**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 – JANUARY 2012**
**ROUND 1 ANALYTIC GEOMETRY: ANYTHING**

**ANSWERS**

A) ______________________

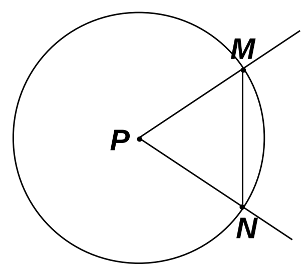
B) ( __________ , __________ )

C) ( __________ , __________ )

A) The area of an ellipse is given by the formula $\pi ab$, where $a$ and $b$ are the lengths of the semi-major and semi-minor axes, respectively.
Compute the absolute value of the difference between the areas of following conic sections:

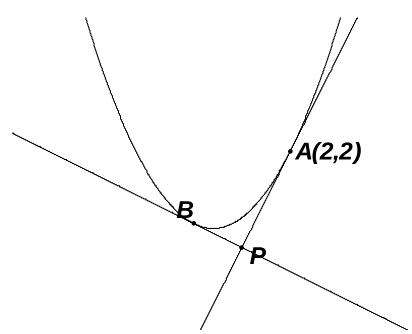
$$x^2 + y^2 = 36$$
$$4x^2 + 9y^2 = 36$$

B) Given: $C_1 = \{(x, y) \mid (x+1)^2 + (y-2)^2 = 64\}$
$m\angle P = 60°$, $M(a, b)$ and $N(a, c)$ lie on $C_1$, where $a > 0$ and $b > c$.




Determine the ordered pair $(a, b)$.

C) Lines are drawn tangent to the parabola $y = \frac{1}{2}x^2$. If $x = a$, the tangent line has equation $ax - y = \frac{a^2}{2}$. The tangent line through $A(2, 2)$ and the tangent line at $B$ are perpendicular and intersect at point $P$. Find the coordinates of point $P$.



**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012**
**ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING**

**ANSWERS**

A) ______________________________

B) ______________________________

C) ______________________________

A) The product of two positive integers *a* and *b* is 210. Determine the minimum value of *a* + *b*.

B) If $x^3y - xy^3 = 91$ and $x^3y^2 - x^2y^3 = 105$, compute the numerical value of $\frac{1}{x} + \frac{1}{y}$.

C) The quadratic trinomial $4x^2 - 17x + A$ factors as $(ax - b)(cx - d)$, where *A* is a positive integer and *a*, *b*, *c* and *d* are also positive integers. Determine all possible values of *A*.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012**
**ROUND 3 TRIG: EQUATIONS WITH A REASONABLE NUMBER OF SOLUTIONS**

**ANSWERS**

A) ______________________________

B) ______________________________

C) ______________________________

***** All answers should be expressed in <u>degrees</u> over the specified intervals. *****

A) Solve for $x$ over $-90° < x < 0°$. $\sin(3x) = -\frac{1}{2}$

B) Solve for $x$, where $0° \le x < 360°$. $3(\cot x + \csc x) = 2\sin x$

C) Solve for $x$ over $0° < x < 180°$. $\cos(2x) - (1 - \sin^2 x) < -0.5$

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012**
**ROUND 4 ALG 2: QUADRATIC EQUATIONS**

**ANSWERS**

A) ______________________________

B) ______________________________

C) ______________________________

A) $A$ is a <u>positive</u> root of $x^2 - 3Ax + B = 0$. Compute $A$, if $B = 72$.

B) Solve for $x$. $$\frac{x+2}{x-2} - \frac{x-3}{x+3} = \frac{7}{5}$$

C) Find the sum of all integral values of $m$ for which the equation $9 + 2mx = 4x - x^2$ will have <u>no</u> real roots.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012**
**ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS**

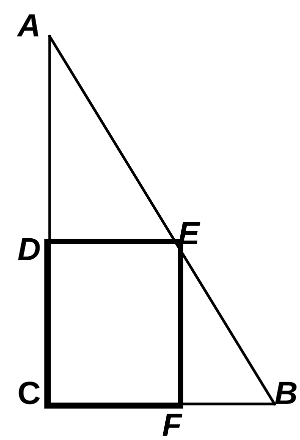
**ANSWERS**

A) ______________________________

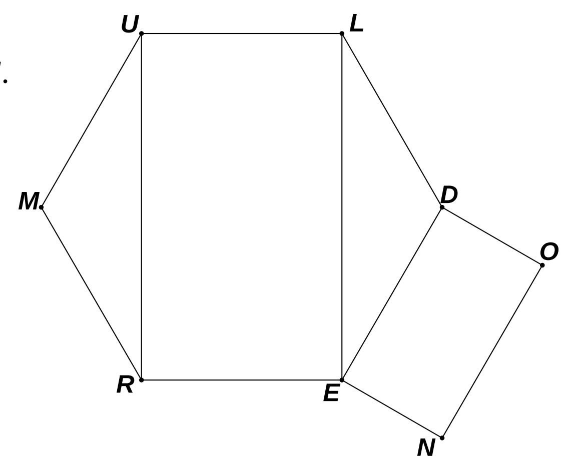
B) ______________________________

C) ______________________________

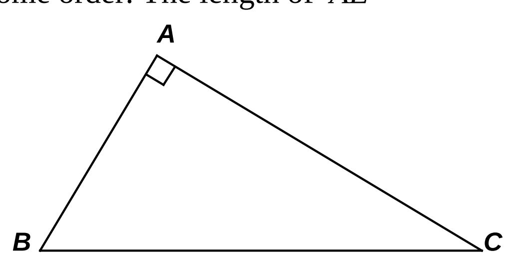
A) $DEFC$ is a rectangle with vertices on the sides of right triangle $ABC$ as indicated in the diagram at the right. If $(DE, DC, DA) = (2, 3, k)$, find a <u>simplified</u> expression for $FB$ in terms of $k$.




B) Given: Regular hexagon $MULDER$ with $MR = 2$.
$DONE$ is a rectangle, where $DO < DE$.
$DONE$ is similar to, but not congruent to, $RULE$.
Compute $DO$.




C) In $\Delta ABC$, $AB = 9$, $BC = 15$, $\overline{BA} \perp \overline{CA}$, $D$ is the <u>midpoint</u> of $\overline{AB}$ and $E$ lies on $\overline{AC}$.
The triangle with vertices $A$, $D$ and $E$ is similar to $\Delta ABC$, in some order. The length of $\overline{AE}$ can be expressed as a simplified ratio of positive integers, $\frac{a}{b}$.
Compute all possible <u>sums</u> $a + b$.



**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012**
**ROUND 6 ALG 1: ANYTHING**

**ANSWERS**

A) ________________________________

B) ( ______________ , ______________ )

C) ________________________ seconds

A) 10W30 oil protects a car engine for air temperatures $T$ which satisfy the inequality

$$|T - 55| \le 50.$$

If $T_{\text{low}}$ and $T_{\text{high}}$ denote the low and high air temperatures for safe range of operation, compute $T_{\text{low}} + T_{\text{high}}$.

B) Carol starts at 1000 and counts down by 7s. Carol's list starts out 1000, 993, 986,…
Tarah starts at 100 and counts up by 13s. Tarah's list starts out 100, 113, 126, …
Let $C$ and $T$ denote numbers in Carol's and Tarah's list respectively between 600 and 700.
Compute the ordered pair $(C, T)$, where $|C - T|$ is a minimum.

C) Two walkers, Chris and Christine, initially 12 miles apart, walk towards each other and meet in 1 hour and 20 minutes. Chris and Christine walk at constant, but different rates.
The faster walker walks twice as fast as the slower walker. Suppose they walk at these rates in the same direction, starting from the same point at the same time.
Compute the time (in <u>seconds</u>) when they are exactly ¼ mile apart.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012**
**ROUND 7 TEAM QUESTIONS**

**ANSWERS**

A) ____________________________ D) ____________________________

B) ____________________________ E) ____________________________

C) ____________________________ F) $ _______ . __ __

A) Compute the slopes of both tangent lines to $x^2 + y^2 = 25$ from $P(7, -1)$.

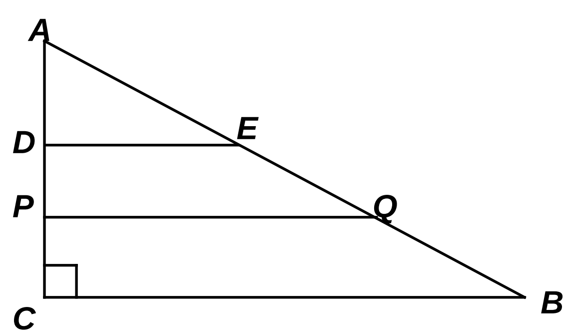
B) $\dfrac{x^4+3x^2+k}{x^2+3}$ is an integer for exactly five positive integer values of $x$.
Determine the <u>smallest</u> possible positive integer value of $k$.

C) Solve for $x$ over $0 \le x < 360°$: $\sin 4x - \cos 2x = 4\sin x\cos x - 1$

D) For positive integer constants $k$, $A$ and $B$, $x^2-(k-4)x+6k-2=(x-A)^2+B^2$ is an identity in $x$.
Find all possible ordered triples $(k, A, B)$.

E) Given: $\overline{AC} \perp \overline{BC}$, $\overline{PQ}$ is a median of trapezoid $BCDE$
$AC = 5$, $BC = 12$ and $AD = k$

Determine the <u>range</u> of values of $k$ for which the perimeter of $PQBC$ is less than or equal to 25.




F) In Maine, at one time, a customer could buy a Frisbee for the selling price of $X, plus a 5% sales tax. However, due to inflation, the vendor raised his selling price by 25% and, adding insult to injury, Maine raised the sales tax to 8%. As a result, the cost to the customer for a Frisbee, including the tax, was an additional 48¢. Compute $X$ (to the nearest cent).

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012 ANSWERS**

**Round 1 Analytic Geometry: Anything**

A) $30\pi$ B) $\left(4\sqrt{3}-1,6\right)$ C) $\left(\frac{3}{4},-\frac{1}{2}\right)$

**Round 2 Alg1: Factoring**

A) 29 B) $\frac{13}{15}$ C) 4, 13, 15 and 18 (in any order)

**Round 3 Trig: Equations**

A) $-10°$, $-50°$ B) 120°, 240° C) $45° < x < 135°$

**Round 4 Alg 2: Quadratic Equations**

A) 6 B) $-\frac{6}{7}$, 7 C) 10

**Round 5 Geometry: Similarity**

A) $\frac{6}{k}$ B) $\frac{2\sqrt{3}}{3}$ C) 7 and 35

**Round 6 Alg 1: Anything**

A) 110 B) (685, 685) C) 300 seconds

**Team Round**

A) $-\frac{4}{3}, \frac{3}{4}$ D) (30, 13, 3), (10, 3, 7), (22, 9, 7)

B) 84 E) $\frac{10}{3} \le k < 5$

C) 0°, 15°, 75°, 180°, 195°, 255° F) \$1.60

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012 SOLUTION KEY**

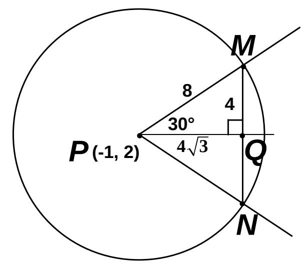
## Round 1

A) The circle has radius 6 and area $36\pi$.

The ellipse $\left(\frac{x^2}{9}+\frac{y^2}{4}=1\right)$ has $a = 3$ and $b = 2 \Rightarrow$ area $= 6\pi$

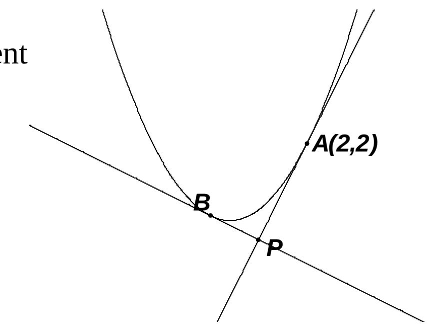
Thus, the difference in areas is $\underline{\mathbf{30\pi}}$.

B) $C_1$ has its center at $(-1, 2)$ and a radius of 8.




Since $\overline{MN}$ is vertical, a horizontal line through $P$ will be the perpendicular bisector of $\overline{MN}$ and the bisector of $\angle P$.
Thus, we have a 30 – 60 – 90 right triangles and $MQ = 4$, $PQ = 4\sqrt{3}$ and the coordinates of $M$ must be $\underline{\left(\mathbf{4\sqrt{3}-1, 6}\right)}$.

C) The tangent through (2, 2) has the equation $2x - y = 2$.




If the coordinates of $B$ are $\left(b, \frac{1}{2}b^2\right)$, then the equation of the tangent through $B$ is $bx - y = \frac{b^2}{2}$. This line has slope $b$ and is perpendicular to $2x - y = 2$, so $b = -1/2$.

The coordinates of $B$ are then $\left(-\frac{1}{2}, \frac{1}{8}\right)$.

This second line then has equation $-\frac{1}{2}x - y = \frac{\left(-\frac{1}{2}\right)^2}{2}$ or $x + 2y = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$

and both equations simplify to $4x + 8y = -1$.

Solving $\begin{cases} 2x - y = 2 \\ 4x + 8y = -1 \end{cases}$, $x = \frac{3}{4}, y = -\frac{1}{2}$ and $\underline{P\left(\mathbf{\frac{3}{4}, -\frac{1}{2}}\right)}$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012 SOLUTION KEY**

**Round 2**

A) Note: 210 = 210(1) (sum: 211), 105(2) (sum: 107), 3(70) (sum: 73) The value of the sum $a + b$ decreases as the difference between the factors decreases.
Thus, we are looking for the pair of factors of 210 with the minimum difference.
The prime factorization of 210 is 2·3·5·7. Pairing the outer and the inner factors, we have 14 and 15. Clearly, 1 is the minimum possible difference (unless 210 were a perfect square which it is not). Thus, the minimum value of $a + b$ = **<u>29</u>**.

B) $$\frac{x^3y-xy^3}{x^3y^2-x^2y^3}=\frac{xy\left(x^2-y^2\right)}{x^2y^2(x-y)}=\frac{xy(x+y)\cancel{(x-y)}}{x^2y^2\cancel{(x-y)}}=\frac{x+y}{xy}=\frac{1}{y}+\frac{1}{x}=\frac{91}{105}=\mathbf{\frac{13}{15}}$$

C) Certainly, $(x-4)(4x-1)$ gives the proper lead coefficient and coefficient of the middle term, so $A = 4$ works. But there are many possible factorizations to examine.
How do we approach the search systematically? How can we limit the search?
First, notice that the lead coefficient either factors as 4·1 or 2·2.
Since filling in the blanks in (2$x$ + ___ )(2$x$ – ___ ) with integers would always produce an even coefficient in the middle term, our search is limited to products of the form (4$x$ − ___ )($x$ − ___ ).
One of the solutions will always be a <u>positive</u> <u>*integer*</u>.
Our first factorization has established an upper limit for integer solutions, since $(x-5)$ requires a second factor of $(4x+3)$ and $d = -3$ violates the condition of positive constants.

Thus, we try products ($x$ – 1) (4$x$ – ___ ), ($x$ – 2)(4$x$ – ___ ) and ($x$ – 3)(4$x$ – ___ ).
FOILing out these products, each can produce a middle coefficient of 17, since the outer product simply makes up the difference. The fill-ins are 13, 9 and 5 respectively.
Therefore, there are 4 possibilities for the product $A$, namely $A$ = **<u>13, 18</u>** and **<u>15</u>**, plus $A$ = **<u>4</u>** that we had found originally (Answers allowed in any order).

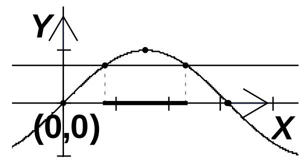
**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012 SOLUTION KEY**

## Round 3

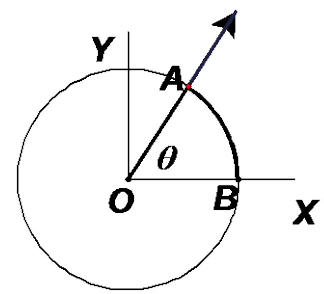
A) $3x = 330° + 360n$ or $210° + 360n \Rightarrow x = 110° + 120n$ or $70° + 120n$
$n = -1 \Rightarrow x = \underline{\mathbf{-10°, -50°}}$

B) $3(\cot x + \csc x) = 3\left(\dfrac{\cos x + 1}{\sin x}\right) = 2\sin x \quad (x \neq 0° + 180n)$
$\Rightarrow 3\cos x + 3 = 2\sin^2 x = 2 - 2\cos^2 x$
$\Rightarrow\ 2\cos^2 x + 3\cos x + 1 = (2\cos x + 1)(\cos x + 1) = 0$
$\Rightarrow\ \cos x = -\dfrac{1}{2}, -1 \Rightarrow$ **<u>120°, 240°</u>**, (180° is extraneous.)

C) $(\cos^2 x - \sin^2 x) - 1 + \sin^2 x < -0.5 \Rightarrow \cos^2 x - 1 < -0.5 \Rightarrow \sin^2 x > 0.5$
Since $\sin x$ is always positive over the specified interval,
we have $\sin x > \dfrac{\sqrt{2}}{2}$ and we know that $\sin 45° = \dfrac{\sqrt{2}}{2}$.
Appealing to the graph of $y = \sin x$, we have $\underline{\mathbf{45° < x < 135°}}$.




[In the graph above, $x$ is measured in radians, so the first point of intersection occurs between 0 and 1 $\left(45° = \dfrac{\pi}{4}^{rad} \approx 0.785\,\text{radians}\right)$, the second point of intersection occurs between 2 and 3 $\left(135° = \dfrac{3\pi}{4}^{rad} \approx 2.356\text{ radians}\right)$ and the function intersects the $x$-axis at $\left(180° = \pi^{rad} \approx 3.142\,\text{radians}\right)$.]




For those unfamiliar with radian measure, you may want to discuss the following with your coach or a teammate.
$m\angle AOB = \theta = 1$ radian if and only if the radius of the circle ($\overline{OA}$) has the same length as the intercepted (minor) arc ($\overset{\frown}{AB}$). Imagine a series of concentric circles, with center at $O$. In all cases, the intercepted arc has the same length as the radius and the central angle measures 1 radian! So to keep things simple, consider the unit circle (radius 1).
The circumference of the unit circle is $2\pi$.
Thus, $2\pi$ copies of central angle $AOB$ complete 1 revolution or 360°.
Dividing by 2, we have the equivalence $\boxed{180° = \pi^{\text{ rad}}}$ or $\boxed{1^{\text{rad}} = \dfrac{180}{\pi}°}$ $(\approx 57.296° \approx 57°17'44.8'')$.
Radian measure of angles allows us to use the same scale <u>on both axes</u> in the graph of $y = \sin x$ above!

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012 SOLUTION KEY**

## Round 4

A) Given: $x^2 - 3Ax + B = 0$ and $A$ is a <u>positive</u> root.
Since the coefficient of $x$ is $-3A$, the roots must be $A$ and $2A$, to make a sum of $3A$.
Since $B$ must be the product of the roots and $A$ is positive, we have $2A^2 = 72 \Rightarrow A =$ **<u>6</u>**.

Alternate Solution: (Brute Force)
Over integers, 72 factors as either 8·9, 6·12, 3·24, 2·36, or 1·72
Considering the possible factorizations and the coefficients of the middle term, we have
$3A$ = 17, 18, 27, 38 and 73. Only $3A$ = 18 produces a value of $A$ which is also a root.

B) $5[(x+2)(x+3)-(x-2)(x-3)] = 7(x-2)(x+3)$

$5\left[\left(x^2+5x+6\right)-\left(x^2-5x+6\right)\right] = 7\left(x^2+x-6\right)$

$5(10x) = 7x^2 + 7x - 42$

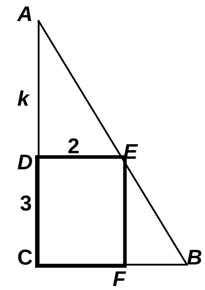
$7x^2 - 43x - 42 = (7x+6)(x-7) = 0 \Rightarrow x = \underline{\boldsymbol{-\frac{6}{7}, 7}}$

C) $9 + 2mx = 4x - x^2 \Leftrightarrow x^2 + (2m-4)x + 9 = 0$
Using the quadratic formula, no real roots $\Leftrightarrow$ a negative discriminant
Therefore,
$(2m-4)^2 - 36 < 0 \Leftrightarrow \left[2(m-2)\right]^2 - 36 < 0 \Leftrightarrow (m-2)^2 - 9 < 0 \Leftrightarrow m^2 - 4m - 5 < 0 \Leftrightarrow (m-5)(m+1) < 0$

The critical values are +5 and $-1$ and the product is negative in between.
Thus, the integer values are 0, 1, 2, 3 and 4, resulting in a sum of **<u>10</u>**.

Note: $m = -1 \Rightarrow x^2 - 6x + 9 = (x-3)^2 = 0$ which has one real root of +3.
$m = 5 \Rightarrow x^2 + 6x + 9 = (x+3)^2 = 0$ which has one real root of $-3$.

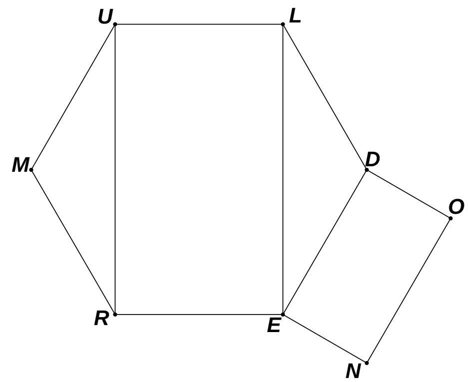
**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012 SOLUTION KEY**



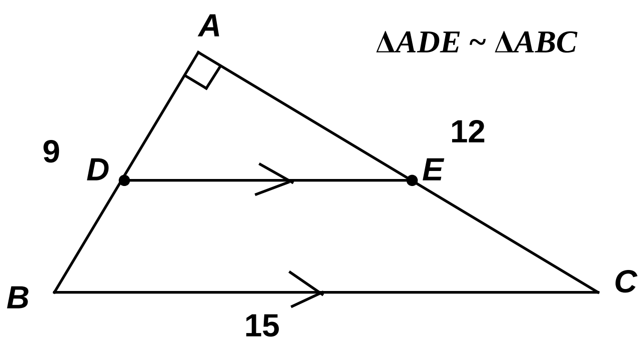

## Round 5

A) $\Delta ADE \sim \Delta EFB \Rightarrow \frac{AD}{DE} = \frac{EF}{FB} \Rightarrow \frac{k}{2} = \frac{3}{FB} \Rightarrow FB = \underline{\mathbf{\frac{6}{k}}}$

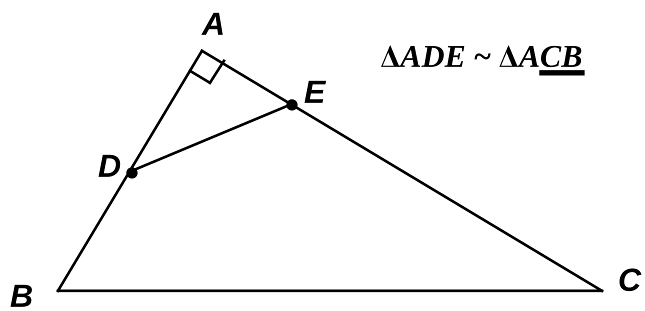
B) Drop a perpendicular from $M$ to $\overline{UR}$, intersecting $\overline{UR}$ in point $Q$, forming two 30 – 60 – 90 right triangles.
$MR = 2 \Rightarrow QU = \sqrt{3}$, $UR = 2\sqrt{3}$
Rectangles whose lengths and widths are in the same ratio are similar.




Thus, $\frac{\text{short}}{\text{long}} = \frac{UL}{UR} = \frac{DO}{DE} \Rightarrow \frac{2}{2\sqrt{3}} = \frac{DO}{2} \Rightarrow DO = \frac{2}{\sqrt{3}} = \underline{\mathbf{\frac{2\sqrt{3}}{3}}}$




**$\Delta ADE \sim \Delta ABC$**




**$\Delta ADE \sim \Delta A\underline{CB}$**

C) Right angle at $A$ and $(AB, BC) = (9, 15) \Rightarrow AC = 12$.
If $\overline{DE} \parallel \overline{BC}$ ($\Delta ADE \sim \Delta ABC$), then $E$ is also a midpoint and $AE = 6 = \frac{6}{1} \Rightarrow a + b = \underline{\mathbf{7}}$.

If $\Delta ADE \sim \Delta ACB$ (<u>Note</u>: $D$ now corresponds to $C$, and $E$ to $B$), then $\frac{AD}{AC} = \frac{AE}{AB} \Rightarrow$

$\frac{4.5}{12} = \frac{AE}{9} \Rightarrow \frac{9}{24} = \frac{3}{8} = \frac{AE}{9} \Rightarrow AE = \frac{27}{8}$
$\Rightarrow a + b = \underline{\mathbf{35}}$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012 SOLUTION KEY**

## Round 6

A) $|T-55| \le 50 \Rightarrow -50 \le T - 55 \le +50 \Rightarrow 5 \le T \le 105 \Rightarrow (T_{low}, T_{high}) = (5, 105) \Rightarrow$ **<u>110</u>**

B) The numbers in Carol's list are generated by $1000 - 7k$.
The numbers in Tarah's list are generated by $100 + 13k$.
Let's see if the lists share any common numbers. For the same value of $k$, does
$1000 - 7k = 100 + 13k$ ?
Since $900 = 20k$ , for $k = 45$, the lists do share a common number.
Multiplying by 7 is easier than by 13, so we substitute in $100 - 7k$.
The common number is $1000 - 7(45) = 1000 - 315 = 685$.
Thus $(C, T) =$ **<u>(685, 685)</u>**.

<u>Note</u>: Since the least common multiple of 7 and 13 is 91, other numbers common to both lists will be of the form $685 + 91n$, for any integer $n$. For example, $n = \pm 1$ results in 594 and 776.
$594 = 100 + 13(38) = 1000 - 7(58)$ and $776 = 100 + 13(52) = 1000 - 7(32)$
The $39^{th}$ number in Carol's list and the $59^{th}$ number in Tarah's list is 594
The $53^{rd}$ number in Carol's list and the $33^{rd}$ number in Tarah's list is 776
Between 600 and 700, only 685 is common to both lists and $|C - T| = 0$, the minimum value for the absolute value of a difference.

C) Suppose the walkers rates are $x$ and $2x$.
You may argue later whether the faster walker is Chris or Christine.
When travelling in the same direction, they are separating at $(2x - x) = x$ mph.
$RT = D \Rightarrow x$ (in mph) $\cdot$ T (in hrs) = $\frac{1}{4}$ mi. or T (in hrs) = $\frac{1}{4x}$
$x \cdot \frac{4}{3} + 2x \cdot \frac{4}{3} = 12 \Leftrightarrow 12x = 36 \Leftrightarrow x = 3$. Thus, $T = \frac{1}{12}$ hr. = 5 minutes = **<u>300</u>** seconds.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 4 - JANUARY 2012 SOLUTION KEY**

## Team Round

A) Let $m$ denote the slope. Then: $mx - y = 7m - (-1) = 7m + 1$ or $mx + (-1)y - (7m + 1) = 0$

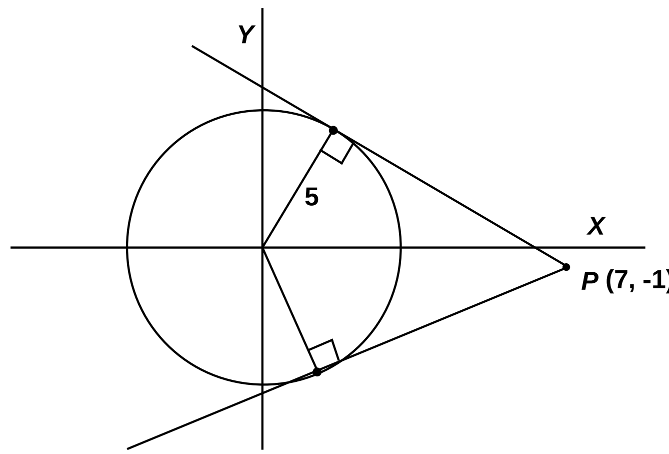
Applying the point to distance formula $\dfrac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$ (from $P(h, k)$ to $Ax + By + C = 0$), we have $\dfrac{|m(0) + (-1)0 - (7m+1)|}{\sqrt{m^2+1}} = 5$

$(7m+1)^2 = 25(m^2+1)$

$49m^2 + 14m + 1 = 25m^2 + 25$

$24m^2 + 14m - 24 = 2(3m+4)(4m-3) = 0$

$m = \underline{\mathbf{-\frac{4}{3}, \frac{3}{4}}}$




B) Since $\dfrac{x^4 + 3x^2 + k}{x^2 + 3} = x^2 + \dfrac{k}{x^2 + 3}$, we see that $x^2 + 3$ must be a factor of $k$, if the original quotient is to be an integer.

Plan: List the values of $x^2 + 3$ for consecutive positive integer values of $x$ until we find the smallest value in the list that is divisible by exactly <u>four</u> values preceding it.
$x = 1, 2, 3, \ldots. \Rightarrow$ divisors: *<u>4</u>*, *<u>7</u>*, *<u>12</u>*, 19, *<u>28</u>*, 39, 52, 67, **<u>84</u>**, …
Since $k = 84$ is divisible by each of the underlined values, $k = 84$ is the smallest value for which the given quotient has integral values for exactly five positive integer values of $x$, namely, 1, 2, 3, 5 and 9.

**Team Round – continued**

C) $\sin 4x - \cos 2x = 4\sin x\cos x - 1 \Leftrightarrow 2\sin 2x\cos 2x - \cos 2x = 2\sin 2x - 1$

$\Leftrightarrow \cos 2x\left(2\sin 2x - 1\right) = 2\sin 2x - 1$

$\Leftrightarrow \left(\cos 2x - 1\right)\left(2\sin 2x - 1\right) = 0$

$\Leftrightarrow \cos 2x = 1$ or $\sin 2x = 1/2$

$\Leftrightarrow 2x = 0° + n\cdot 360°$ or $2x = \begin{cases} 30° + n\cdot 360° \\ 150° + n\cdot 360° \end{cases}$

Dividing by 2 and letting $n = 0, 1 \Rightarrow x =$ **<u>0°, 180°, 15°, 195°, 75°, 255°</u>**.
(Answers in any order.)

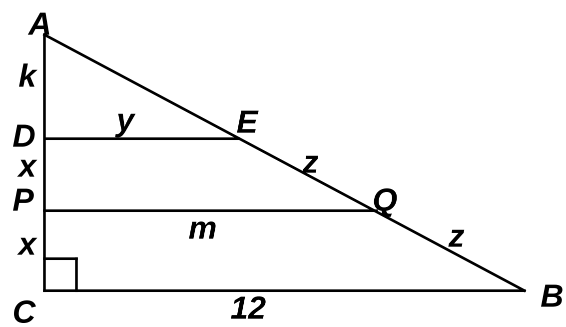
D) $\begin{cases} (1)\ \ k - 4 = 2A \\ (2)\ \ 6k - 2 = A^2 + B^2 \end{cases}$ Substituting $2A + 4$ for $k$ in (2), we have $12A + 22 = A^2 + B^2$.

Completing the square, $\left(A^2 - 12A \underline{+36}\right) = \left(A - 6\right)^2 = 22 - B^2 \underline{+36} = 58 - B^2$

For positive integer values of $B$ ($1 \le B \le 7$), $58 - B^2$ must evaluate to a perfect square.
When $B = 3$, we have $(A - 6)^2 = 49 \Rightarrow A = 6 \pm 7 \Rightarrow A = 13$ (–1 is rejected).
When $B = 7$, we have $(A - 6)^2 = 9 \Rightarrow A = 6 \pm 3 \Rightarrow A = 3, 9$
For all other values of $B$, $58 - B^2$ is not a perfect square.
Thus, there are three solutions **<u>(30, 13, 3)</u>**, **<u>(10, 3, 7)</u>** and **<u>(22, 9, 7)</u>**.
(The ordered triples may be listed in any order.)

E) Since $\overline{PQ}$ is a median of trapezoid $BCDE$, $x > 0$ and, consequently $k < 5$.

Since $AC = 5$, $x = \dfrac{5-k}{2}$




$\Delta ADE \sim \Delta ACB \Rightarrow \dfrac{k}{5} = \dfrac{y}{12} = \dfrac{AE}{13} \Rightarrow \begin{cases} y = \dfrac{12k}{5} \\ AE = \dfrac{13k}{5} \end{cases}$

Therefore, $m = \dfrac{y+12}{2} = \dfrac{\frac{12k}{5}+12}{2} = \dfrac{12(k+5)}{10}$ and

$z = \dfrac{13 - \frac{13k}{5}}{2} = \dfrac{13(5-k)}{10}$. The perimeter of $PQBC = x + m + z + 12 \le 25 \Rightarrow$

$\dfrac{5-k}{2} + \dfrac{12(k+5)}{10} + \dfrac{13(5-k)}{10} \le 13 \Rightarrow 25 - 5k + 12k + 60 + 65 - 13k \le 130 \Rightarrow 20 \le 6k \Rightarrow k \ge \dfrac{10}{3}$

Combining the restrictions, $\underline{\boldsymbol{\dfrac{10}{3} \le k < 5}}$.

**Team Round – continued**

F) Suppose the price on a Frisbee is \$X.
Adding a 5% sales tax, the original cost to the customer was $X + 0.05X = 1.05X$ (dollars).
The current cost is $1.08 \cdot 1.25X$ (dollars) – an 8% sales tax added to the increased selling price.
The difference is \$0.48.

$1.08(1.25X) - 1.05X = 0.48$

$\Leftrightarrow X\left(108(1.25) - 105\right) = 48$

$\Leftrightarrow X\left(108\left(\frac{5}{4}\right) - 105\right) = 48$

$\Leftrightarrow X(135 - 105) = 48$

Thus, $X = \frac{48}{30} = \frac{8}{5} = 1.6$ and the original selling price of the Frisbee was exactly **\$1.60**.