MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2012
ROUND 1 COMPLEX NUMBERS (No Trig)

ANSWERS

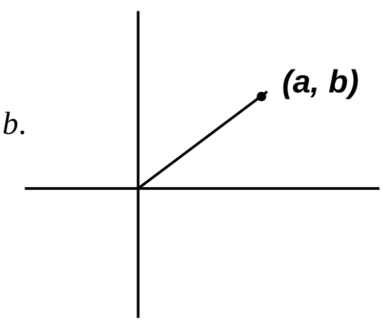
A) (____________ , ____________)

B) ______________________________

C) (____________ , ____________)

A) Determine the ordered pair of integers (a, b) for which $a+bi=(2-i)(a-i)$.

B) Let $z = a + bi$, where a and b are nonzero integers and $a > b$.
For how many ordered pairs (a, b) does $|z| = 5$?



$|z|$ denotes the absolute value of the complex number,
i.e. its distance from the origin.
Specifically, $|a+bi|=\sqrt{a^2+b^2}$.

C) For complex numbers z_1 and z_2, we have $\begin{cases} z_1^2+z_2^2=-41-6i \\ (2-i)z_1z_2=-15-20i \end{cases}$.

Find all possible values of z_1+z_2.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2012
ROUND 2 ALGEBRA 1: ANYTHING

ANSWERS

A) ______________________________ lbs

B) ______________________________

C) None 1 2 3 4

A) My current weight this morning is 195 lbs.
If I were to gain 10 lbs, I would be 5 lbs over my ideal weight W.
If instead I were to lose x lbs, I would be 9 lbs under my ideal weight W.
How much will I weigh tomorrow, if I gain x lbs today?

B) Given: x and y are integers for which $\begin{cases} x+y=17 \\ xy>42 \end{cases}$.
What is the maximum possible value of $16x + y$?

C) My book report contains 6 double-sided pages consecutively numbered 1 to 12.
(For example, pages 3 and 4 are printed on opposite sides of the same sheet.)
On my way to class, my report was swept away by a strong gust of wind.
I was only able to find three of the sheets.
Circle "None" or the number(s) of all the statements which must be true.

1) The sum of the page numbers on the lost sheets could be 24
2) The sum of the page numbers on the lost sheets could be 49
3) The sum of the page numbers on the lost sheets must be at least 21 and at most 57.
4) The sum of the page numbers on the sheets I found could equal the sum of the page numbers on the lost sheets.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2012
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

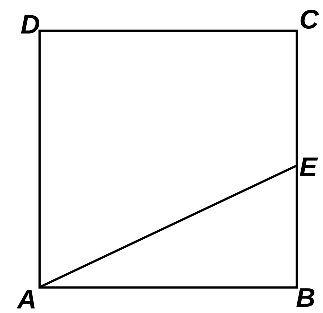
ANSWERS

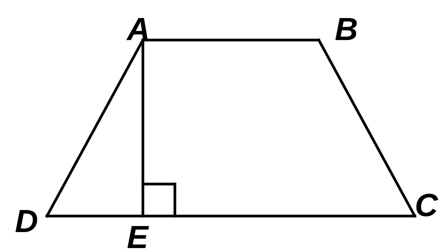
A) ____________ : ____________

B) ___________________________ units^2

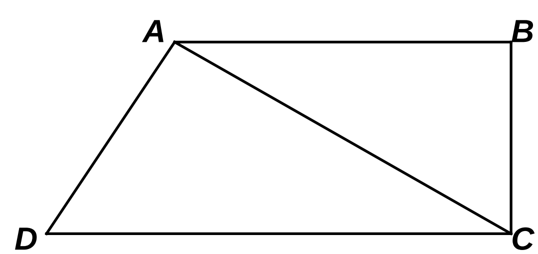
C) ___________________________ units^2

A) *ABCD* is a square with side 4.
The ratio of the area of *ADCE* to the area of *ABE* is 7 : 1.
Compute *BE* : *CE*.





B) In isosceles trapezoid *ABCD*, $\overline{AB} \parallel \overline{CD}$,
$AB = AE = 12$ and the perimeter = 60.
If the sides of the trapezoid have integer length,
compute the area of *ABCD*.



C) In quadrilateral *ABCD*,
$\overline{AB} \perp \overline{BC}$, $\overline{DA} \perp \overline{AC}$,
$AC = 45$, $AD : AB = 7 : 9$, and
$$\frac{\text{area}(\Delta ABC)}{\text{area}(\Delta ADC)} = \frac{27}{35}.$$
Compute the area of *ABCD*.
Do <u>not</u> assume *ABCD* is a trapezoid.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2012
ROUND 4 ALG 1: FACTORING AND ITS APPLICATIONS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) How many factors of 126 are also factors of 132?

B) Given: $x^3y - xy^3 = 12$ and $x^3y^2 - x^2y^3 = 15$

Compute $\frac{1}{x} + \frac{1}{y}$.

C) Solve for x over the reals. $(x+1)(x-6)(x+3) - (x+1)^2(x-2) > 0$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2012
ROUND 5 TRIG: FUNCTIONS OF SPECIAL ANGLES

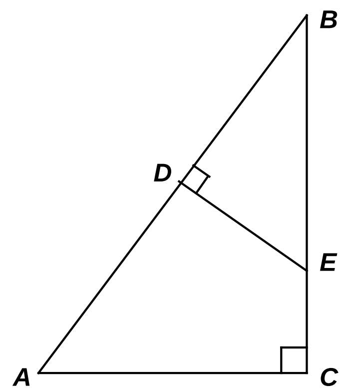
ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) In ΔABC, the measures of the angles are $x^\circ, \frac{x}{5}^\circ$ and $\frac{3x}{10}^\circ$. Compute $2\cos^2\left(x^\circ\right)-1$.



B) Given: $\cos\left(\angle BED\right)=\frac{3}{5}$

Compute $\cot^2 A+\cos\left(\angle DEC\right)$.

C) Suppose A = 60°. There are many ordered pairs (k, B) of relatively prime positive integers that are solutions of $k\tan\left(\frac{A}{2}\right)=\sin 2A+B\sin A$. Of those pairs compute the <u>smallest prime</u> value of the sum k + B.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2012
ROUND 6 PLANE GEOMETRY: ANGLES, TRIANGLES AND PARALLELS

ANSWERS

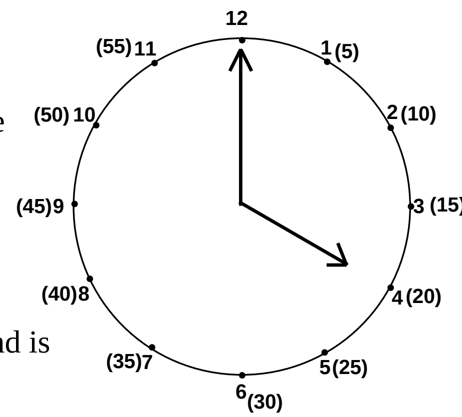
A) ______________________ °

B) ______________________ °

C) ______________________ °

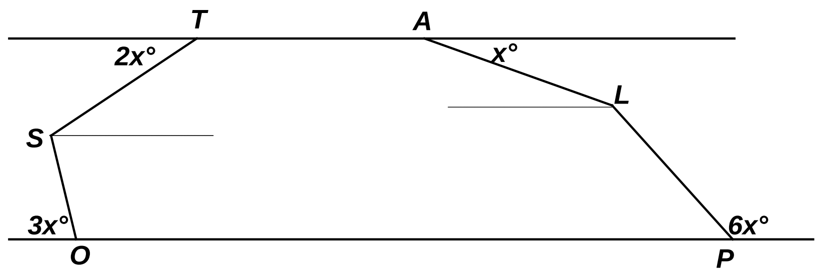
A) In a pentagon *MAGIC*, ∠s *M*, *A* and, *G* are congruent and∠s *I* and *C* are congruent.
All angle measures are positive integers.
Determine the <u>smallest</u> possible measure (in degrees) of ∠*C*.

B) On an analog clock, what is the degree measure of the <u>acute</u> angle between the minute and hour hand at 4:21 AM?



Note: In the diagram, the numbers 1 - 12 represent hours.
The time in the diagram is 4:00 (AM or PM).
The numbers in parentheses indicate minutes past the hour. For example, at 4:15, the minute hand points at the 3 and the hour hand is pointing somewhere <u>between</u> the 4 and the 5.

C) $\overleftrightarrow{TA} \parallel \overleftrightarrow{OP}$. In convex hexagon *POSTAL*, all interior angles have integer measure and $m\angle L > m\angle TAL$. Compute the <u>maximum</u> sum of the measures of the largest and smallest angles in *POSTAL*. Reminder: The diagram is <u>not</u> necessarily drawn to scale.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2012
ROUND 7 TEAM QUESTIONS
ANSWERS

A) (______ , ______ , ______) D) ____________________________

B) ____________________________ E) _______________________ units

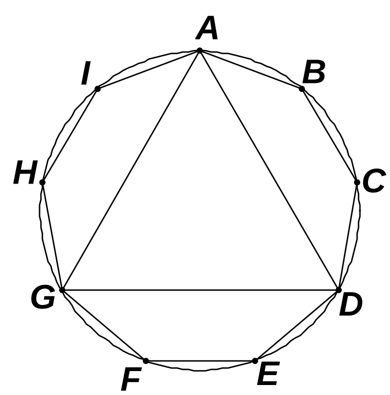
C) (____________ , ____________) F) _______________________ units

A) Let $Z = \cfrac{1}{i - \cfrac{1}{i - \cfrac{1}{i - \cfrac{1}{i - \dots \cfrac{1}{i - \cfrac{1}{i}}}}}}$, where the complex fraction contains k instances of i and $k > 2$.

For example, for $k = 2$, $Z = \cfrac{1}{\boldsymbol{i} - \boxed{\cfrac{1}{\boldsymbol{i}}}}$.

For some minimum value of k this expression simplifies to $-\frac{A}{B}i$, where A and B are positive integers and A is a perfect square ($A \neq 1$). Determine the ordered triple (k, A, B).

B) Dick, Joe and Norm are practicing for a big math contest. They are very competitive and equally talented and on a set of 100 practice questions, each was able to correctly answer 60 questions and no question stumped all three mathletes.
A question is defined to be <u>*hard*</u> if exactly one mathlete got it right.
A question is defined to be <u>*easy*</u> if all three mathletes got it right.
Some questions are neither easy nor hard.
There were k more *hard* questions than *easy* questions. Compute k.



C) In a regular nonagon *ABCDEFGHI*, ΔADG has area $36\sqrt{3}$.
The area of the nonagon is $k\sin\theta°$. Find the ordered pair (k, θ°), where both k and θ are positive integers and θ is acute.

D) For how many integer values of x between 0 and 100 inclusive, does the quotient $\dfrac{8x+4}{\frac{2}{x+1}+\frac{14}{x-3}}$ produce an integer value?

E) In polar coordinates, the equation $r = \cos\theta + \sqrt{3}\sin\theta$ defines a circle which passes through the origin. $\theta = 30°$ and $\theta = 60°$ defines lines through the origin which make angles of 30° and 60° respectively, measured counterclockwise from the positive x-axis. Let B and C be the points in the first quadrant where these lines intersect the circle. Compute the distance between B and C.

F) <u>Scalene</u> triangle *ABC* has sides of integer length.
$\overline{AD}$ is the altitude to side $\overline{BC}$.
If $AB = 12$ and $m\angle BAD = 30°$, compute <u>all</u> possible perimeters of ΔABC.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2012 ANSWERS

Round 1 Algebra 2: Complex Numbers (No Trig)

A) $(1, -3)$ B) 4 C) $-2+7i,\ 2-7i$

Round 2 Algebra 1: Anything

A) 199 B) 212 C) 2 and 3

Round 3 Plane Geometry: Area of Rectilinear Figures

A) 1 : 3 B) 204 C) 1116

Round 4 Algebra 1: Factoring and its Applications

A) 4 B) $\frac{4}{5}$ C) $-8 < x < -1$

Round 5 Trig: Functions of Special Angles

A) $-\frac{1}{2}$ B) $-\frac{3}{80}$ (also acceptable: $\frac{-3}{80}, \frac{3}{-80}$) C) 19

Round 6 Plane Geometry: Angles, Triangles and Parallels

A) 3 B) 4.5° C) 205

Team Round

A) (12, 144, 233)

B) 20

C) (216, 40)

D) 49

E) 1

F) 60 and 72 (Both are required.)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2012 SOLUTION KEY

Round 1

A) FOILing, $a+bi=(2-i)(a-i)=2a-2i-ai-1=(2a-1)+(-2-a)i$

Equating the real and imaginary coefficients, $\begin{cases} a=2a-1 \\ b=-2-a \end{cases} \Rightarrow (a, b) =$ **<u>(1, –3)</u>**

B) The only nonzero integers for which $\sqrt{a^2+b^2}=5$ are ±3 and ±4, four choices for *a* and four choices for *b*.
Thus, sixteen ordered pairs are possible, but since *a* > *b*, of these only (4, 3), (4, –3), (3, –4) and (–3, –4) are acceptable.
Thus, there are only **<u>4</u>** ordered pairs.

C) Given: $\begin{cases} {z_1}^2+{z_2}^2=-41-6i \\ (2-i)z_1z_2=-15-20i \end{cases}$

Dividing $(2-i)$ and multiplying by 2, the second equation gives us

$$2z_1z_2=2\left(\frac{-15-20i}{2-i}\right)\cdot\frac{2+i}{2+i}=2\left(\frac{-5(3+4i)(2+i)}{5}\right)=2(-6-3i-8i+4)=-4-22i$$

Adding to the first equation, ${z_1}^2+{z_2}^2+2z_1z_2=(-41-6i)+(-4-22i)$

$\Leftrightarrow (z_1+z_2)^2=-45-28i$

Since the sum of two complex numbers is a complex number, let $z_1+z_2=a+bi$, for real numbers *a* and *b*.

Then: $\begin{array}{l} a^2-b^2=-45 \\ 2ab=-28 \Rightarrow ab=-14 \end{array}$

Clearly, *a* and *b* have opposite signs.
The ordered pairs (2, –7) and (–2, 7) satisfy both equations and $z_1+z_2=$ <u>$\mathbf{2-7}\boldsymbol{i}$</u>, <u>$\mathbf{-2+7}\boldsymbol{i}$</u> .

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2012 SOLUTION KEY

Round 2

A) My ideal weight W must be 195 + 10 – 5 = 200 lbs.
$195 - x = 200 - 9 \Rightarrow x = 4$
Thus, my new weight tomorrow is 195 + 4 = **199**.

B) $\begin{cases} x+y=17 \\ xy>42 \end{cases} \Rightarrow x(17-x)>42 \Rightarrow x^2-17x+42=(x-3)(x-14)<0 \Rightarrow 3<x<14.$
The largest value occurs when x is as large as possible, namely 16(13) + 4 = 208 + 4 = **212**.

An equivalent problem in the language of bases would have been:

Let $N = \underline{x}\,\underline{y}$ be a two-digit base 16 integer.
(x is the most significant digit and y is the least significant digit.)
In base 10, $x + y = 17$ and $xy > 42$.
What is the largest possible value of N in base 10?

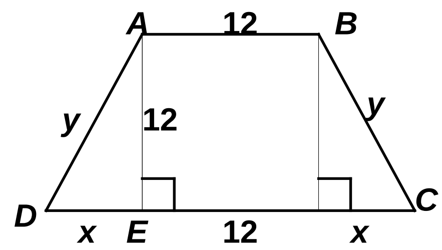
The base 16 number would be represented as $16x + y$ in base 10.
Recall in base 10 there are 10 digits, the largest being 9 (1 less than the base).
Similarly, in base 16 there are 16 digits and the largest is equivalent to 15 in base 10.
They are usually represented 0, 1, 2, … ,9 , A = 10, B = 11, … F = 15.
So D = 13 is a legal digit in base 16.
Base 16 (commonly called hexadecimal) is frequently used in computer science to represent computer addresses. On a 64-bit operating system, a legal address might be 0000 4A73 FFB1 8660. Spaces added for legibility only. Who knows, the first letter you type in your next homework assignment might be stored there!
N's largest value is $D4_{(16)} = 16(13) + 4 = \mathbf{212_{(10)}}$.

C) The possible sums for the 6 sheets are: 3, 7, 11, 15, 19 and 23
All of these are odd numbers.
Adding any three of these numbers will result in an odd total; therefore 1) is false.
The minimum sum of lost pages numbers is (3 + 7 + 11) = 21; the maximum (15 + 19 + 23) = 57; therefore 3) is true.
If I lost 7, 19 and 23, my total would be 49; therefore 2) is true.
The total of the 6 numbers is 78.
If lost and found totals are equal, then they must both be 39.
The units digits of my 6 numbers include all the odd digits and 3 occurs twice.
Since I desire a sum ending in 9, I have 2 possibilities: numbers ending in 3,7 and 9 or 3, 5 and 1.
Thus, the only possibilities are: 3, 7, 19 7, 19, 23 3, 11, 15 or 11, 15, 23
None of these total 39, so 4) is false.
Therefore, the only true statements are **2 and 3**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2012 SOLUTION KEY

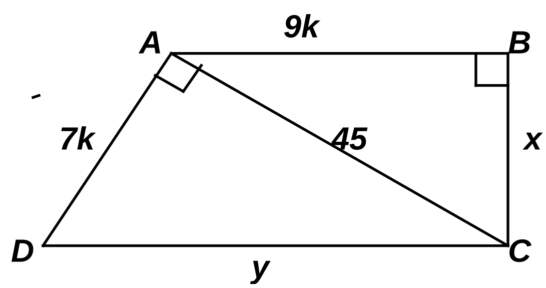
Round 3

A) Let $BE = x$. Then: $\dfrac{16-2x}{2x}=\dfrac{8-x}{x}=\dfrac{7}{1}\Rightarrow x=1\Rightarrow BE:CE=$ **<u>1 : 3</u>**.



B) $2(x + y + 12) = 60 \Rightarrow x + y = 18$
From knowledge of common Pythagorean triples,
$(x, y) = (5, 13)$ or noting that
$x^2+12^2=y^2\Rightarrow x^2+144=(18-x)^2\Rightarrow 144=324-36x\Rightarrow x=5$

The area of $ABCD$ is $\frac{1}{2}(12)(12+22)=6(34)=\underline{\mathbf{204}}$.



C) $\dfrac{\frac{1}{2}x(9k)}{\frac{1}{2}7k(45)}=\dfrac{27}{35}\Rightarrow\dfrac{x}{35}=\dfrac{27}{35}\Rightarrow x=27$
In ΔABC, $(27, 9k, 45) = 9(3, k, 5) \Rightarrow$
$k = 4 \Rightarrow AD = 28$
Thus, the area of $ABCD$ =
$\frac{1}{2}\cdot 36\cdot 27+\frac{1}{2}\cdot 28\cdot 45=18(27)+14(45)=18(27+35)=\underline{\mathbf{1116}}$

Note that finding the value of y was <u>not</u> necessary.
By Pythagorean Theorem, we could have computed the value of DC. It is actually 53.
Note also that $ABCD$ is <u>not</u> a trapezoid. If it were a trapezoid, the length of the altitude from A to $\overline{DC}$ (call it $\overline{AE}$) would be 27.
ΔADC has sides 28, 45 and 53.
Thus, as a right triangle, the area of ΔADC is $\frac{1}{2}\cdot 28\cdot 45$ or $\frac{1}{2}\cdot AE\cdot 53$, implying
$AE=\sqrt{495}=3\sqrt{55}\neq 27$.
Thus, $ABCD$ is definitely <u>not</u> a trapezoid!

MASSACHUS0ETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2012 SOLUTION KEY

Round 4

A) $126 = 6 \cdot 21 = 2 \cdot 3^2 \cdot 7$
$132 = 11 \cdot 12 = 2^2 \cdot 3 \cdot 11$
The common prime factors are 2 and 3, so the common factors are 1, 2, 3 and 6
$\Rightarrow$ **<u>4</u>** common factors.

B) $x^3y - xy^3 = xy\left(x^2 - y^2\right) = 12$
$x^3y^2 - x^2y^3 = x^2y^2\left(x - y\right) = 15$
$\dfrac{xy\left(x^2 - y^2\right)}{x^2y^2\left(x - y\right)} = \dfrac{12}{15} \to \dfrac{x+y}{xy} = \dfrac{1}{x} + \dfrac{1}{y} = \underline{\boldsymbol{\dfrac{4}{5}}}$

C) $(x+1)(x-6)(x+3) - (x+1)^2(x-2) = \left(x+1\right)\left(\left(x^2 - 3x - 18\right) - \left(x^2 - x - 2\right)\right)$
$= \left(x+1\right)\left(-2x - 16\right) = -2(x+1)(x+8) > 0$
Dividing through by -2, $(x+1)(x+8) < 0$
The critical values are -1 and -8.
Both factors are negative for $x < -8$, positive for $x > -1$ and in between -1 and -8, they have opposite signs. Therefore, the product is negative for <u>$\boldsymbol{-8 < x < -1}$</u>.

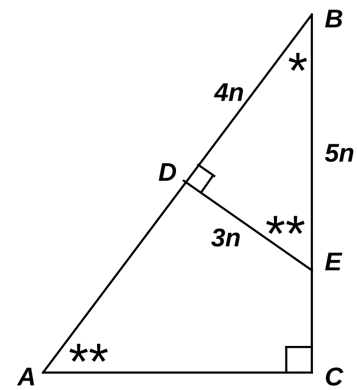
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2012 SOLUTION KEY

Round 5

A) $x+\frac{x}{5}+\frac{3x}{10}=180 \Leftrightarrow 15x=180(10) \Rightarrow x=120$

Recognizing the double angle formula, or plugging directly into $2\cos^2(x)-1$, we have

$2\cos^2(120)-1=2\left(-\frac{1}{2}\right)^2-1=\underline{\boldsymbol{-\frac{1}{2}}}$.



B) Since $\angle A \cong \angle BED$ and $\cos(\angle BED)=\frac{3}{5}$, $\cot A=\frac{3}{4}$.

Since $\angle DEC$ is the supplement of $\angle BED$, $\cos(\angle DEC)=-\frac{3}{5}$.

$\Rightarrow \cot^2 A+\cos(\angle DEC) = \frac{9}{16}-\frac{3}{5}=\frac{9\cdot 5-3\cdot 16}{16\cdot 5}=\underline{\boldsymbol{-\frac{3}{80}}}$

($\frac{-3}{80}$ and $\frac{3}{-80}$ are also acceptable.)

C) Substituting for A, $k\left(\frac{\sqrt{3}}{3}\right)=\frac{\sqrt{3}}{2}+B\left(\frac{\sqrt{3}}{2}\right) \Rightarrow \frac{k}{3}=\frac{1+B}{2} \Rightarrow 2k=3(1+B)$

B must be odd, since the product on the left side is even.
B = 1 $\Rightarrow$ k = 3, but 3 + 1 = 4 is not prime
B = 3 $\Rightarrow$ k = 6, but 6 + 3 = 9 is not prime
B = 5 $\Rightarrow$ k = 9, but 5 + 9 = 14 is not prime
B = 7 $\Rightarrow$ k = 12 and 7 + 12 = 19 is prime and $gcf(12,7)=1$
(i.e. 12 and 7 are relatively prime integers). Thus, (k, B) = (12, 7) $\Rightarrow$ <u>**19**</u>.

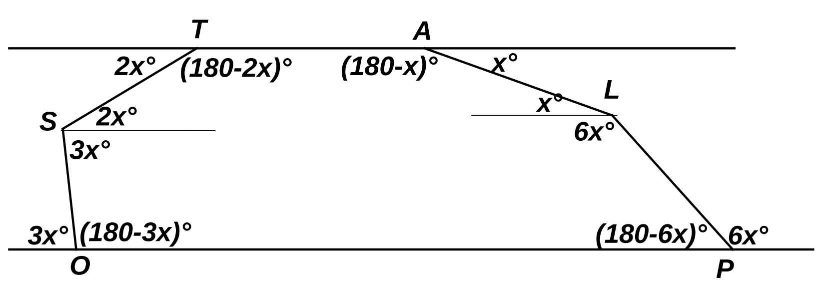
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2012 SOLUTION KEY

Round 6

A) Let $m\angle M = m\angle A = m\angle G = a°$ and $m\angle I = m\angle C = b°$.
Then: $3a + 2b = 3(180) = 540$
$b = 1 \Rightarrow 538/3$ fails
$b = 2 \Rightarrow 536/3$ fails
$b = \underline{\mathbf{3}} \Rightarrow 534/3 = 178$ Bingo!

B) The minute hand makes a complete revolution (or turns through 360°) in 1 hour, whereas the hour hand takes 12 hours to make a complete revolution.
Since the minute hand turns 12 times faster than the hour hand, in one minute the minute hand turns through $\frac{360°}{60} = 6°$ and the hour hand turns through $\frac{1}{2}°$.
At 4:21, the minute hand has turned though 126° (6° per minute measured from the top of the hour). The hour hand has turned $\frac{1}{12}$ as far, namely through $\frac{1}{12}(126°) = 10.5°$ or (130.5° measured from the top of the hour). Thus, the minute hand has passed the hour hand and the degree measure of angle between the hands is $130.5 - 126 = \underline{\mathbf{4.5}}$.

C) Draw a line through S parallel to $\overleftrightarrow{TA}$. Two pairs of alternate interior angles are formed, one pair measuring $2x°$ and the other pair measuring $3x°$. Thus, $m\angle S = 5x$. Similarly, $m\angle L = 7x$.
Since the other 4 angles are supplements of the marked angles, they have measures of $180 - x$, $180 - 2x$, $180 - 3x$ and $180 - 6x$.
The largest angle must be either $\angle L$ or $\angle TAL$.
To guarantee $\angle L$ is the largest, we require $7x > 180 - x$ or $x > 22.5$
Since $m\angle L = 7x < 180$ and x is an integer, $x \le 25$.
Thus, we must examine angle measures for $x = 23$, 24 and 25.
Let $x = 23$. The 6 angles in hexagon *POSTAL* (in increasing order) measure
(P) <u>42°</u>, (O)111°, (S) 115°, (T) 134°, (A) 157° and (L) <u>161°</u>$\Rightarrow$ 203°.
For the other possible values of x, the smallest angle will be P $(180 - 6x)$ and the largest will be L $(7x)$. As x increases by 1, $m\angle P$ decreases by 6 and $m\angle Q$ increases by 7, changing the net total by +1, producing additional totals of 204 and 205.
The required sum is 30 + 175 = **<u>205</u>**.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2012 SOLUTION KEY

Team Round

A) Examining complex fractions with increasingly more instances of i, starting with k = 1, look for a pattern.

$\underline{1}\ i$: $\boxed{\frac{1}{i}} = \frac{1}{i}\cdot\frac{i}{i} = -1i = -\frac{1}{1}i$

$\underline{2}\ i$ s: $\frac{1}{i-\boxed{\frac{1}{i}}} = \frac{1}{i-(-1i)} = \frac{1}{2i} = -\frac{1}{2}i$

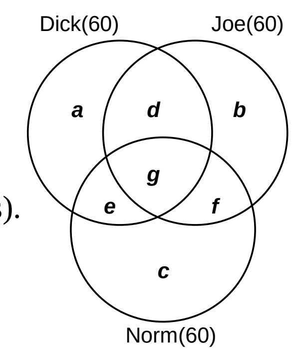
$\underline{3}\ i$ s: $\frac{1}{i-\boxed{\frac{1}{i-\frac{1}{i}}}} = \frac{1}{i-\left(-\frac{1}{2}i\right)} = \frac{1}{\frac{3}{2}i} = -\frac{2}{3}i$

$\underline{4}\ i$ s: $\frac{1}{i-\boxed{\frac{1}{i-\frac{1}{i-\frac{1}{i}}}}} = \frac{1}{i-\left(-\frac{2}{3}i\right)} = \frac{1}{\frac{5}{3}i} = -\frac{3}{5}i$

Each time the denominator of the current term becomes the numerator of the next term and the denominator of the next term is the sum of the numerator and the denominator of the previous term.
Shades of the Fibonacci sequence, 1, 1, 2, 3, 5, 8, 13, …!
Continue the sequence until we reach a perfect square, 21, 34, 55, 89, ***<u>144</u>***, 233

Thus, the 12th term (the complex fraction with 12 is) will be $-\frac{144}{233}i$;

hence, (k, A, B) = **<u>(12, 144, 233)</u>**.



B) Consider the Venn diagram at the right.
The circles contain the questions the mathletes answered correctly.
a, b and c: answered correctly by exactly 1 mathlete (i.e. HARD questions).
d, e, and f were answered questions correctly by exactly 2 mathletes.
g were questions answered correctly by all 3 mathletes (i.e. EASY questions).
Knowing they each answered 60 questions correctly, we have

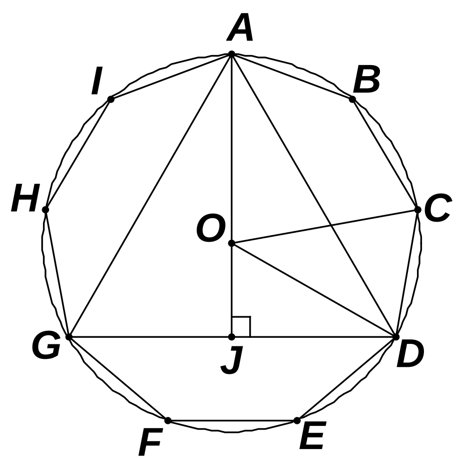
$$\begin{cases} a+d+e+g=60 \\ b+d+f+g=60 \Rightarrow (a+b+c)+2(d+e+f)+3g=180 \\ c+e+f+g=60 \end{cases}$$

Thus, HARD + 2(d + e + f) + 3EASY = 180
Rearranging the terms, we have $(a+b+...+g)+(d+e+f)+2g=180$
Since the first sum represents all the questions, we have 100 + (d + e + f) + 2EASY = 180 or (d + e + f) = 80 – 2EASY
Substituting, HARD + 2(80 – 2EASY) + 3EASY =180 ⇒ HARD – EASY = 20 ⇒ k = **<u>20</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2012 SOLUTION KEY

Team Round

C) ΔADG is equilateral and its area is $\dfrac{s^2\sqrt{3}}{4} = 36\sqrt{3} \Rightarrow s = 12$.



Since ΔDOJ is a 30-60-90 right triangle,
$JD = 6$, $OJ = 2\sqrt{3}$, $OD = 4\sqrt{3}$

$m\measuredangle COD = \dfrac{360°}{9} = 40°$

Using $A(\Delta) = \dfrac{1}{2}ab\sin\theta$, area(ΔCOD) =

$\dfrac{1}{2}\left(4\sqrt{3}\right)^2 \sin 40° = 24\sin 40°$

Therefore, the area of the nonagon is $9\left(24\sin 40°\right)$

Since both k and θ are positive integers and θ is acute, (k, θ°) = **<u>(216, 40)</u>**.

D) Of the 101 integers in the given range 3 must be excluded, since it causes division by zero.
Simplifying the complex fraction, we have

$$\frac{8x+4}{\dfrac{2}{x+1}+\dfrac{14}{x-3}} = \frac{4(2x+1)}{\dfrac{2(x-3)+14(x+1)}{(x+1)(x-3)}} = \frac{4(2x+1)}{\dfrac{16x+8}{(x+1)(x-3)}} = 4(2x+1)\cdot\frac{(x+1)(x-3)}{8(2x+1)} = \frac{(x+1)(x-3)}{2}$$

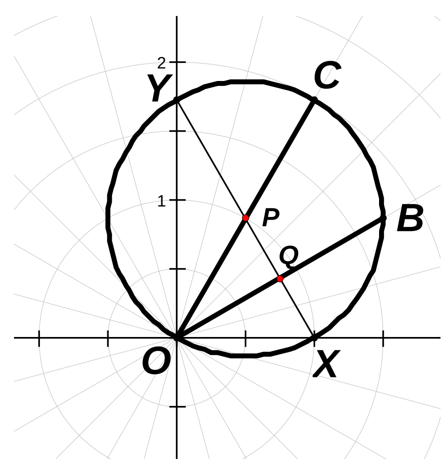
Clearly, the numerator must be even and this happens if and only if x is odd.
Between 0 and 100 inclusive, there are 51 even integers and 50 odd integers.
Thus, the quotient is integral for **<u>49</u>** values of x.

E) $\theta = 0 \Rightarrow r = 1+\sqrt{3}\cdot 0 = 1$

$\theta = 90 \Rightarrow r = 0+\sqrt{3}\cdot 1 = \sqrt{3}$

Thus, in the Cartesian coordinate system, where points are located with x- and y-coordinates, $X(1,0)$, $Y\left(0,\sqrt{3}\right)$, ΔYOX is a 30-60-90 right triangle, and $XY = 2$.

The circle is circumscribed about ΔYOX and $\overline{XY}$ is a diameter and the midpoint of $\overline{XY}$ is the center of the circle.



Team Round
E) – continued

In polar coordinates $B(r, 30°)$. $r = \frac{\sqrt{3}}{2} + \sqrt{3} \cdot \frac{1}{2} = \sqrt{3}$

Knowing $OB = \sqrt{3}$, drop a perpendicular from B to $\overrightarrow{OX}$ forming a 30-60-90 right triangle.
The horizontal side is $\frac{3}{2}$, the vertical side is $\frac{\sqrt{3}}{2}$, so the (*x*, *y*) coordinates of *B* are $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$.

In polar coordinates $C(r, 60°)$. $r = \frac{1}{2} + \sqrt{3} \cdot \frac{\sqrt{3}}{2} = 2$

Knowing $OC = 2$, drop a perpendicular from C to $\overrightarrow{OX}$ forming a 30-60-90 right triangle.
The horizontal side is 1, the vertical side is $\sqrt{3}$, so the (*x*, *y*) coordinates of *C* are $\left(1, \sqrt{3}\right)$.

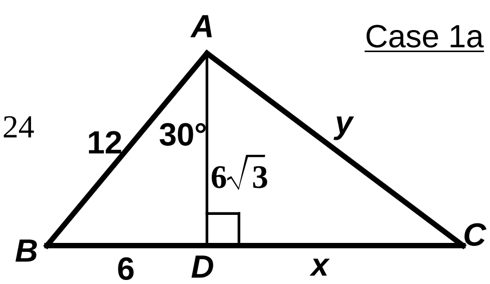
Applying the distance formula, we have $BC = \sqrt{\left(\frac{3}{2} - 1\right)^2 + \left(\frac{\sqrt{3}}{2} - \sqrt{3}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \underline{\mathbf{1}}$.

F) Using the Pythagorean Theorem on ΔADC, we have

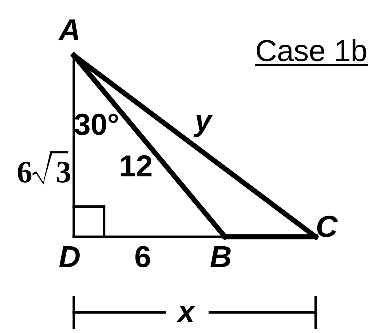
$$y^2 - x^2 = \left(6\sqrt{3}\right)^2 = 108$$

Since *x* and *y* are integers, we examine the possible factorizations of 108.

$$\begin{cases} y - x = \;\; 1 \quad \boxed{2} \quad 3 \quad 4 \quad \boxed{6} \quad 9 \quad 12 \\ y + x = 108 \quad \boxed{54} \quad 36 \quad 27 \quad \boxed{18} \quad 12 \quad 9 \end{cases}$$



Adding, only two factorizations give integer results: 2*y* = 56 or 24
Case 1: $y = 28 \Rightarrow x = 26$ Case 2: $y = 12 \Rightarrow x = 6$
The second result gives us an equilateral triangle of side 12.
The first result gives us a scalene triangle with sides 12, 28 and 32, resulting in a perimeter of **<u>72</u>**.



If *B* were reflected over $\overline{AD}$, the diagram would also satisfy the given conditions and *BC* = 26 – 6 = 20, resulting in a perimeter of 12 + 28 + 20 = **<u>60</u>**.

<u>FYI</u>:

In case 1, using the Law of Cosines, $\cos(\angle BAC) = \frac{12^2 + 28^2 - 32^2}{2 \cdot 12 \cdot 28} = -\frac{1}{7}$
and $\angle BAC$ must be obtuse (approx. 98°).
In case 2, as the supplement of $\angle ABD$, $m\angle ABC = 120°$ and this is verified by the Law of Cosines,
$\cos(\angle ABC) = \frac{12^2 + 20^2 - 28^2}{2 \cdot 12 \cdot 20} = -\frac{1}{2}$.