{10} = 0$

$\Rightarrow x = \frac{18 \pm \sqrt{18^2 - 32(10)}}{2\sqrt{10}} = \frac{18 \pm 2}{2\sqrt{10}} = \sqrt{10}$ (rejected) or $\frac{8}{\sqrt{10}} = \underline{\mathbf{\frac{4\sqrt{10}}{5}}}$.

Alternative Solution #1 (Norm Swanson)

$AC = 3$ and $\frac{AM}{AC} = \frac{1}{3} \Rightarrow AM = 1$, so ΔABM is isosceles. $m\angle MAC = 90 - (180 - 2B) = 90 - 2B$

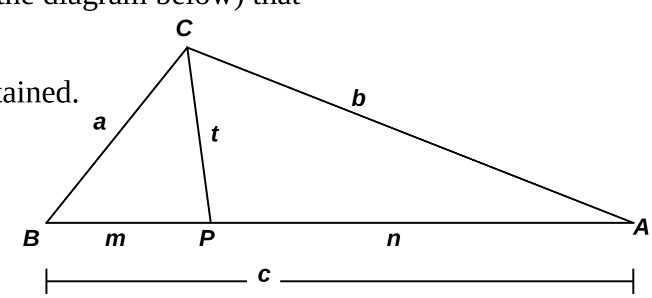
$\cos\left(2B - 90\right) = \cos\left(90 - 2B\right) = \sin 2B = 2\sin B \cos B = 2\left(\frac{3}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right) = \frac{3}{5}$

Using the Law of Cosines on ΔMAC,

$MC^2 = 1^2 + 3^2 - 2\cdot 1\cdot 3\cdot \frac{3}{5} = 10 - \frac{18}{5} = \frac{32}{5} = \frac{16\cdot 10}{25} \Rightarrow MC = \underline{\mathbf{\frac{4\sqrt{10}}{5}}}$

Alternative solution #2 applies Stewart's Theorem to ΔABC.
Stewart Theorem states that if a segment is drawn from the vertex of <u>any</u> triangle to <u>any</u> point on the opposite side (with lengths as indicated in the diagram below) that

$$\boxed{a^2n + b^2m = t^2c + cmn}.$$



It is left to you to check that the same result is obtained.
The proof requires some basic trig and some heavy algebraic lifting, but is not out of reach.
You might want to try deriving it on your own or peeking at the end of this solution key.

Hint: Use Law of Cosines on ΔBPC and ΔCPA.

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Team Round - continued

C) Let x and y denote the number of postcard and first-class stamps respectively.
$19x + 29y = 19x + 29(40 - x) = 920 \Rightarrow -10x = 920 - 1160 \Rightarrow x = 24, y = 16$
Current cost 24(28) + 16(44) = 672 + 704 = 1376¢ = <u>**$13.76**</u>

D) By long division, $N = \dfrac{10x}{x+10} = 10 - \dfrac{100}{x+10}$
Clearly, N is an integer if and only if $x + 10$ is a factor of 100.
We must consider both positive and negative factors of 100.
100 has 9 positive and 9 negative divisors. [±(1, 2, 4, 5, 10, 20, 25, 50, 100)]
Positive factors of 100 are obtained by letting $x = -9, -8, -6, -5, 0, 10, 15, 40$ and 90.
N is positive for the last four x-values. Thus, N = <u>*5*, *6*, *8* and *9*</u>, resulting in a total of <u>28</u>.

Negative factors of 100 are obtained by letting $x = -11, -12, -14, -15, -20, -30, -35, -60$ and -110.
We get 9 values for N: <u>*110*, *60*, *35*, *30*, *20*, *15*, *14*, *12* and *11*</u>. A total of <u>307</u>.
Note that except for the sign, the list of x-values read backwards is the list of N-values.
The smallest N-value in our list is 5.
Could N assume an integer value smaller than this, say 4?
$N = 4 \Rightarrow \dfrac{10x}{x+10} = 4 \Rightarrow 10x = 4x + 40$, which is not solvable for <u>integer</u> x.
Similarly, N = 3, 2 and 1 fail.
Our double check confirms that the 13 N-values we found are the only possible integer ones.
Their sum is <u>**335**</u>.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2011 SOLUTION KEY

Team Round - continued

E) $|2x + 1| < x - c$ is equivalent to $-x + c < 2x + 1 < x - c$ which in turn is equivalent to the compound condition $-x + c < 2x + 1$ and $2x + 1 < x - c$

Thus, $x > \frac{c-1}{3}$ and $x < -1 - c \Rightarrow \frac{c-1}{3} < x < -1 - c$.

For this to make any sense at all, we require that $\frac{c-1}{3} < -1 - c \Rightarrow c - 1 < -3 - 3c \Rightarrow c < -\frac{1}{2}$

$c = -1 \Rightarrow$ open interval $\left(-\frac{2}{3}, 0\right)$ - no integer solutions

$-2 \Rightarrow \left(-\frac{3}{3}, 2\right) = (-1, 1) \Rightarrow$ 1 integer solution

$-3 \Rightarrow \left(-\frac{4}{3}, 2\right) \Rightarrow$ 3 integer solutions

$-4 \Rightarrow \left(-\frac{5}{3}, 3\right) \Rightarrow$ 4 integer solutions

$-5 \Rightarrow \left(-\frac{6}{3}, 4\right) \Rightarrow$ 5 integer solutions

$-6 \Rightarrow \left(-\frac{7}{3}, 5\right) \Rightarrow$ 7 integer solutions

Clearly, the solution is unique. Build a table of c-values and n, the corresponding number of solutions for values of c immediately preceding a jump of 2 in the number of solutions.

As c decreases by 3, n increases by 4. $n = 17 \Rightarrow c = \underline{\mathbf{-14}}$.

Check: $n = -14 \Rightarrow (-5, 13) \Rightarrow -4, \ldots, -1, 0, 1, \ldots, 12$

c	***n***
-2	1
-5	5
-8	9
-11	13
-14	17

F) The "Program" searches for twin primes (primes differing by 2) for which the larger is 3 more than a multiple of 4.

Note the twin prime pair (11, 13) is **<u>not</u>** added into the total since the roles of p and q are reversed. The smaller prime is 3 more than a multiple of 4 and the larger prime is 1 more than a multiple of 4.

For $m = 24$, $p = 97$ and $q = 99$ (not prime) and the "program" makes one more pass.

$m = 25 \Rightarrow p = 101$, $q = 103$ (both are primes) and the loop is exited.

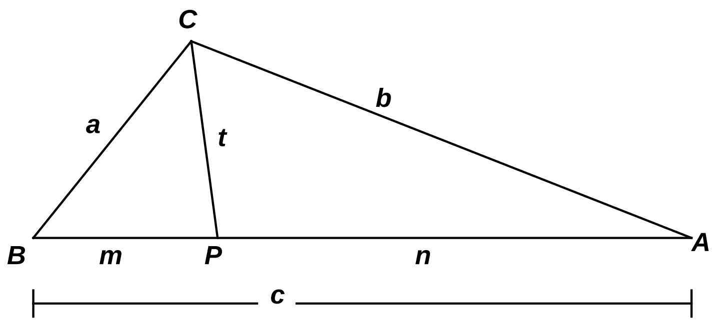
T is increased for (5, 7), (17, 19), (29, 31), (41, 43) and (101, 103)

$\Rightarrow T = 12 + 36 + 60 + 84 + 204 = \underline{\mathbf{396}}$.

Stewart's Theorem

If a segment is drawn from the vertex of <u>any</u> triangle to <u>any</u> point on the opposite side (with lengths as indicated in the diagram below) then

$$\boxed{a^2n+b^2m=t^2c+cmn}.$$



Using Law of Cosines on ΔBPC , $a^2 = t^2 + m^2 - 2tm\cos(\measuredangle BPC)$.

Using Law of Cosines on ΔCPA , $b^2 = t^2 + n^2 - 2tn\cos(\measuredangle CPA)$.

BUT $\measuredangle BPC$ and $\measuredangle CPA$ are supplementary and
$\cos(\measuredangle CPA) = \cos(180 - \measuredangle BPC) = -\cos(\measuredangle BPC)$

Therefore, the two equations become $\begin{cases} a^2 = t^2 + m^2 - 2tm\cos(\measuredangle BPC) \\ b^2 = t^2 + n^2 + 2tn\cos(\measuredangle BPC) \end{cases}$

The plan is to eliminate the last terms in each equation by multiplying the first equation by *n*, the second equation by *m*, and then adding the two equations.

$$\left(a^2n + b^2m\right) = n\left(t^2 + m^2\right) + m\left(t^2 + n^2\right)$$

$$\Rightarrow \left(a^2n + b^2m\right) = t^2\left(m+n\right) + \left(nm^2 + mn^2\right)$$

$$\Rightarrow \left(a^2n + b^2m\right) = t^2\left(m+n\right) + mn\left(m+n\right)$$

$$\Rightarrow \boxed{a^2n+b^2m=t^2c+cmn}$$

Q.E.D. – That's all folks!
Powerful medicine indeed – when the problem involves triangles and nothing else seems to apply, try Stewart's Theorem.