MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 – DECEMBER 2012
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

ANSWERS

A) _______________________________

B) _______________________________

C) _______________________________

A) Compute the sine of the smaller acute angle in a right triangle whose hypotenuse has length 37 and whose long leg has length 35.

B) In square *ABCD*, points *E* and *F* lie on $\overline{AD}$ and $\overline{AB}$ respectively such that *AE* : *DE* = 2 : 1 and *AF* : *FB* = 2 : 1. Compute $\cos\left(\angle FCE\right)$.

C) In ΔABC, $\frac{BC}{AC}=\frac{2}{\sqrt{7}}$. Compute $7\cos^2 A-4\cos^2 B$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012
ROUND 2 ARITHMETIC/NUMBER THEORY

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) Compute the absolute value of the cube of the difference between the unit digits of 2^{71} and 3^{44}.

B) In a certain positive integer base *b*, $314_{(b)}$ is twice $132_{(b)}$. Compute *b*.

C) Determine the 117th natural number that is divisible by 2 or 3, but <u>not</u> divisible by either 4 or 6.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES

ANSWERS

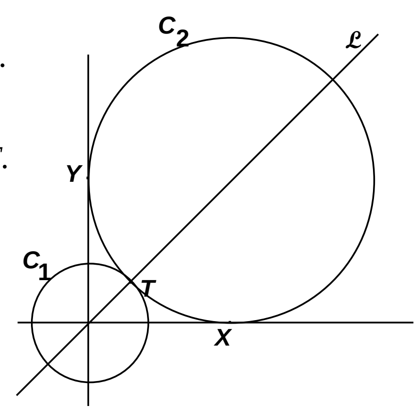
A) k = ______________________________

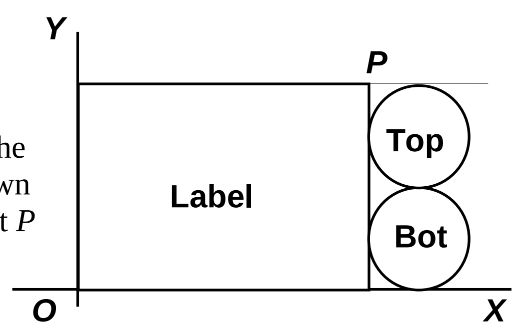
B) a = ______________________________

C) (_________ , _________ , ________)

A) The lines $y = mx + 1$ and $y = \frac{2x}{5} - m$ intersect at the point $(6, k)$. Determine the value of k.

B) Let circle $C_1 = \left\{(x, y) \mid x^2 + y^2 = 36\right\}$ and line $\mathcal{L} = \left\{(x, y) \mid y = x\right\}$. Circle C_2 has its center on $\mathcal{L}$ outside of C_1 and is tangent to the x-axis at $X(a, 0)$, the y-axis at $Y(0, b)$ and circle C_1 at point T. Compute the value of a.





C) When removed, the label on a cylindrical can is a rectangle. Suppose the height (H) of the can is 4 times the radius (r) of the base. The label is placed in quadrant 1 of the xy-plane as shown in the diagram at the right. The distance from point O to point P can be expressed in terms of H and r in simplest form as $\frac{\sqrt{A\pi^2 + B}}{C} \frac{H^2}{r}$, where A, B and C are positive integers. Compute the ordered triple (A, B, C).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

ANSWERS

A) ________________________________

B) ________________________________

C) (________ , ________ , ________)

A) Compute the real value(s) of x for which $8^x = \sqrt[3]{\dfrac{2}{4^x}}$.

B) The real number a is the x-intercept of the function $f(x) = -3 + 4\log_{16} x$.

Compute $\left(\dfrac{\log_2 a^8}{\log_4 a^2}\right)^{\frac{1}{3}}$.

C) Given: $y_1 = 2\left(4^x\right)$, $y_2 = \dfrac{8^{x+2}}{4}$

For $x = a\log_2 b + c$, where a, b and c are integers and b is as small as possible, $y_2 = 81y_1$.
Compute the ordered triple (a, b, c).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

ANSWERS

A) k = ____________________________

B) (_____________ , ____________)

C) _________________________________

A) 20% of A plus 60% of B equals 100% of B.
30% of B plus 10% of A is equivalent to k% of A. Compute k.

B) According to Newton's law of universal gravitation, the force of attraction (F) between two bodies varies directly with the product of the masses (m_1 and m_2) and inversely with the square of the distance (d) between them. The actual calculations could get quite messy, so here we use some simplistic measurements.
Suppose F_1 = 0.004, when (m_1, m_2, d) = (2, 4, 12).
Let k be the proportionality constant.
Let F_2 be the force between two bodies when (m_1, m_2, d) = (3, 6, 8).
Compute the ordered pair (k, F_2).

C) Let P be the difference between the cubes of two consecutive integers.
Let Q be the difference between the squares of two consecutive integers.
If $P : Q = 13 : 1$, compute the <u>smallest</u> possible sum of the larger cube and the smaller square.

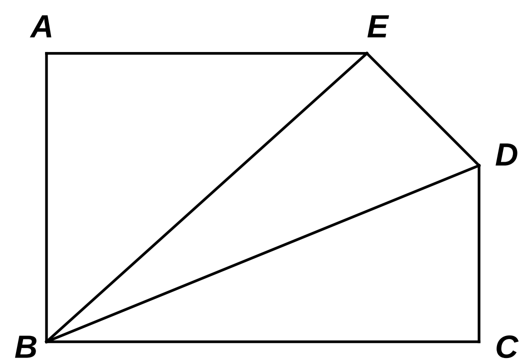
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

ANSWERS

A) ______________________

B) ______________________ °

C) ______________________ °



A) Given: Right angles at A, B and C
$AE = 9$, $BC = 12$, and $DC = DE = 5$
Compute $BE + BD - AB$.

B) The interior angles of pentagon P have degree-measures of x^2, x^2, $13x + 100$, 120 and 170.
Compute the <u>sum</u> of the measures of the <u>largest</u> two angles in P.

C) Given: $ABCD$ is a square, E and F are points on $\overline{AB}$ and $\overline{BC}$ respectively.
$AE = CF$. K is the point of intersection of $\overline{AF}$ and $\overline{CE}$.
$m\angle BAF = 40°$
Compute $m\angle EKF$.

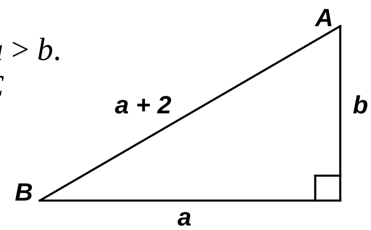
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012
ROUND 7 TEAM QUESTIONS
ANSWERS

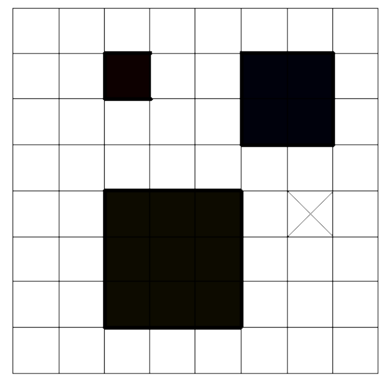
A) ____________________ D) (__________ , __________)

B) ____________________ E) ____________________

C) ____________________ F) ____________________

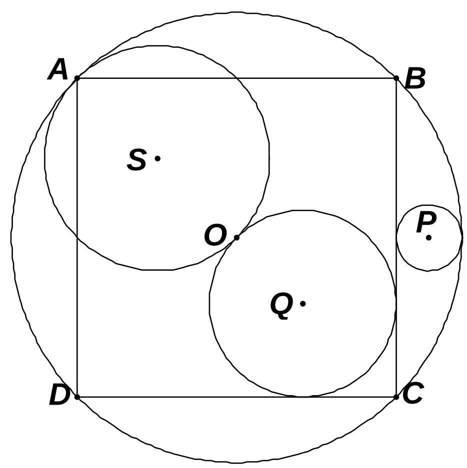
A) Given: a, b are relatively prime integers, $a > b$.
Compute the minimum perimeter of ΔABC in which $(\cos\angle A) < \dfrac{1}{10}$.





B) In the 8 x 8 grid at the right there are squares of three different sizes that do not contain the "X".
Consider all possible squares on this grid from 1 x 1 through 8 x 8 inclusive. How many of these squares do not contain the "X" ?

C) Let R denote the radius of circle O which is circumscribed about square $ABCD$.
Let (r_1, r_2, r_3) be the radii of the circles centered at P, Q and S respectively.
Circle Q and S are tangent at O.
Circle Q is also tangent to two sides of the square.
Circle P is externally tangent to square $ABCD$ and internally tangent to circle O.
If $r_1 + r_2 + r_3 = kR$, compute k as a simplified fraction.



D) Given: $f(x) = \begin{cases} 2\cdot 4^x + A + 3 & \text{, if } x \ge 2 \\ A\log_4 x + B & \text{, if } 0 < x < 2 \end{cases}$ is a piecewise function and $A + B = 17$.
Compute the ordered pair of integers (A, B) for which this function is continuous at $x = 2$.

E) Given: $x = 2$, $y < 0$, $z > 0$ and $\dfrac{x+y}{z} = \dfrac{y+z}{x} = \dfrac{x+z}{y}$
Compute the largest possible value of y.

F) Compute the maximum number of sides in a regular polygon in which the number of diagonals is less than the degree measure of an interior angle.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012 ANSWERS

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

A) $\frac{12}{37}$ B) $\frac{3}{5}$ C) 3

Round 2 Arithmetic/Elementary Number Theory

A) 343 B) 5 C) 350

Round 3 Coordinate Geometry of Lines and Circles

A) $\frac{11}{5}$ B) $6\left(\sqrt{2}+1\right)$ C) $\left(1,4,8\right)$

Round 4 Alg 2: Log and Exponential Functions

A) $\frac{1}{11}$ B) 2 C) $(4, 3, -3)$

Round 5 Alg 1: Ratio, Proportion or Variation

A) 25 B) $\left(\frac{9}{125},\frac{81}{4000}\right)$ or $(0.072, 0.02025)$ C) 225

Round 6 Plane Geometry: Polygons (no areas)

A) $4+9\sqrt{2}$ B) 348 C) 170

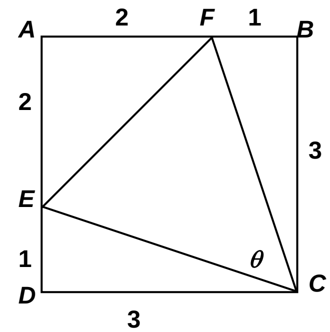
Team Round

A) 840 D) $(-12, 29)$

B) 166 E) -3

C) $\frac{3\sqrt{2}}{4}$ F) 19

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012 SOLUTION KEY



Round 1

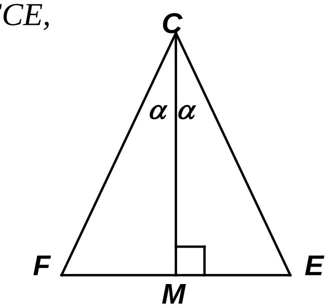
A) The smaller angle is opposite the shorter leg which we find by using the Pythagorean Theorem, $37^2 = 35^2 + x^2 \Rightarrow x^2 = 37^2 - 35^2$.
Recognizing this as the difference of perfect squares, we avoid the need to square these numbers.

$$37^2 - 35^2 = (37+35)(37-35) = 72 \cdot 2 = 144 \Rightarrow x = 12$$

Thus, SOHCAHTOA $\Rightarrow \sin\theta = \underline{\mathbf{\dfrac{12}{37}}}$.

B) With no loss of generality, assume the side of square *ABCD* is 3. Using the Pythagorean Theorem, the sides of ΔFCE are easily determined. Using the Law of Cosines on ΔFCE,

$$8 = 10 + 10 - 2 \cdot 10\cos\theta \Rightarrow \cos\theta = \frac{12}{20} = \underline{\mathbf{\frac{3}{5}}}.$$



Alternate solution:
Recognizing that ΔEFC is isosceles, draw the perpendicular bisector $\overline{CM}$ of the base $\overline{EF}$, where *M* is the midpoint of $\overline{EF}$. $\overline{CM}$ bisects $\angle FCE$.
Appling the Pythagorean Theorem, $CF = \sqrt{10},\ EF = \sqrt{8} = 2\sqrt{2}$

$$CM = \sqrt{8},\ \sin\alpha = \frac{\sqrt{2}}{\sqrt{10}},\ \cos\alpha = \frac{\sqrt{8}}{\sqrt{10}}\quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{8}{10} - \frac{2}{10} = \underline{\mathbf{\frac{3}{5}}}.$$

C) Using the Law of Sines $\left(\dfrac{BC}{\sin A} = \dfrac{AC}{\sin B}\right)$, we have $\dfrac{BC}{AC} = \dfrac{\sin A}{\sin B} = \dfrac{2}{\sqrt{7}}$

Squaring both sides and substituting for $\sin^2$, $\dfrac{\sin^2 A}{\sin^2 B} = \dfrac{1-\cos^2 A}{1-\cos^2 B} = \dfrac{4}{7}$

Cross multiplying, $7 - 7\cos^2 A = 4 - 4\cos^2 B \Leftrightarrow 7\cos^2 A - 4\cos^2 B = \underline{\mathbf{3}}$.
Amazingly, for a right triangle the answer is still 3.
If ΔABC is a right triangle, *A* cannot be the right angle. ($\overline{BC}$ would be the hypotenuse and $2 \not> \sqrt{7}$.)

If *B* is the right angle, then $AB = \sqrt{3}$ and $7\left(\dfrac{\sqrt{3}}{\sqrt{7}}\right)^2 - 4(0)^2 = 7 \cdot \dfrac{3}{7} - 0 = \underline{\mathbf{3}}$.

If *C* is the right angle, then $AB = \sqrt{11}$ and $7\left(\dfrac{\sqrt{7}}{\sqrt{11}}\right)^2 - 4\left(\dfrac{2}{\sqrt{11}}\right)^2 = 7 \cdot \dfrac{7}{11} - 4 \cdot \dfrac{4}{11} = \dfrac{49-16}{11} = \underline{\mathbf{3}}$.

Round 1 – continued

Alternate solutions/Generalizations to 1C (Norm Swanson):

Assume ΔABC is isosceles with $AB = AC = \sqrt{7}$ and $BC = 2$. Then: $\cos B = \dfrac{1}{\sqrt{7}} = \sin\left(\dfrac{A}{2}\right)$

Using the double angle formula, $\cos A = 1 - 2\sin^2\left(\dfrac{A}{2}\right) = 1 - 2\left(\dfrac{1}{7}\right) = \dfrac{5}{7} \Rightarrow 7\left(\dfrac{25}{49}\right) - 4\left(\dfrac{1}{7}\right) = \dfrac{21}{7} = \underline{\mathbf{3}}$.

Or

Consider the "collapsed" triangle ABC where $A = B = 0°$ and $C = 180°$.
Then: $7\cos^2 A - 4\cos^2 B = 7\cos^2 0° - 4\cos^2 180° = 7(1) - 4(1) = \underline{\mathbf{3}}$

Try proving this generalization:
In any triangle ABC, where $\dfrac{BC}{AC} = \dfrac{n}{m}$, $m^2\cos^2 A - n^2\cos^2 B = m^2 - n^2$

This is equivalent to the identity $\dfrac{b\cos A - a\cos B}{b-a} \cdot \dfrac{b\cos A + a\cos B}{b+a} = 1$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012 SOLUTION KEY

Round 2

A) The unit digits of both the powers of 2 and the powers of 3 repeat in cycles of 4.

$2^1 = \underline{2}, 2^2 = \underline{4}, 2^3 = \underline{8}, 2^4 = 1\underline{6}, 2^5 = 3\underline{2}$ For any positive integer k, 2^{4k} has a unit digit of 6.

$3^1 = \underline{3}, 3^2 = \underline{9}, 3^3 = 2\underline{7}, 3^4 = 8\underline{1}$ For any positive integer k, 3^{4k} has a unit digit of 1.

$2^{71} = \left(2^4\right)^{17} \cdot 2^3 = (\ldots 6)^{17} \cdot 8$.

Since the powers of a number ending in 6 always end in 6, 2^{71} ends in 8.

$3^{44} = \left(3^4\right)^{11} = (\ldots 1)^{11}$.

Since the powers of a number ending in 1 always end in 1, 3^{44} ends in 1.

$\left|(8-1)^3\right| = 7^3 = \underline{\mathbf{343}}$

B) $314_{(b)} = 3b^2 + 1b + 4$ and $132_{(b)} = 1b^2 + 3b + 2$

Therefore, $3b^2 + b + 4 = 2\left(b^2 + 3b + 2\right) \Leftrightarrow b^2 - 5b = b(b-5) = 0$ and b = **<u>5</u>**.

or, alternately, with digits of 1,2,3 and 4, the base must be at least 5. By trial and error, the first trial works.

$314_{(5)} = 3(25) + 5 + 4 = 84$
$132_{(5)} = 25 + 3(5) + 2 = 42$ and 84 = 2(42), so b = **<u>5</u>**.

C) Consider the first 12 natural numbers. 1 <u>2</u> <u>3</u> 4 5 6 7 8 <u>9</u> <u>10</u> 11 12

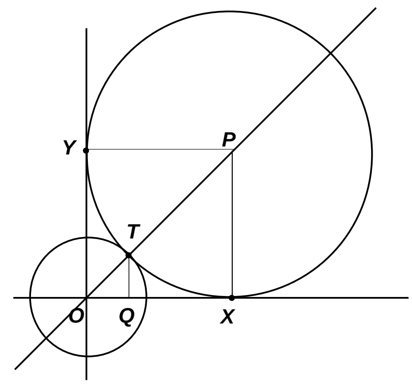
Exactly four of them are divisible by 2 or 3, but not 4 or 6, namely the underlined values.
Since 12 is the least common multiple of 2, 3, 4 and 6, it follows that
(N + 12) satisfies the required conditions if and only if N does. Thus, in the second block of 12 natural numbers, 14, 15, 21 and 22 satisfy the divisibility requirements.
In each block of 12 natural numbers, the four numbers satisfying the required conditions will always be the second, third, ninth and tenth numbers.
Since 117 = 29 · 4 + 1, the first 29 blocks will contain 116 numbers satisfying the divisibility requirements and the 117th natural number will be in the 30th block.
The first block ends in 12 = 12(1), the second in 24 = 12(2). The Nth block ends in 12N.
Thus, the last number in the 29th block is 29·12 = 348. The required number is the second number in the next block, namely **<u>350</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012 SOLUTION KEY

Round 3

A) Substituting the coordinates of the point of intersection into both equations,

$$\begin{cases} k = 6m+1 \\ k = \frac{2}{5}\cdot 6 - m \end{cases} \Rightarrow 6m+1 = \frac{12}{5} - m \Rightarrow 7m = \frac{7}{5} \Rightarrow m = \frac{1}{5} \text{ and } k = \underline{\mathbf{\frac{11}{5}}}$$



B) Note that m$\angle POX$ = 45°, so *OXPY* is a square and $a = b$.

$OT = 6 \Rightarrow OQ = 3\sqrt{2}$

Let $PX = PY = r$

$$\Delta TOQ \sim \Delta POX \Leftrightarrow \frac{TO}{PO} = \frac{QO}{XO} \Leftrightarrow$$

$$\frac{6}{r+6} = \frac{3\sqrt{2}}{r} \Rightarrow 2r = r\sqrt{2} + 6\sqrt{2} \Rightarrow$$

$$r = \frac{6\sqrt{2}}{2-\sqrt{2}} = \frac{6\sqrt{2}\left(2+\sqrt{2}\right)}{4-2} = 3\sqrt{2}\left(2+\sqrt{2}\right) = 6\sqrt{2}+6 = \underline{\mathbf{6\left(\sqrt{2}+1\right)}}$$

Alternate solutions:

Consider isosceles right ΔPOX. $OT = 6 \Rightarrow a\sqrt{2} = a + 6 \Rightarrow a = \frac{6}{\sqrt{2}-1}$ and the same result follows or, using the P.T. and the quadratic formula, $a^2 + a^2 = (a+6)^2 \Rightarrow a^2 - 12a - 36 = 0$

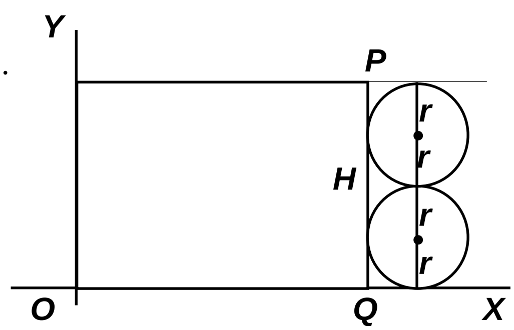
$$\Rightarrow \frac{12 \pm \sqrt{144 - 4(-36)(-1)}}{2} = \frac{12 \pm \sqrt{144(2)}}{2} = 6 + 6\sqrt{2} \quad \left(6 - 6\sqrt{2} < 0, \text{ rejected}\right)$$

C) $H = PQ = 4r$. Since *OQ* is the circumference of a base of a cylinder, $OQ = 2\pi r$ and $OP^2 = (2\pi r)^2 + \left(4r\right)^2 = 4r^2\left(\pi^2 + 4\right)$.

Thus, $OP = 2r\sqrt{\pi^2 + 4} \Rightarrow \left(A, B\right) = (1, 4)$

$$\frac{H^2}{Cr} = 2r \Leftrightarrow \frac{\left(4r\right)^2}{Cr} = 2r \Leftrightarrow 2Cr^2 = 16r^2 \Rightarrow C = 8$$

$(A, B, C) = \underline{\mathbf{(1, 4, 8)}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012 SOLUTION KEY

Round 4

A) $8^x = \sqrt[3]{\dfrac{2}{4^x}} \Leftrightarrow 2^{3x} = \left(\dfrac{2^1}{2^{2x}}\right)^{\frac{1}{3}} = \left(2^{1-2x}\right)^{\frac{1}{3}} = 2^{\frac{1-2x}{3}} \Leftrightarrow 3x = \dfrac{1-2x}{3} \Rightarrow 9x = 1-2x \Rightarrow x = \underline{\mathbf{\dfrac{1}{11}}}$.

Alternate solution:

Cubing both sides, $8^{3x} = \dfrac{2}{4^x} \Rightarrow 2^{9x} = \dfrac{2^1}{2^{2x}} = 2^{1-2x} \Rightarrow 9x = 1-2x \Rightarrow x = \underline{\mathbf{\dfrac{1}{11}}}$.

B) Since the x-intercept of $y = -3 + 4\log_{16} x$ is determined by letting $y = 0$, we have

$\log_{16} a = \dfrac{3}{4} \Leftrightarrow a = 16^{\frac{3}{4}} = 2^3 = 8$.

$$\left(\frac{\log_2 a^8}{\log_4 a^2}\right)^{\frac{1}{3}} = \left(\frac{\log_2 8^8}{\log_4 8^2}\right)^{\frac{1}{3}} = \left(\frac{\log_2\left(2^3\right)^8}{\log_4\left(4^3\right)}\right)^{\frac{1}{3}} = \left(\frac{24}{3}\right)^{\frac{1}{3}} = \underline{\mathbf{2}}$$

C) $y_2 = 81y_1 \Rightarrow y_2 = 81\left(2\left(4^x\right)\right) = 81\left(2^{2x+1}\right)$. Also, $y_2 = \dfrac{8^{x+2}}{4} = \dfrac{2^{3x+6}}{2^2} = 2^{3x+4}$

Thus, $81\left(2^{2x+1}\right) = 2^{3x+4}$

$\Rightarrow 81 = \dfrac{2^{3x+4}}{2^{2x+1}} = 2^{x+3} \Rightarrow x+3 = \log_2 81 = 2\log_2 9 = 4\log_2 3$

b as small as possible $\Rightarrow x = 4\log_2 3 - 3 \Rightarrow$ **<u>(4, 3, –3)</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012 SOLUTION KEY

Round 5

A) $.2A+.6B=B \Leftrightarrow 2A+6B=10B \Leftrightarrow A=2B$

$.3B+.1A=.3\left(\frac{A}{2}\right)+.1A=\left(\frac{3}{20}+\frac{1}{10}\right)A=\frac{1}{4}A \Rightarrow k=\underline{\mathbf{25}}$.

Alternately, pick a convenient value of A, say 100.
Applying the first condition, $20+.6B=B \Rightarrow 20=.4B \Leftrightarrow B=50$.
Remember k% is equivalent to k parts out of 100, i.e. $\frac{k}{100}$.

Applying the second condition, $10+15=\frac{k}{100}\cdot 100 \Rightarrow k=\underline{\mathbf{25}}$.

B) The variation rule is $F=\frac{km_1m_2}{d^2}$.

$F_1=0.004=\frac{4}{1000}=\frac{1}{250}$ Substituting, $\frac{1}{250}=k\cdot\frac{8}{144}=\frac{k}{18}\Rightarrow k=\frac{9}{125}$ (or 0.072)

For the second scenario, $F_2=\frac{\frac{9}{125}\cdot 3\cdot 6}{64}=\frac{81}{4000}$ (or 0.02025)

Thus, $(k,F_2)=\underline{\left(\mathbf{\frac{9}{125},\frac{81}{4000}}\right)}$ or $\underline{(\mathbf{0.072, 0.02025})}$. Of course, the lead zeros are optional.

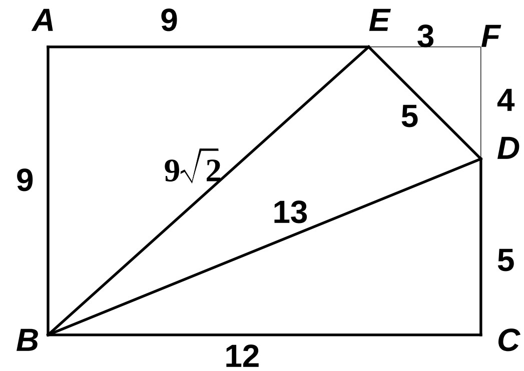
C) $\frac{(N+1)^3-N^3}{(A+1)^2-A^2}=\frac{3N^2+3N+1}{2A+1}=\frac{3N(N+1)+1}{2A+1}$

Note the numerator, as 3 times the product of two consecutive integers, plus 1 must be odd.
Therefore, the denominator must also be odd.
A = 1 $\Rightarrow$ denom = 3$\Rightarrow$ numerator = 39 (impossible $3N^2+3N-38=0$ has no integer solutions)
A = 3 $\Rightarrow$ denom = 7$\Rightarrow$ numerator = 91
($3N^2+3N-90=3\left(N^2+N-30\right)=3(N+6)(N-5)=0$)
Thus, for N = 5 and A = 3, the larger cube is 216 and the smaller square is 9 $\Rightarrow$ **<u>225</u>**.
As A increases, so does N. Therefore we have the smallest possible sum.

Alternate solution:
The differences Q between consecutive squares 1, 4, 9, 16, 25, … are 3, 5, 7, 9,… .
The differences P between consecutive cubes 1, 8, 27, 64, 125, … are 7, 19, 37, 61, …
Notice: In this second sequence, the differences between consecutive terms are 12, 18, 24, …, an amount that is increasing by 6. Thus, the next term is 61 + 30 = 91 = 13(7). Since we are looking for a term in the P sequence which is 13 times a term in the Q sequence, we have the first occurrence and $6^3 + 3^2$ = 216 + 9 = **<u>225</u>**.

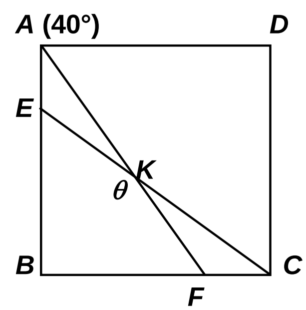
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012 SOLUTION KEY

Round 6



A) "Completing" the rectangle, we recognize two common right triangles, 3-4-5 and 5-12-13.
The required value is $\left(13+9\sqrt{2}\right)-9=\underline{\mathbf{4+9\sqrt{2}}}$.

B) The sum of the degree-measures of the 5 angles in any pentagon = (5 – 2)180 = 540.
Thus, in P, we have $2x^2+(13x+100)+120+170=540$.
$\Leftrightarrow 2x^2+13x-150=0$
$\Leftrightarrow (2x+25)(x-6)=0 \Leftrightarrow x=-\dfrac{25}{2}, 6$
$x=-\dfrac{25}{2}$ is rejected, because ($13x$ + 100) becomes negative.
$x=6 \Rightarrow$ 36, 36, 178, 120, 170 $\Rightarrow$ largest sum = **<u>348</u>**.



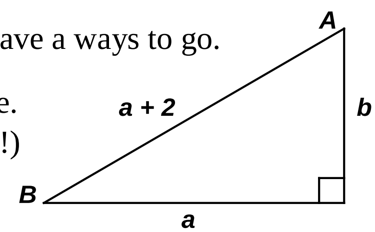
C) Let $m\angle EKF=\theta°$. $m\angle AFB=50°$
$AE = CF \Rightarrow BE = BF$.
Since $\Delta BEC \cong \Delta BFA$ (SAS), $m\angle BCE=40°$, $m\angle BEC=50°$
Thus, θ = 360 – 90 – 2(50) = **<u>170</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2012 SOLUTION KEY

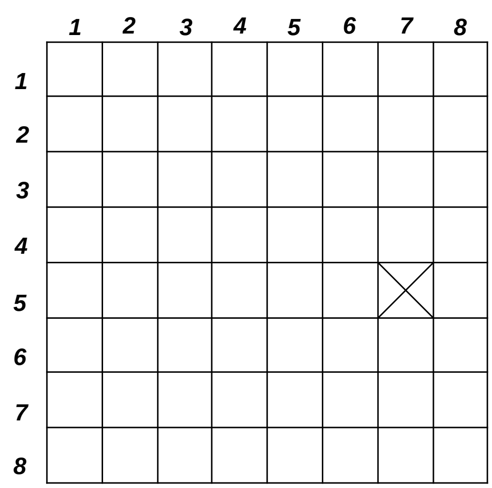
Team Round

b	a	c
6	8	10
8	15	17
10	24	26
12	35	37
14	48	50

A) Since $\cos A = \dfrac{b}{a+2}$, we could simply examine Pythagorean Triples, where the long leg and hypotenuse differ by 2. However, looking for a pattern could take a while. Let's take a different tact. Applying the Pythagorean Theorem,
$a^2 + b^2 = (a+2)^2 \Rightarrow b^2 = 4(a+1) \Rightarrow b = 2\sqrt{a+1}$
Since b must be an integer, a + 1 must be a perfect square
So, we only need consider a-values like 3, 8, 15, 24, 35, …. and the b-values are easy to compute, as are the hypotenuses (simply add 2 to a).



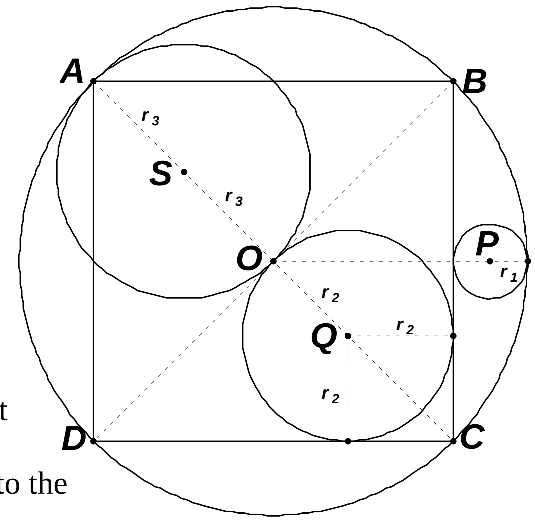
$a + 1 = 6^2 = 36 \Rightarrow a = 35 \Rightarrow$ 35-12-37 $\Rightarrow \cos A = \dfrac{12}{37} > \dfrac{12}{120} = 0.1$, so we have a ways to go.
$16^2 = 256 \Rightarrow a = 255 \Rightarrow b = 32$, $a + 2 = 257 \Rightarrow \cos A > 0.1$, so we continue.
$18^2 = 324 \Rightarrow a = 323 \Rightarrow b = 36$, $a + 2 = 325 \Rightarrow \cos A > 0.1$ (getting close!)
$19^2 = 361 \Rightarrow a = 360$ $b = 38$, $a + 2 = 362 \Rightarrow \cos A > 0.1$ (closer).
$20^2 = 400 \Rightarrow a = 399$ $b = 40$, $a + 2 = 401 \Rightarrow \cos A < 0.1$ (Bingo!).
Thus, the minimum perimeter is **840**.



B) Counting the squares of all possible sizes:
1 x 1: 64
A 2 x 2 square can have its upper left cell in any column, except column 8, and in any row, except row 8.
Thus, there are 7 x 7 = 49 squares of side 2.
3 x 3: 36 4 x 4: 25 5 x 5: 16
6 x 6: 9 7 x 7: 4 8 x 8: 1
Counting those that do not contain the "X":
1 x 1 squares $\Rightarrow$ 63
Of the 49 2 x 2 squares only 4 contain the "X" $\Rightarrow$ 45
The "X" would have to be in the UL, UR, LL or LR cell of the 2 x 2 square and all these squares fit on the grid.
Of the 36 3 x 3 squares only 6 contain the "X" $\Rightarrow$ 30
The "X" could only be in the center or rightmost columns of a 3 x 3 square.
Of the 25 4 x 4 squares only 8 contain the "X" $\Rightarrow$ 17
The "X" could only be in the rightmost two columns of a 4 x 4 square.
Of the 16 5 x 5 squares only 8 contain the "X" $\Rightarrow$ 8
Of the 9 6 x 6 squares 6 contain the "X" $\Rightarrow$ 3
All the 7 x 7 and 8 x 8 squares contain the "X".
The total "X"-less squares is then **166**.

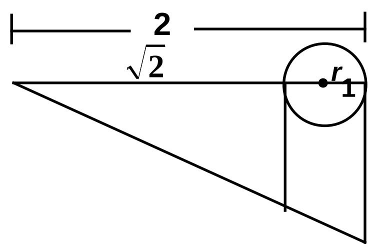
Team Round - continued

C) Let $R = OB = 2$ and $AS = r_3 = 1$.
Then $r_1 + r_2 + r_3 = kR \Leftrightarrow r_1 + r_2 + 1 = 2k$.



Since $m\angle OBC = 45°$ and the side of square $ABCD$ is $2\sqrt{2}$,

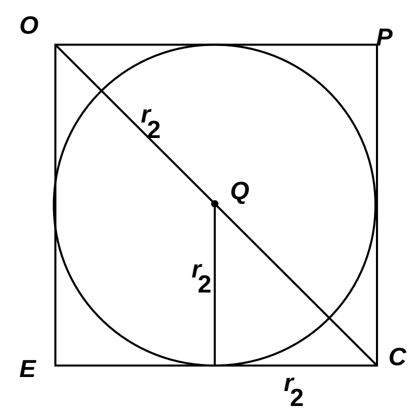
$r_1 = \dfrac{2-\sqrt{2}}{2} = 1 - \dfrac{\sqrt{2}}{2}$



Finding r_2 is the hardest.
Recall that the incenter of a triangle (as the intersection point of the angle bisectors) is equidistant from the three vertices.
The radius of the inscribed circle (center at Q) is equivalent to the area of the triangle (ΔBCD) divided by its semi-perimeter.

Since the area of the square is 8, we have $r_2 = \dfrac{4}{\dfrac{\left(4\sqrt{2}+4\right)}{2}} = \dfrac{2}{\sqrt{2}+1} \cdot \dfrac{\sqrt{2}-1}{\sqrt{2}-1} = 2\left(\sqrt{2}-1\right)$

Thus, $2k = \left(1 - \dfrac{\sqrt{2}}{2}\right) + 2\left(\sqrt{2}-1\right) + 1 \Rightarrow k = \underline{\mathbf{\dfrac{3\sqrt{2}}{4}}}$.



If you were unfamiliar with the relationship of the radius of the inscribed circle in a triangle and the area/perimeter of the triangle, consider the cutout diagram of the lower right corner of the overall diagram.

$OC = 2 \Rightarrow QC = r_2\sqrt{2} \Rightarrow r_2 + r_2\sqrt{2} = 2$

Solving, $r_2 = \dfrac{2}{\sqrt{2}+1} \cdot \dfrac{\sqrt{2}-1}{\sqrt{2}-1} = 2\left(\sqrt{2}-1\right)$ and the result follows.

Team Round - continued

D) For $x = 2$, the top level of the function rule applies and $f(2) = A + 35$.
The piecewise function is defined by the logarithmic component to the left of the vertical line $x = 2$ and by the exponential component to the right. As x approaches 2 from the left, $f(x)$ approaches

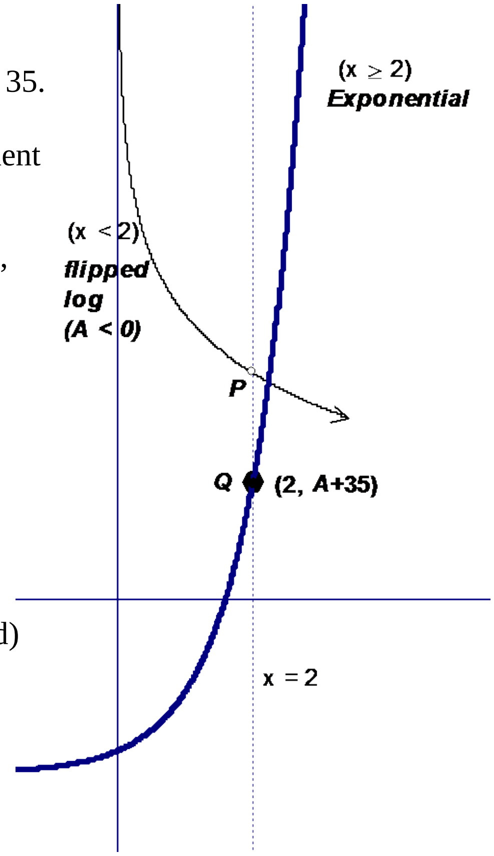
$A\log_4(2) + B = \dfrac{A}{2} + B$. If the function is to be continuous at $x = 2$,

then these function values must be equal, namely $A + 35 = \dfrac{A}{2} + B$.

Combining with $A + B = 17$, we have

$\dfrac{A}{2} + (17 - A) = A + 35 \Rightarrow A + 34 - 2A = 2A + 70 \Rightarrow A = -12$

Thus, (A, B) = **<u>(–12, 29)</u>**.
Graphically, since $A < 0$, the logarithmic piece is flipped.
Each piece is itself continuous, but there is a gap at $x = 2$
To close the gap the logarithmic piece must be translated (dropped) so that point P (the hole) coincides with the endpoint Q.



E) Given: $x = 2$, $y < 0$, $z > 0$ and $\dfrac{x+y}{z} = \dfrac{y+z}{x} = \dfrac{x+z}{y}$

Substituting for x, we have $\dfrac{2+y}{z} = \dfrac{y+z}{2} = \dfrac{2+z}{y}$

Cross multiplying the first two fractions, $4 + 2y = yz + z^2 \Leftrightarrow z^2 + yz - 2(y+2) = 0$
Since this is a quadratic equation in z, using the QF,

$$z = \frac{-y \pm \sqrt{y^2 + 8(y+2)}}{2} = \frac{-y \pm \sqrt{(y+4)^2}}{2} = \frac{-y \pm (y+4)}{2}$$

Thus, $z = 2, -(y + 2)$.
Case 1:

$z = 2 \Rightarrow \dfrac{2+y}{2} = \dfrac{y+2}{2} = \dfrac{2+2}{y} = \dfrac{4}{y}$ which is satisfied if $y^2 + 2y - 8 = (y+4)(y-2) = 0$

$\Rightarrow y = 2, -4 \Rightarrow (x, y, z) = \cancel{(2,2,2)}$ or $(2,-4,2)$ The first solution is rejected, since $y > 0$.

Case 2:
$z = -(y + 2) > 0 \Rightarrow y < -2$ and solution must be of the form $(2, y, -(y + 2))$.
Picking y as large as possible, we have $(2, -3, 1)$.
Thus, the maximum value of y is **<u>–3</u>**.

Team Round - continued

F) We require that $\frac{n(n-3)}{2} < \frac{180(n-2)}{n}$.

Since $n > 0$, we can cross multiply. $n^2(n-3) < 360(n-2) \Leftrightarrow n^3 - 3n^2 - 360n + 720 < 0$

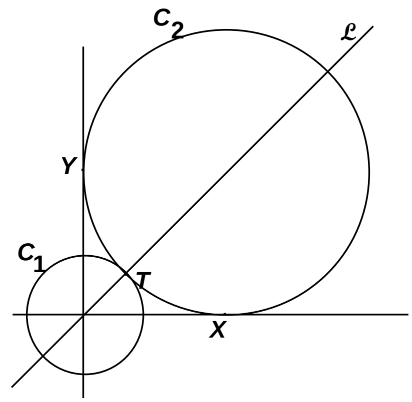
Using direct or synthetic substitution, we want the smallest n that satisfies the inequality.

$$\begin{array}{r|rrrrl} & 1 & -3 & -360 & 720 & \\ \hline 20 & 1 & 17 & -20 & >0 & \text{(20 sides fails)} \\ 19 & 1 & 16 & -56 & <0 & (\underline{\mathbf{19}}\text{ sides works}) \end{array}$$

Check: 19 sides: $\frac{19(16)}{2} = 152$ diagonals / $\frac{180(17)}{19} = 161^+$ degrees $\quad (152 < 161^+)$

20 sides: $\frac{20(17)}{2} = 170$ diagonals / $\frac{180(18)}{20} = 162$ degrees $\quad (170 \not< 162)$

Addendum:

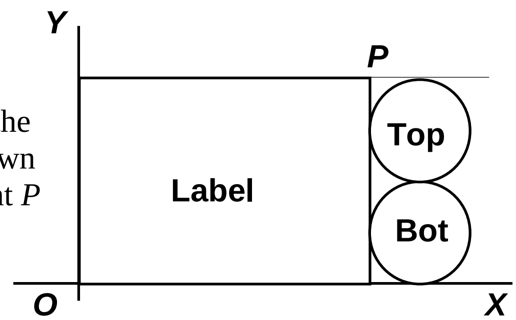


The original contest had two appeals in round 3

B) The original question was

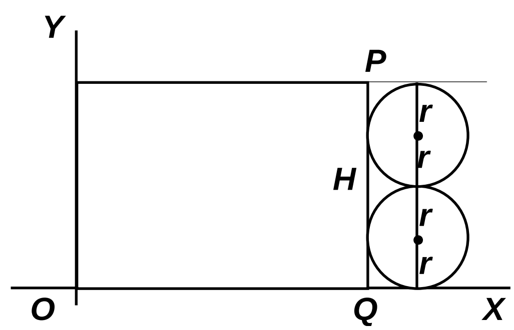
Let circle $C_1 = \{(x, y) \mid x^2 + y^2 = 36\}$ and line $\mathcal{L} = \{(x, y) \mid y = x\}$.
Circle C_2 has its center on $\mathcal{L}$ and is tangent to the x-axis at $X(a, 0)$, the y-axis at $Y(0, b)$ and circle C_1 at point T.
Compute the value of a.

In the second line the phrase <u>outside of C_1</u> was omitted and since there is a circle inside of C_1 which satisfies the verbally stated conditions of the problem, $6\left(\sqrt{2}-1\right)$ was also accepted.



C) The original question was

When removed, the label on a cylindrical can is a rectangle. Suppose the height (H) of the can is 4 times the radius (r) of the base. The label is placed in quadrant 1 of the xy-plane as shown in the diagram at the right. The distance from point O to point P can be expressed in terms of H and r in simplest form as $A\sqrt{B}\dfrac{H^2}{r}$, where A and B are positive constants and B is expressed in terms of π. Compute the ordered pair (A, B).



Since it was perfectly logical for a student to proceed
$OP^2 = H^2 + OQ^2 = H^2 + (2\pi r)^2$

Substituting for r, $H^2 + H^2 \cdot \dfrac{\pi^2}{4} = H^2\left(1 + \dfrac{\pi^2}{4}\right)$

$OP = H\sqrt{1+\dfrac{\pi^2}{4}}$ and $B = 1+\dfrac{\pi^2}{4}$

Now $\dfrac{AH^2}{r} = H \Rightarrow AH^2 = Hr = \dfrac{H^2}{4} \Rightarrow A = \dfrac{1}{4}$

An alternate answer of $\left(\dfrac{1}{4}, \dfrac{\pi^2}{4}+1\right)$ was also accepted.