MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2012
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

ANSWERS

A) ______________________________

B) ______________ : ______________

C) ______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Let $f(x)=\begin{cases} 4x+5 & \text{for } x>2 \\ 10 & \text{for } -2<x\le 2 \\ 3x-4 & \text{for } -10\le x\le -2 \end{cases}$ Compute: $f\left(f(2)\right)+f\left(f(-2)\right)+f(0)$

B) $f(x)$ is a function of minimum degree with integer coefficients and zeros of -1, $\sqrt{6}$ and i. If the lead coefficient is positive and the greatest common factor of the coefficients is 1, compute the ratio $2f(-2):3f(3)$.

C) Given $f(x) = \dfrac{x+1}{x-1}$ Express $f(2x)$ in terms of $f(x)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2012
ROUND 2 ARITHMETIC / NUMBER THEORY

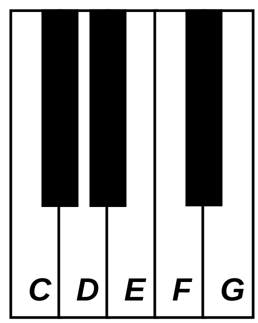
ANSWERS

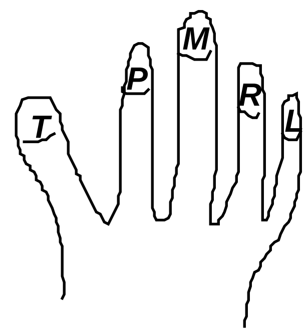
A) (____________ , ____________)

B) ______________________________

C) (____________ , ____________)

******* NO CALCULATORS ON THIS ROUND *******





A) Engelbert practiced proper finger position on the piano with his right hand. Starting with his thumb he plays middle C, followed immediately by DEFGFED, using his other fingers as follows: pointer, middle, ring, “pinky” (i.e. little), ring, middle and pointer.
Starting again with the thumb he continues to repeat the exercise (ad nauseum).
Let *F* denote the finger used –
T (thumb), P (pointer), M (middle), R (ring), L (“pinky”, i.e. little).
Let *N* denote the note being played.
Determine the ordered pair (*F*, *N*) when he plays his 2012^{th} note and gets to stop for lunch.

B) Determine all primes between 300 and 500 ending in 7 whose digit sum is a multiple of 11?

C) A spinner has 13 equally spaced positions, numbered 1 through 13 clockwise.
Pointer *A* is initially pointing at 3 and moves clockwise 7 positions every second.
Pointer *B* is initially pointing at 3 and moves counterclockwise 5 positions every second.
Pointer *C* is initially pointing at 3 and moves clockwise 2 positions every second.
Let (a, b, c) denote the numbers being referenced by the pointers *A*, *B* and *C* at one second intervals and $S(n) = a + b + c$, after *n* seconds have elapsed.
For example, at $n = 1$, $(a, b, c) = (10, 11, 5)$ and $S(1) = 26$.
Compute (n, m), where m = the maximum value of $S(n)$ and n is as small as possible.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2012
ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

ANSWERS

A) ________________________________

B) ________________________________

C) (________ , ________ , ________)

******* NO CALCULATORS ON THIS ROUND *******

A) Given: $Arc\tan(0.75) = Arc\sin(N)$
Compute N.

B) Compute: $\cos\left(Cos^{-1}\left(-\frac{3}{5}\right) - Tan^{-1}\left(-\frac{40}{9}\right)\right)$

C) $\left(2\sin 7\frac{1}{2}^\circ \cos 7\frac{1}{2}^\circ\right)\left(1-2\sin^2 37\frac{1}{2}^\circ\right) = \frac{A-\sqrt{B}}{C}$ Compute the ordered triple (A, B, C).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2012
ROUND 4 ALG 1: WORD PROBLEMS

ANSWERS

A) _________________________ mph

B) ______________________________

C) ______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) A train traveled 300 miles in t hours. When the speed of the train was increased by 5 mph, it covered 20 more miles in the same amount of time. Find the faster rate of the train.

B) There are three types of coins in the fountain, pennies, dimes and quarters worth \$6.66. If there are 156 pennies, what is the maximum number of coins in the fountain?

C) A men can complete a job in B days. If C men are taken from the original crew, how many more days will be needed to complete the original job? Give your answer as a simplified expression in terms of A, B and C.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2012
ROUND 5 PLANE GEOMETRY: CIRCLES

ANSWERS

A) ______________________________

B) ______________________________

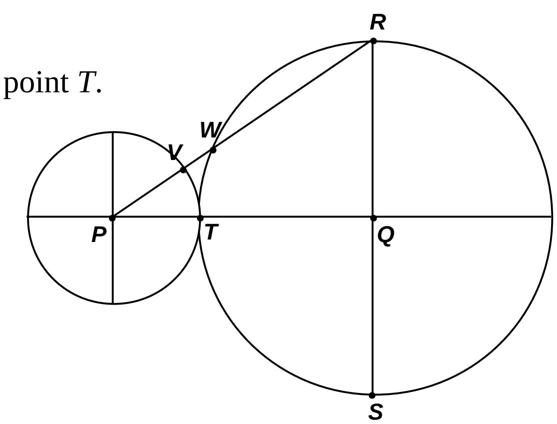
C) ______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Four congruent circles are inscribed in a square. If the circumference of each is 12π, compute the area of the circle circumscribed about this square?

B) Point P is in the exterior of circle O.
Q is the point on circle O closest to P.
R is the point on circle O farthest from P.
If $PQ = 4$ and $PR = 20$, compute the length of a tangent to circle O from point P.

C) Circle P with radius 1 is tangent to circle Q with radius 3 at point T.
Diameter $\overline{RS}$ is perpendicular to the line of centers at point Q. $\overleftrightarrow{PR}$ intersects the circles at points V and W.
Compute VW.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2012
ROUND 6 ALG 2: SEQUENCES AND SERIES

ANSWERS

A) ______________________________

B) ______________ : ______________

C) ______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) $4x+1,\ 7x,\ 8x+3$ form an arithmetic progression.
Find the sum of the first 30 terms of this progression.

B) The following three terms $(x + 2)$, $(4x + 2)$ and $(12x + 6)$ are the second, third and fourth terms of a geometric sequence. When 12 is added to the middle term, this sequence of three terms becomes the first three terms of an arithmetic sequence. Compute the ratio of the fifth term of the arithmetic sequence to the sixth term of the geometric sequence

C) Given: $t_n = -2, \frac{4}{3}, -\frac{8}{9}, \frac{16}{27}, \ldots$, $A = \sum_{n=1}^{\infty}(t_n)$ and $B_n = \sum_{k=1}^{n}(1-i)^k$, where $i = \sqrt{-1}$.

Compute $\frac{A}{B_3}$ as a reduced quotient.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2012
ROUND 7 TEAM QUESTIONS
ANSWERS

A) ____________________________ D) ____________________________

B) ____________________________ E) ____________________________ °

C) (________ , ________ , ________) F) ____________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Given: $f(x) = 2x^4 + x^3$, $f(h(x)) = 32x^4 - 56x^3 + 36x^2 - 10x + 1$
If $h(x) = Ax + B$, where A and B are integer constants, compute $h^{-1}(3)$.

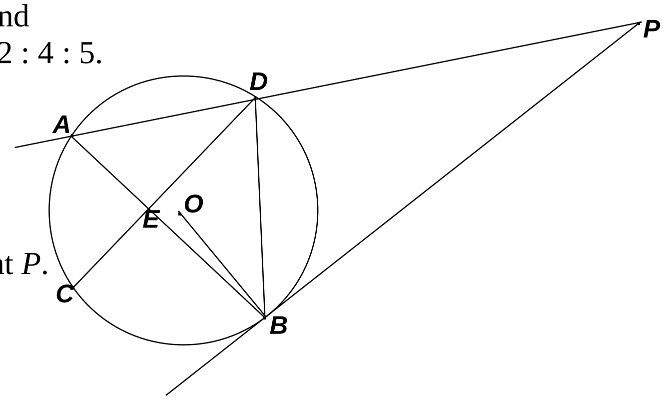
B) Let M(b) be the base 10 representation of the minimum natural number in base b that has a digit sum greater than 10. For example, $M(10) = 29$ and $M(3) = 122222_{(3)} = 845_{(10)}$.
Compute $\sum_{b=4}^{b=9} M(b)$. Recall: $\sum$ is the summation symbol. (Ex: $\sum_{x=1}^{x=4} x^2 = 1+4+9+16 = 30$)

C) Compute the ordered triple (A, B, C) for which the following equation is an identity, for all values of x for which both sides of the equation are defined.

$$\frac{2\tan x\left(1-\tan^2 x\right)}{\left(1+\tan^2 x\right)^2} = A + B\sin\left(Cx\right)$$

D) In a special summer session of Hogwarts School of Witchcraft and Wizardry, three courses were offered to new students: Charms (C), Potions (P) and Flying (F).
Every student chose to take at least one course and some chose to take multiple courses.
Let XY denote taking both course X and course Y.
Let $X + Y$ denote taking either course X or course Y (or both).
Let $n(X)$ denote the number of students signed up for course X.

Given: $n(C) = 30$, $n(C + P + F) = 116$, $n(CPF) = 6$ and
$n(CP) : n(CF) : n(PF) = n(C) : n(P) : n(F) = 2 : 4 : 5$.
Compute the largest possible number of students who could have signed up just for flying.



E) $\overline{AB}$ and $\overline{CD}$ are chords in circle O that intersect at point E. A secant line through points A and D and a line tangent to the circle at point B intersect at point P.
If $m\angle DBA = m\angle ADC + 10°$, $m\angle P = 5°$ and $m\angle AED : m\angle BED = 4 : 5$, compute $m\angle EBO$.

F) The sum of an infinite geometric progression with first term a and common multiplier r, is one more than the sum of its first two terms. If $2 \le a \le 6$, compute all possible values of r.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2012 ANSWERS

Round 1 Alg 2: Algebraic Functions

A) 21 B) 1 : 18 C) $f(2x) = \dfrac{3f(x)+1}{f(x)+3}$

Round 2 Arithmetic/ Number Theory

A) (R, F) B) 317 C) (5, 29)

Round 3 Trig Identities and/or Inverse Functions

A) 0.6 (or $\frac{3}{5}$) B) $-\dfrac{187}{205}$ C) (2, 3, 4)

Round 4 Alg 1: Word Problems

A) 80 mph B) 204 C) $\dfrac{BC}{A-C}$

Round 5 Geometry: Circles

A) 288π B) $4\sqrt{5}$ C) 0.4 (or $\frac{2}{5}$)

Round 6 Alg 2: Sequences and Series

A) 2445 B) $\dfrac{17}{81}$ C) $\dfrac{3-15i}{65}$

Team Round

A) 2

B) 476

C) $\left(0, \frac{1}{2}, 4\right)$

D) 40

E) 20°

F) $-1 < r \le -\frac{1}{2}$ or $\frac{1}{3} \le r \le \frac{1}{2}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2012 SOLUTION KEY

Round 1

A) $f\left(f(2)\right)+f\left(f(-2)\right)+f(0)=f(10)+f(-10)+f(0)$ = 45 + (-34) + 10 = **<u>21</u>**.

B) $f\left(x\right)=(x+1)\left(x^2-6\right)\left(x^2+1\right)$

$$\frac{2f(-2)}{3f(3)}=\frac{2\left(-1\cdot-2\cdot5\right)}{3\left(4\cdot3\cdot10\right)}=\frac{20}{360}=\underline{\mathbf{\frac{1}{18}}}$$

C) Replace $f(x)$ with y, i.e. let $y = \dfrac{x+1}{x-1}$ and solve for x in terms of y.

$\Rightarrow xy - y = x + 1$. Solving for $x \Rightarrow y + 1 = xy - x = x(y - 1) \Rightarrow x = \dfrac{y+1}{y-1}$

Thus, $f(2x) = \dfrac{2x+1}{2x-1} = \dfrac{2\frac{y+1}{y-1}+1}{2\frac{y+1}{y-1}-1} = \dfrac{2(y+1)+(y-1)}{2(y+1)-(y-1)} = \dfrac{3y+1}{y+3} = \underline{\mathbf{\dfrac{3f(x)+1}{f(x)+3}}}$

Round 2

A) The notes are being played in repeating blocks of 8 in the sequence CDEFGFED.
$\dfrac{2011}{8}=251,\quad r=4$.
Thus, Engelbert completed 251 blocks and stops on the 4th note in the 252nd block
$\Rightarrow (F, N) =$ **<u>(R, F)</u>**.

B) The 3-digit number must be of the form $3x7$ or $4x7 \Rightarrow$ digit sum is $10 + x$ or $11 + x$.
For integers in the 300s, we must test only 317.
For integers in the 400s, we must test only 407
Since 407 = 11(37), we need only rigorously test 317.
We must test for divisibility only by primes smaller than $\sqrt{317}$. Since $19^2 = 361 > 317$, the divisors to be tested are: 2, 3, 5, 7, 11, 13 and 17.
The first three divisors are easily eliminated.
$7 \Rightarrow r = 2$ $11 \Rightarrow r = 9$ $13 \Rightarrow r = 5$ $17 \Rightarrow r = 11$
Thus, **<u>317</u>** is prime.

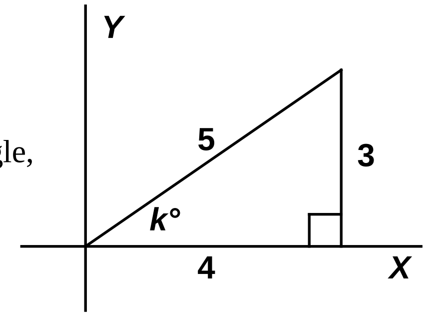
C) Each pointer cycles through the 13 numbered positions, before returned to #3.

Pointer A:	3	10	4	11	5	12	6	13	7	1	8	2	9	| 3
Pointer B:	3	11	6	1	9	4	12	7	2	10	5	13	8	| 3
Pointer C:	3	5	7	9	11	13	2	4	6	8	10	12	1	| 3
Sums:		26	17	21	25	$\boxed{29}$	20	24	15	19	23	27	18	

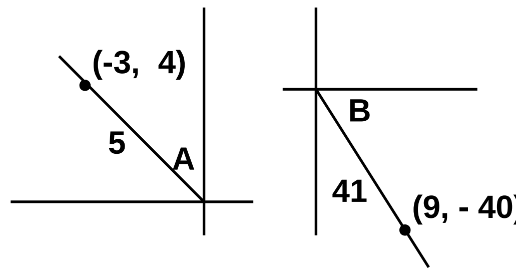
Thus, the maximum sum is 29 and it occurs first for $n = 5 \Rightarrow$ **<u>(5, 29)</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2012 SOLUTION KEY

Round 3



A) Think 3 – 4 – 5 right triangle!
$Arc\tan(0.75)$ is the measure of the smaller acute angle,
namely $k°$, since $\tan(k°) = \frac{y}{x} = \frac{OPP.}{ADJ.} = 0.75 = \frac{3}{4}$.
$\sin(k°) = \underline{\mathbf{\frac{3}{5}}}$ (or $\underline{\mathbf{0.6}}$) Recall: SOH-CAH-TOA



B) $\cos(A - B) = \cos A\cos B + \sin A\sin B =$
$\left(-\frac{3}{5}\right)\left(\frac{9}{41}\right)+\left(\frac{4}{5}\right)\left(-\frac{40}{41}\right)=\frac{-27-160}{205} = \underline{\mathbf{-\frac{187}{205}}}$

C) $\left(2\sin 7\frac{1}{2}°\cos 7\frac{1}{2}°\right)\left(1-2\sin^2 37\frac{1}{2}°\right) = \sin 15°\cdot\cos 75° =$

$\sin^2 15° = \left(\sin(45-30)\right)^2 = \left(\sin 45\cos 30 - \sin 30\cos 45\right)^2 = \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2$

$= \frac{8-2\sqrt{12}}{16} = \frac{8-4\sqrt{3}}{16} = \frac{2-\sqrt{3}}{4} \Rightarrow (A, B, C) = \underline{\mathbf{(2, 3, 4)}}$.

Round 4

A) $\frac{300}{t}+5=\frac{320}{t} \Rightarrow 300 + 5t = 320 \Rightarrow t = 4$.

Thus, the faster rate is $\frac{320}{4} = \underline{\mathbf{80}}$ mph

B) Let D and Q denote the number of dimes and quarters respectively. Then:
$10D + 25Q = 666 – 156 = 510 \Rightarrow 2D + 5Q = 102$
$(D, Q) = (1, 20), (6, 18), \ldots, (46, 2), (51, 0)$
Since there are 3 types of coins in the fountain (51, 0) is rejected and the maximum number of coins in the fountain is 156 + 46 + 2 = **<u>204</u>**.

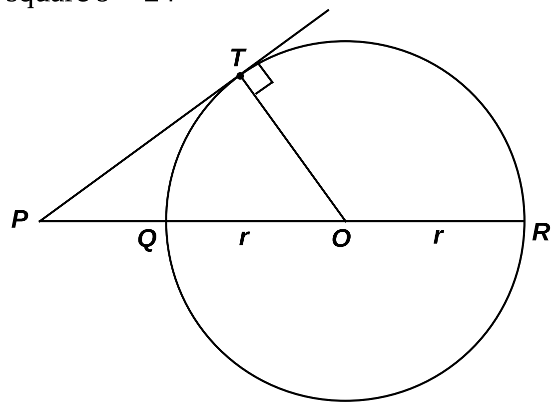
C) Let X denote the number of days it would take for $(A – C)$ men to complete the job. The time it takes to complete a job is inversely proportional to the size of the works force. The larger the workforce, the less time it takes to complete the job. Thus, $\frac{A}{A-C}=\frac{X}{B} \Rightarrow X = \frac{AB}{A-C}$

Additional days $= x – B = \frac{AB}{A-C} - B = \frac{AB-B(A-C)}{A-C} = \underline{\mathbf{\frac{BC}{A-C}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2012 SOLUTION KEY

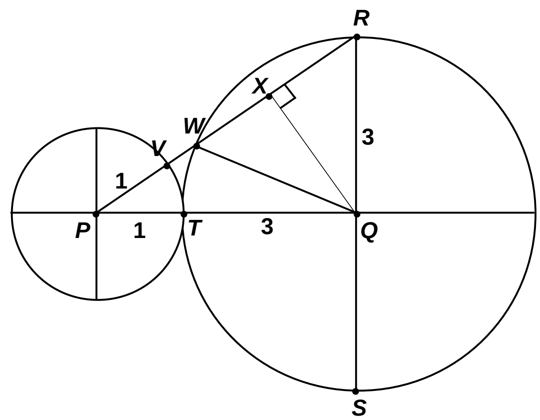
Round 5

A) $C = 12\pi \Rightarrow$ radius $r = 6 \Rightarrow$ diameter $d = 12 \Rightarrow$ side of square $s = 24$
$\Rightarrow$ diagonal of the square = $24\sqrt{2}$
$\Rightarrow$ radius of c.c. circle = $12\sqrt{2}$ $\Rightarrow$ area = $\underline{\mathbf{288\pi}}$



B) Draw $\overleftrightarrow{PO}$. This line intersects the circle twice.
The point between P and O is the closet point, Q.
The other point of intersection is point R.
$PR = PQ + QR \Rightarrow 20 = 4 + QR$
Since QR is a diameter, the radius of circle O is 8.
Using the Pythagorean Theorem on ΔTPO,
$PT^2 + 8^2 = 12^2 \Rightarrow PT = \sqrt{80} = \underline{\mathbf{4\sqrt{5}}}$

C) ΔRQP is a 3 – 4 – 5 right triangle (with area 6).
Thus, $\frac{1}{2}(QX)(PR) = 6 \Rightarrow \frac{1}{2} \cdot QX \cdot 5 = 6 \Rightarrow QX = \frac{12}{5}$.
Note since $\Delta RXQ \sim \Delta RQP$, ΔRXQ is a scaled version of a 3 – 4 – 5 triangle.
$\left(\frac{12}{5}, \_\_, 3\right) = \frac{1}{5}(12, \_\_, 15) = \frac{3}{5}(4, \_\_, 5)$
Rather than grinding out the Pythagorean Theorem for ΔRXQ, we see that $XR = \frac{3}{5}(3) = \frac{9}{5} = 1.8$
Therefore, $WR = 3.6$ and $VW = 5 - 1 - 3.6 = \underline{\mathbf{0.4}}$



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2012 SOLUTION KEY

Round 6

A) $8x+3-7x=7x-(4x+1) \Rightarrow 2x = 4 \Rightarrow x = 2$
AP: 9, 14, 19, …
$a = 9$, $d = 5$ and $n = 30$ $S_{30} = \dfrac{30}{2}\left(2(9)+(30-1)5\right)=15(18+29\cdot 5) = 15(163) = \underline{\mathbf{2445}}$

B) The first three terms of the AP are $(x + 2)$, $(4x + 14)$ and $(12x + 6)$.
The common difference $d = (4x + 14) - (x + 2) = (12x + 6) - (4x + 14) \Rightarrow 3x + 12 = 8x - 8$
$\Rightarrow x = 4$ and $d = 24$. The AP is 6, 30, 54, … and the GP is 6, 18, 54, … .
The required ratio is $(54 + 48) : (54\cdot 9) = 102 : 486 =$ **<u>17 : 81</u>**.

C) t_n generates a geometric sequence, where $r = -\dfrac{2}{3}$ and $a = -2$.

Since $|r| < 1$, the infinite geometric series converges to $\dfrac{a}{1-r}=\dfrac{-2}{1+\frac{2}{3}}=-\dfrac{6}{5}$.

$B_3=(1-i)^1+(1-i)^2+(1-i)^3=1-i+(-1-2i+1)+(-2i-2)=-1-5i$.

$\dfrac{A}{B_3}=-\dfrac{6}{5}\cdot\dfrac{1}{-1-5i}\cdot\dfrac{-1+5i}{-1+5i}=\dfrac{6-30i}{5(26)}=\underline{\dfrac{\mathbf{3-15}\boldsymbol{i}}{\mathbf{65}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2012 SOLUTION KEY

Team Round

A) $f(x)=2x^4+x^3=x^3(2x+1)$

Thus, $f(h(x))=\left(h(x)\right)^3\left(2h(x)+1\right)=\left(Ax+B\right)^3(2Ax+(2B+1))$.

Expanding $\left(Ax+B\right)^3(2Ax+(2B+1))$, the lead coefficient would be $2A^4$ and the constant term would be $B^3(2B+1)$. Therefore, $2A^4=32\Rightarrow A=\pm 2$ and

$B^3(2B+1)=1\Leftrightarrow 2B^4+B^3-1=0\Rightarrow B=-1$

Consequently, $h(x)=2x-1$ or $-2x+1$.

However, checking the other coefficients of $f(h(x))$, only $A=2$ produces the correct coefficients for x^3, x^2 and x and $h(x)=2x-1$ only.

Thus, $h^{-1}(x)=\dfrac{x+1}{2}$ and $h^{-1}(3)=\dfrac{4}{2}=\underline{\mathbf{2}}$.

B) Base 4: $2333_{(4)}=2(4^3)+(4^3-1)=3(64)-1=191$
Base 5: $344_{(5)}=3(5^2)+(5^2-1)=99$
Base 6: $155_{(6)}=36+30+5=71$
Base 7: $56_{(7)}=35+6=41$
Base 8: $47_{(8)}=32+7=39$
Base 9: $38_{(9)}=27+8=35$
Total : **<u>476</u>**

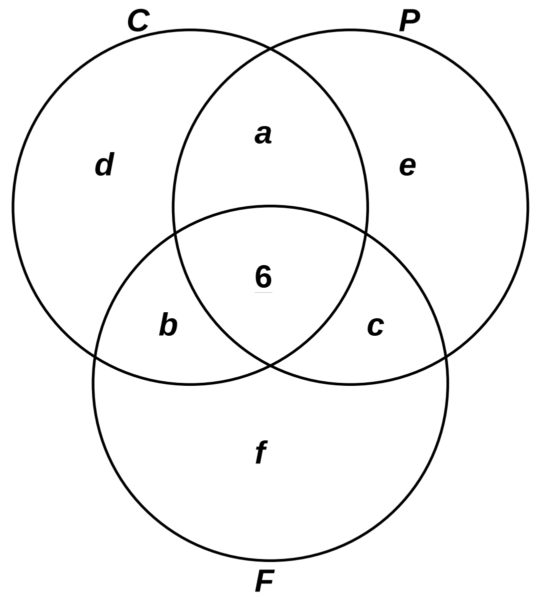
C) $$\frac{2\tan x\left(1-\tan^2 x\right)}{\left(1+\tan^2 x\right)^2}=\frac{2\dfrac{\sin x}{\cos x}\left(\dfrac{\cos^2 x-\sin^2 x}{\cos^2 x}\right)}{\sec^4 x}=2\sin x\left(\frac{\cos^2 x-\sin^2 x}{\cos^3 x}\right)\cos^4 x$$

$$=2\sin x\cos x\left(\cos^2 x-\sin^2 x\right)=\sin 2x\cos 2x=\frac{1}{2}\left(2\sin 2x\cos 2x\right)=\frac{1}{2}\sin 4x$$

Thus, (A, B, C) = $\underline{\left(\mathbf{0},\dfrac{\mathbf{1}}{\mathbf{2}},\mathbf{4}\right)}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2012 SOLUTION KEY

Team Round - continued



D) (A classic Venn Diagram problem)

$$\frac{a+6}{b+6}=\frac{2}{4}=\frac{1}{2} \Leftrightarrow 2a+12=b+6 \Leftrightarrow b=2a+6$$

$$\frac{a+6}{c+6}=\frac{2}{5} \Leftrightarrow 5a+30=2c+12 \Leftrightarrow c=\frac{5a+18}{2}$$

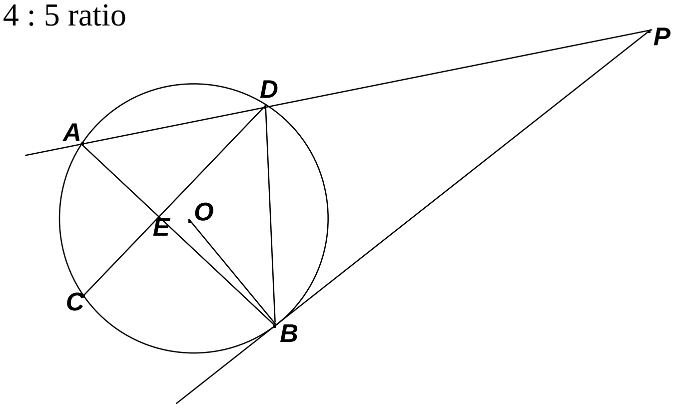
To insure that c is an integer, a must be even.
Since the total number of students involved is 116, to maximize f, we must minimize a.
Let $n(C) = 2N$, $n(P) = 4N$ and $n(F) = 5N$.
If $a = 2$, $(b, c) = (10, 14)$

$$\begin{cases} d+18=2N \\ e+22=4N \\ 30+f=5N \end{cases} \text{ and}$$

$$(2+10+14)+d+e+f+6=116 \Leftrightarrow d+e+f=84$$

Consequently,

$$\Rightarrow (2N-18)+(4N-22)+(5N-30)=84 \Leftrightarrow 11N=154 \Leftrightarrow N=14 \Rightarrow f=70-30=\underline{\mathbf{40}}.$$



E) Since angles AED and BED are supplementary, a 4 : 5 ratio implies m$\angle AED$ = 80° and m$\angle BED$ = 100°.
Let m$\angle ADC$ = x° and m$\angle DBA$ = $(x + 10)$° and $m\left(\overset{\frown}{BD}\right)=y°$.

As arcs subtended by inscribed angles, $m\left(\overset{\frown}{AC}\right)=2x°$ and $m\left(\overset{\frown}{AD}\right)=(2x+20)°$.

As a leftover arc, $m\left(\overset{\frown}{BC}\right)=(340-4x-y)°$

As an angle formed by intersecting chords,

$$m\angle BED=\frac{1}{2}(2x+y)=100 \Leftrightarrow 2x+y=200$$

As an angle formed by a tangent and a secant line $m\angle P=\frac{1}{2}\left((340-2x-y)-y\right)=5$

$$\Leftrightarrow 340-2x-2y=10$$

$$\Leftrightarrow x+y=165$$

Thus, $x = 35$, $y = 130 \Rightarrow m\left(\overset{\frown}{AD}\right)=90° \Rightarrow m\left(\overset{\frown}{ADB}\right)=220°$

If $\overleftrightarrow{BO}$ intersects the circle in point X, then $m\left(\overset{\frown}{AX}\right)=220°-180°=40°$.

As an inscribed angle, m$\angle EBO$ = $\underline{\mathbf{20}}$°.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2012 SOLUTION KEY

Team Round - continued

F) An infinite geometric series converges to a sum $\left(\dfrac{a}{1-r}\right)$ if and only if $|r| < 1$.

Thus, $\dfrac{a}{1-r} = a + ar + 1 = a(1+r) + 1 \Leftrightarrow a = a\left(1-r^2\right) + \left(1-r\right) \Leftrightarrow ar^2 + r - 1 = 0 \Leftrightarrow a = \dfrac{1-r}{r^2}$.

This means $2 \le \dfrac{1-r}{r^2} \le 6 \Leftrightarrow 2r^2 \le 1-r$ and $1-r \le 6r^2$ or $\begin{cases} 2r^2 + r - 1 \le 0 \\ 6r^2 + r - 1 \ge 0 \end{cases}$.

We must take the intersection of these two conditions.

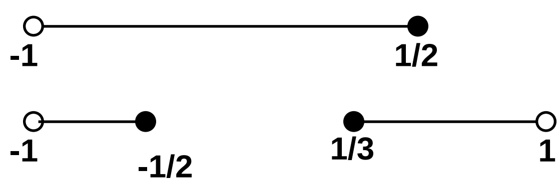
$$\begin{cases} 2r^2 + r - 1 \le 0 \\ 6r^2 + r - 1 \ge 0 \end{cases} \Leftrightarrow \begin{cases} (2r-1)(r+1) \le 0 \\ (2r+1)(3r-1) \ge 0 \end{cases}$$

The first condition requires that $-1 \le x \le \dfrac{1}{2}$, but convergence requires $r \ne -1$; hence,

$-1 < x \le \dfrac{1}{2}$.

The second condition requires $r \le -\dfrac{1}{2}$ or $r \ge \dfrac{1}{3}$, but convergence requires $-1 < r < 1$; hence,

$-1 < r \le -\dfrac{1}{2}$ or $\dfrac{1}{3} \le r < 1$



Carefully taking the intersection, we have $\underline{\boldsymbol{-1 < r \le -\frac{1}{2}} \text{ or } \boldsymbol{\frac{1}{3} \le r \le \frac{1}{2}}}$.