MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2013
ROUND 1 ANALYTIC GEOMETRY: ANYTHING

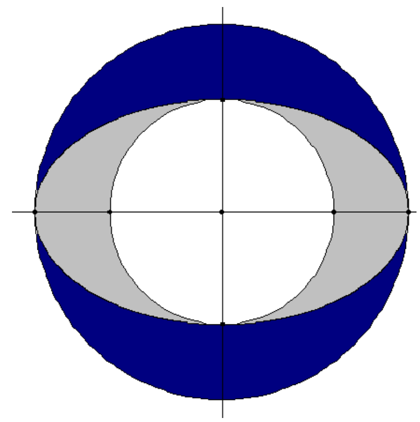
ANSWERS

A) ____________ : ______________

B) Center: (____ , ____) Radius: ____

C) (______ , ______ , ______ , _____)

A) Circle C_1 is tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ at the endpoints of the minor axis.
Circle C_2 is tangent at the endpoints of the major axis.
Compute the ratio of the shaded area outside the ellipse to the shaded area inside the ellipse.



B) The ellipses $2x^2+3y^2-8x+6y-48=0$ and $3x^2+2y^2-12x+4y-52=0$ intersect in four points which lie on a circle. Find the center and radius of this circle.

C) One of the asymptotes for a hyperbola whose transverse (major) axis is parallel to the *y*-axis is $\sqrt{3}x+y=2-3\sqrt{3}$. One of its foci is at (–3, –8). The equation of a hyperbola with axes parallel to the coordinate axes may be written in the form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ or $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.
Compute the ordered quadruple $\left(a^2,b^2,h,k\right)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013
ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) I am thinking of <u>three</u> integers for which $\sqrt[3]{x-3} = x-3$. What are they?

B) Compute <u>all</u> roots of $3\left(x^2-4\right)+x^2\left(x+2\right)=0$.

C) Factor over the integers: $x^2-4y^2+5x+2y+6$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013
ROUND 3 TRIG: EQUATIONS WITH A RESAONABLE NUMBER OF SOLUTIONS

ANSWERS

A) ______________________________ °

B) ______________________________

C) ______________________________

Unless otherwise indicated, list all answers in radian measure.

A) In quadrilateral $ABCD$, $m\angle B = 60°$. If $\sin A = \sin C$, but $A \neq C$, compute $m\angle D$ (in degrees).

B) Solve for x over $0 \leq x < 2\pi$. $\tan x - \cot x = 2\cos x \csc x$

C) Find the <u>number of solutions</u> over $0 \leq x < 2\pi$. $\sin 3x + \sin 5x + \sin 7x = 0$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013
ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Find <u>all</u> positive real numbers with the property that "six times the reciprocal of the number exceeds the number by 5".

B) Solve for *x*.

$$\frac{x+2}{x-1}-3=18\left(\frac{x-1}{x+2}\right)$$

C) The difference between the square of a <u>two-digit</u> natural number and the square of the sum of its digits is 2655.
Compute this <u>unique</u> natural number.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013
ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

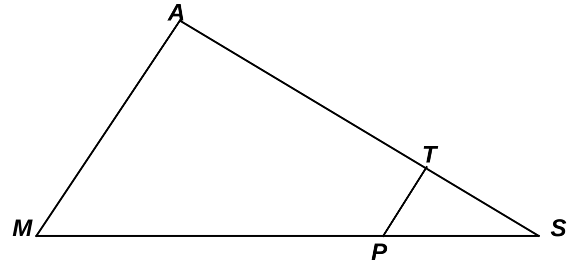
ANSWERS

A) _______________ : ________________

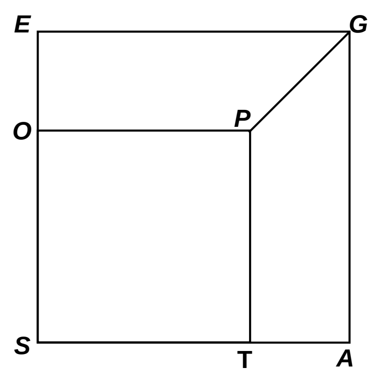
B) ___________________________________

C) ___________________________________

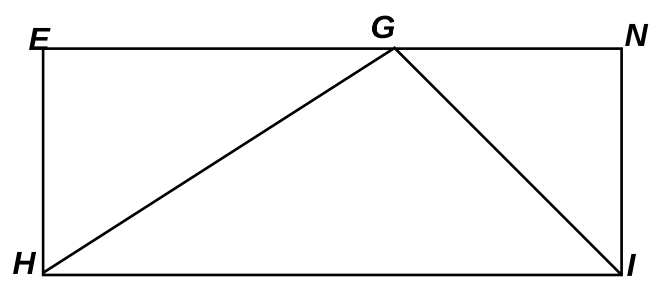
A) Given: area(ΔSTP) = 18, area(trapezoid $TAMP$) = 110
Compute $ST : TA$.



B) *POST* and *SAGE* are squares.
Their areas differ by 80.
$PG = 4\sqrt{2}$.
Compute the area of the larger square.



C) *NEHI* is a rectangle, $HE = 3$, $NE = a$, $GE > NG$
$\Delta HGE \sim \Delta GIN$, area(ΔGHI) = 9.375
Compute *GE*.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013
ROUND 6 ALG 1: ANYTHING

ANSWERS

A) ______________________________ ° C

B) ______________________________

C) ______________________________

A) At $-40°$, the Fahrenheit and Centigrade thermometers register the same temperature. Above that temperature, the absolute value of the Fahrenheit temperature is greater than the absolute value of the Centigrade temperature.

At what Centigrade temperature does a Fahrenheit thermometer register a temperature <u>exactly twice</u> that of a Centigrade thermometer?

Note: <u>An</u> equation relating <u>equivalent</u> Fahrenheit and Celsius temperatures is $C = \frac{5}{9}(F - 32)$.

.

B) Compute the <u>minimum</u> value of $\frac{10}{2x-1}$, if $3x^2 + 2x = 1$.

C) The number of positive integer solutions to $|2x - c| < 10$ is exactly twice the number of negative integer solutions for <u>exactly one</u> positive integer constant c. Compute c.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) (________ , ________)

B) ____________________________ E) ____________________________

C) (___ , ____ , ____ , ____ , ___) F) ____________________________

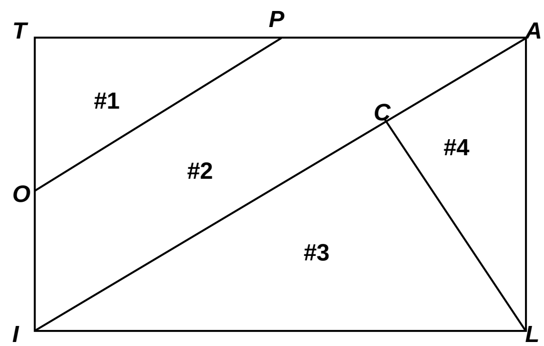
A) The set of points in the xy-plane equidistant from $F(2, 1)$ and the line $x + y = 0$ crosses one of the axes twice. Compute the coordinates of the points of intersection with that axis.

B) Compute <u>all</u> values of x for which $\left(\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{2}}}\right) \div \left(\dfrac{1}{2+\dfrac{1}{2+\dfrac{1}{x}}}\right) = -1$.

C) The curve represented by the parametric equations $\begin{cases} x = 5\cot(t) \\ y = 3\csc(t) \end{cases}$ may be expressed in the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where A, C, D, E and F are integers and $A > 0$. Determine the ordered 5-tuple (A, C, D, E, F).

D) For exactly two irrational values of the constant B, the equation $(2x-3)(Bx-1)=5$ has exactly one real root. Compute the ordered pair (P, Q), where $Q > 0$ and $\dfrac{P}{Q}$ is the reduced <u>rational</u> approximation of the <u>larger</u> value of B obtained by using the closest integer approximation for the simplified radical in the exact value of B.

E) Given: $LATI$ is a rectangle, O and P are midpoints, $\overline{LC} \perp \overline{IA}$, $LI = 5$, $LA = 3$
Compute the ratio of the areas of the four regions, listed from <u>smallest to largest</u>.
Diagram is not necessarily drawn to scale.



F) A fastfood restaurant has 5-piece chicken nuggets and 8-piece chicken nuggets on their value menu. A customer can <u>not</u> order individual chicken nuggets, so, for example, an order for 12 chicken nuggets is not possible. What is the <u>largest</u> number of nuggets that can not be ordered?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013 ANSWERS

Round 1 Analytic Geometry: Anything

A) 5 : 3 B) C: (2, –1) R: 5 C) $(75, 25, -3, 2)$

Round 2 Alg1: Factoring

A) 2, 3, 4 B) $-2, \frac{-3 \pm \sqrt{33}}{2}$ C) $(x+2y+2)(x-2y+3)$

Round 3 Trig: Equations

A) 120 B) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ C) 14

Answers may be listed in any order.

Round 4 Alg 2: Quadratic Equations

A) –6, 1 B) $\frac{1}{4}, \frac{8}{5}$ C) 52

Round 5 Geometry: Similarity

A) 3 : 5 B) 144 C) 4

Round 6 Alg 1: Anything

A) 160 B) –30 C) 3

Team Round

A) $\left(4 \pm \sqrt{6}, 0\right)$ D) (–2, 9)

B) –1, –2 E) 17 : 18 : 50 : 51

C) (9, –25, 0, 0, 225) F) 27

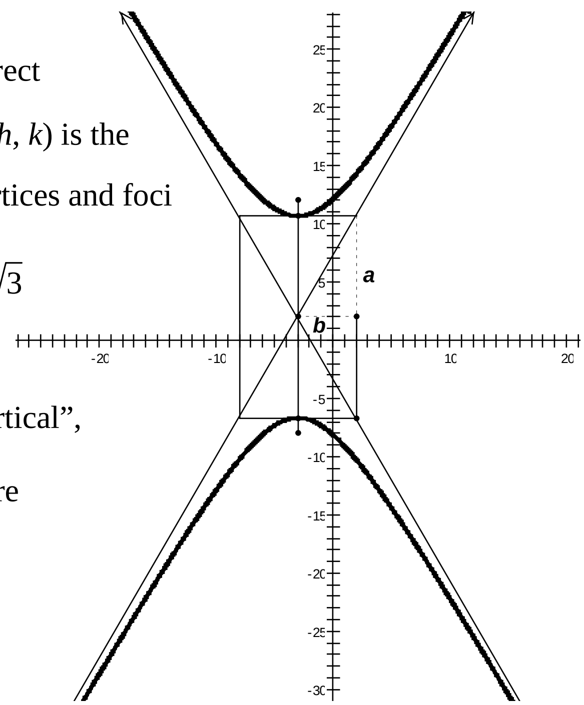
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013 SOLUTION KEY

Round 1

A) The semi-major axis (a) is 5; the semi-minor axis (b) is 3.
The area of the lightly shaded region is $\pi \cdot 3 \cdot 5 - \pi \cdot 3^2 = 6\pi$.
The area of the darkly shaded region is $\pi \cdot 5^2 - \pi \cdot 3 \cdot 5 = 10\pi$.
Thus, the required ratio is 5 : 3.
Convince yourself this ratio is always a : b for any ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$.

B) Adding the equations, we get $5x^2 + 5y^2 - 20x + 10y - 100 = 0 \Leftrightarrow x^2 + y^2 - 4x + 2y - 20 = 0$.
Completing the square,
$\left(x^2 - 4x \underline{+4}\right) + \left(y^2 + 2y \underline{+1}\right) = 20 \underline{+4+1} = 25 \Leftrightarrow (x-2)^2 + (y+1)^2 = 5^2$
The center is **<u>(2, –1)</u>** and the radius is **<u>5</u>**.
<u>FYI</u>: One of the 4 points is (6, 2) and the other three can be found without too much additional effort, using symmetry w.r.t. the center.

C) Since the transverse axis is parallel to the y-axis, the correct format of the equation is $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, where ($h$, k) is the center, a and c are the distance from the center to the vertices and foci respectively. For a hyperbola, $c^2 = a^2 + b^2$.
Rewriting the equation of the asymptote $\sqrt{3}x + y = 2 - 3\sqrt{3}$ in point slope form, we have $y - 2 = -\sqrt{3}(x+3)$ and the center must be at (–3, 2).
Given foci at (–3, –8) and knowing the hyperbola is "vertical", we have $c = 10$.
The slopes of the asymptotes of a "vertical" hyperbola are always $\pm\frac{a}{b}$. Therefore, $\frac{a}{b} = \frac{\sqrt{3}}{1}$ or $a^2 = 3b^2$.
Substituting, $c^2 = 4b^2 \Rightarrow b^2 = 25$.
Thus, $\left(a^2, b^2, h, k\right) = \underline{\mathbf{(75, 25, -3, 2)}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013 SOLUTION KEY

Round 2

A) For what numbers is the cube of the number equal to the number? $\Rightarrow x - 3 = -1, 0, 1 \Rightarrow$ **<u>2, 3, 4</u>**

or cube both sides and factor, $(x-3)=(x-3)^3 \Rightarrow (x-3)\left((x-3)^2-1\right)=0$

$\Rightarrow (x-3)\left(x^2-6x+8\right)=(x-3)(x-2)(x-4)=0 \Rightarrow x =$ **<u>2, 3, 4</u>**

B) Rather than expanding the sum, let's take out the common factor of $(x + 2)$.

$3\left(x^2-4\right)+x^2\left(x+2\right)=\left(x+2\right)\left(3(x-2)+x^2\right)=\left(x+2\right)\left(x^2+3x-6\right)=0$,

implying $x =$ **<u>–2</u>** and factoring the quadratic trinomial, $x=\dfrac{-3\pm\sqrt{9+24}}{2}=\underline{\boldsymbol{\dfrac{-3\pm\sqrt{33}}{2}}}$.

C) Suppose $\left(x+2y+2\right)\left(x-2y+3\right)$ factors as $\left(x+Ay+B\right)\left(x-Ay+C\right)$

The coefficients of the y-term must be the same, but opposite in sign, since there is no xy-term.

Multiplying out, we have $\begin{cases} A^2=4 \\ B+C=5 \\ AC-AB=A(B-C)=2 \\ BC=6 \end{cases}$, implying $A = 2$ or $A = -2$.

$A=2 \Rightarrow (B,C)=(2,3), \quad A=-2 \Rightarrow (B,C)=(3,2)$

In either case, we have $\underline{\boldsymbol{\left(x+2y+2\right)\left(x-2y+3\right)}}$**.**

An alternate solution uses completing the square. Show that

$x^2-4y^2+5x+2y+6=\left(x+\dfrac{5}{2}\right)^2-\left(2y-\dfrac{1}{2}\right)^2$ and the same result follows.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013 SOLUTION KEY

Round 3

A) Since $\sin A = \sin C$, but $A \neq C$, A and C must be supplementary. Let $(A, C) = (\theta, 180 - \theta)$.
Then: $\theta + 60 + (180 - \theta) + m\angle D = 360 \Rightarrow m\angle D = \underline{\mathbf{120}}$.

B) Since $2\cos x \csc x = 2\dfrac{\cos x}{\sin x} = 2\cot x$, we have $\tan x - 3\cot x = 0$
Multiplying by $\tan x$, $\tan^2 x - 3 = 0 \Rightarrow \tan x = \pm\sqrt{3}$ and the result follows.

C) Using the identity $\sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$, $\sin 3x + \sin 7x = 2\sin 5x \cos 2x$
Therefore, $\sin x + \sin 2x + \sin 3x = 0 \Rightarrow \sin 5x\left(2\cos 2x + 1\right) = 0$.
$\sin 5x = 0 \Rightarrow 5x = \begin{cases} 0 + 2n\pi \\ \pi + 2n\pi \end{cases} \Rightarrow 5x = 0 + n\pi \Rightarrow x = 0 + \dfrac{n\pi}{5}$ and as n assumes values from 0 to 9, we get 10 distinct solutions over the specified interval.
$2\cos 2x + 1 = 0 \Rightarrow \cos 2x = -\dfrac{1}{2} \Rightarrow x = \begin{cases} \dfrac{\pi}{6} + n\pi \\ \dfrac{5\pi}{6} + n\pi \end{cases}$ and as n assumes values of 0 and 1 we get 4 more solutions for a total of $\underline{\mathbf{14}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013 SOLUTION KEY

Round 4

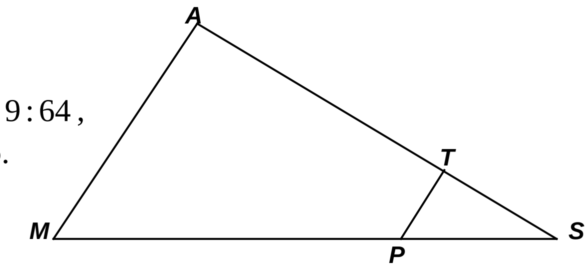
A) $6\left(\frac{1}{x}\right)=x+5 \Leftrightarrow x^2+5x-6=(x+6)(x-1)=0 \Rightarrow x=\underline{\mathbf{-6,\ 1}}$

B) Note: $x \neq 1, -2$
If a solution produces either of these values, they are extraneous.
Let $w = \frac{x+2}{x-1}$. Then $\frac{x+2}{x-1}-3=18\left(\frac{x-1}{x+2}\right) \Rightarrow w-3=18\left(\frac{1}{w}\right)$
$\Rightarrow w^2-3w-18=(w-6)(w+3)=0 \Rightarrow w = 6, -3.$
$\frac{x+2}{x-1}=6 \Rightarrow 6x-6=x+2 \Rightarrow 5x=8 \Rightarrow x=\underline{\mathbf{\frac{8}{5}}}$.
$\frac{x+2}{x-1}=-3 \Rightarrow -3x+3=x+2 \Rightarrow 4x=1 \Rightarrow x=\underline{\mathbf{\frac{1}{4}}}$.

C) Let $N=\underline{X}\,\underline{Y}=10X+Y$. Then:
$N^2=(10X+Y)^2=100X^2+20XY+Y^2$
Subtracting the square of the sum of the digits $\left(X^2+2XY+Y^2\right)$, we have
$99X^2+18XY=9X(11X+2Y)=2655 \Rightarrow X(11X+2Y)=295$
Either X is divisible by 5 or $11X+2Y$ is!
Knowing that both X and Y are single-digit integers,
we try $X = 5$, $55+2Y=59 \Rightarrow Y=2$
Could our unique two-digit number be 52?
To minimize number crunching, we take advantage of the *difference of perfect squares*.
Checking, $52^2-7^2=(52+7)(52-7)=(60-1)(45)=2700-45=2655$ Bingo!

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013 SOLUTION KEY

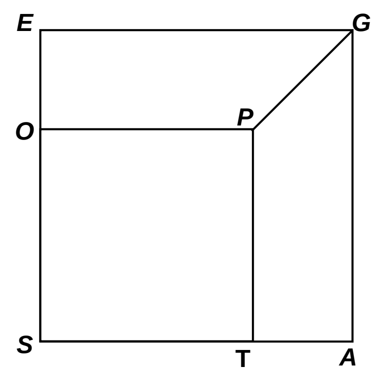
Round 5



A) $\overline{TP} \parallel \overline{AM} \Rightarrow \Delta STP \sim \Delta SAM$

Their areas are in a ratio of $18:(18+110)=18:128=9:64$,
implying their corresponding sides are in a 3 : 8 ratio.

$\dfrac{ST}{SA}=\dfrac{3}{8} \Rightarrow \dfrac{ST}{TA}=\underline{\mathbf{\dfrac{3}{5}}}$



B) Let $ST = x$ and $SA = kx$. Then:

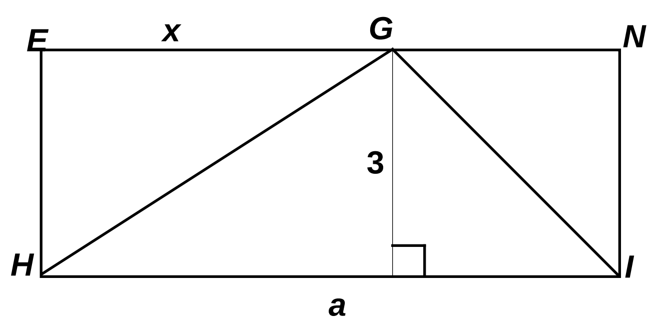
$PG = SG - SP = kx\sqrt{2}-x\sqrt{2}=4\sqrt{2} \Rightarrow x(k-1)=4$

$(xk)^2 - x^2 = 80 \Rightarrow x^2\left(k^2-1\right)=80$

Dividing, $x(k+1)=20$

Therefore, $\dfrac{\cancel{x}(k+1)}{\cancel{x}(k-1)}=\dfrac{20}{4}=5 \Rightarrow k+1=5k-5 \Rightarrow k=\dfrac{3}{2}, x=8$

$\Rightarrow$ area($SAGE$) $= 12^2 = \underline{\mathbf{144}}$.



C) Regardless of the location of point G on $\overline{NE}$,
the area of ΔGHI is half the area of the rectangle.

Therefore, $\dfrac{3a}{2}=9.375=9\dfrac{3}{8}=\dfrac{75}{8} \Rightarrow a=\dfrac{25}{4}=6.25$

Let $GE = x$. Then:

$\Delta HGE \sim \Delta GIN \Rightarrow \dfrac{GE}{IN}=\dfrac{HE}{GN}$

$\Rightarrow \dfrac{x}{3}=\dfrac{3}{a-x}=\dfrac{3}{\frac{25}{4}-x} \Rightarrow \dfrac{25}{4}x - x^2 = 9 \Rightarrow 4x^2-25x+36=(4x-9)(x-4)=0 \Rightarrow x=\dfrac{9}{4}=2.25 \text{ or } 4$

Since we were given that $GE > NG$, $GE = \underline{\mathbf{4}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013 SOLUTION KEY

Round 6

A) Substituting $2C$ for F and solving, $C=\frac{5}{9}(2C-32) \Leftrightarrow 9C=10C-160 \Leftrightarrow C=\underline{\mathbf{160}}$.

Check: $160 \stackrel{?}{=} \frac{5}{9}(320-32)$ $\qquad \frac{5}{9}(288)=5\cdot 32 \stackrel{\checkmark}{=} 160$

B) $3x^2+2x=1 \Leftrightarrow (3x-1)(x+1)=0 \Rightarrow x=\frac{1}{3},\ -1$

Substituting, $x=\frac{1}{3} \Rightarrow \frac{10}{2x-1}=-30$ and $x=-1 \Rightarrow -\frac{10}{3}$. Thus, the minimum value is **<u>–30</u>**.

C) $|2x-c|<10 \Leftrightarrow -10<2x-c<+10$. Isolating x, we have $\frac{c-10}{2}<x<\frac{c+10}{2}$.

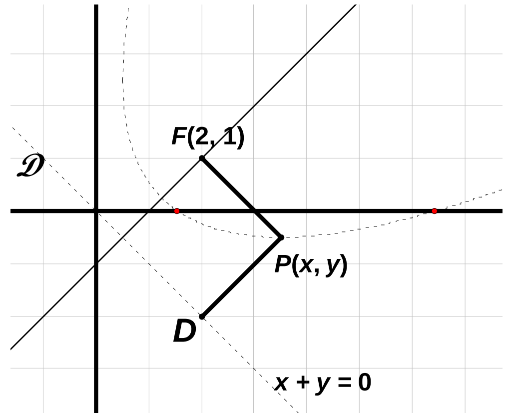
For values of $c \geq 10$, there are no negative solutions.
Thus, we examine positive integer values of $c < 10$.

$c = 9 \Rightarrow -\frac{1}{2}<x<\frac{19}{2} \Rightarrow$ 0 negative and 9 positive integer solutions.
$c = 8 \Rightarrow -1<x<9 \Rightarrow$ 0 negative and 8 positive integer solutions.
$c = 7 \Rightarrow -\frac{3}{2}<x<\frac{17}{2} \Rightarrow$ 1 negative and 8 positive integer solutions.

Continuing,… c = **<u>3</u>** $\Rightarrow -\frac{7}{2}<x<\frac{13}{2}$ and we have 3 negative solutions, namely x = –1, –2,–3, and 6 positive solutions, namely x = 1, …, 6.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2013 SOLUTION KEY

Team Round



A) The set of points satisfying the given conditions is a parabola, but the axis of symmetry is neither vertical nor horizontal.
The directrix $\mathcal{D}$ is $y = -x$ and the axis of symmetry is $y = x$. The graph is shown at the right, but the sketch is only included to reinforce the algebra. Applying the point-to-point and point-to-line distance formulas gets us the equation we need.

$$PF = PD \Rightarrow \sqrt{(x-2)^2+(y-1)^2} = \frac{|x+y+0|}{\sqrt{1^2+1^2}}$$

Squaring both sides, $2\left((x-2)^2+(y-1)^2\right) = |x+y|^2 = (x+y)^2$

$\Rightarrow 2x^2-8x+2y^2-4y+10 = x^2+2xy+y^2$

$\Rightarrow \left(x^2-8x\right)+\left(y^2-4y\right) = 2xy-10$ Completing the square, we have

$\left(x^2-8x+16\right)+\left(y^2-4y+4\right) = 2xy-10+16+4$

$\Rightarrow (x-4)^2+(y-2)^2 = 2(xy+5)$

Letting $x = 0 \Rightarrow 16+(y-2)^2 = 10$ confirms what is clear from the sketch, namely that the graph does not cross the y-axis. Letting $y = 0 \Rightarrow (x-4)^2+4 = 10 \Rightarrow x = 4\pm\sqrt{6}$ and the coordinates of the x-intercepts are $\underline{\left(\mathbf{4\pm\sqrt{6},0}\right)}$.

Extra challenge:
You might want to try solving for y in terms of x.
You should get $y = x+2\pm\sqrt{6-(x-4)^2+(x+2)^2}$.
Aaaargh!! For the contest director, this was necessary in order to draw the sketch above, since the available software did not have the capability of plotting implicit functions.

Team Round - continued

B) The first complex fraction simplifies to $\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{2}}}=\dfrac{1}{x+\dfrac{2}{2x+1}}=\dfrac{2x+1}{2x^2+x+2}$.

The second complex fraction simplifies to $\dfrac{1}{2+\dfrac{1}{2+\dfrac{1}{x}}}=\dfrac{1}{2+\dfrac{x}{2x+1}}=\dfrac{2x+1}{5x+2}$.

Thus, we require that $\dfrac{5x+2}{2x^2+x+2}=-1$

$\Leftrightarrow 2x^2+x+2=-5x-2$

$\Leftrightarrow 2x^2+6x+4=2(x+1)(x+2)=0$

$\Leftrightarrow x=\underline{\mathbf{-1,-2}}$

C) $\sin^2(t)+\cos^2(t)=1 \Rightarrow 1+\cot^2(t)=\csc^2(t)$

$x^2=25\cot^2(t)$ and $y^2=9\csc^2(t)$ Subtracting, $\dfrac{x^2}{25}-\dfrac{y^2}{9}=\cot^2 t-\csc^2 t=-1$

Thus, $9x^2-25y^2=-225$ or $9x^2-25y^2+225=0 \Rightarrow (A, C, D, E, F)=$ $\underline{\mathbf{(9, -25, 0, 0, 225)}}$

D) $(2x-3)(Bx-1)=5 \Leftrightarrow 2Bx^2-(2+3B)x-2=0$

To insure exactly one root, we set the discriminant equal to zero.

$b^2-4ac=\left(-(2+3B)\right)^2-4(2B)(-2)=0 \Rightarrow 9B^2+28B+4=0$

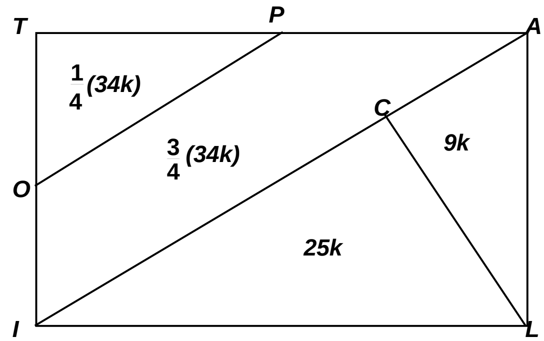
$$\Rightarrow B=\frac{-28\pm\sqrt{28^2-4(36)}}{18}=\frac{-28\pm\sqrt{4^2\left(7^2-9\right)}}{18}=\frac{-28\pm 8\sqrt{10}}{18}=\frac{-14\pm 4\sqrt{10}}{9}$$

The larger of the two values is $\dfrac{-14+4\sqrt{10}}{9}$. Substituting, 3 for $\sqrt{10}$, we have

$\dfrac{-14+4(3)}{9}=\dfrac{-2}{9}\Rightarrow (P,Q)=\underline{\mathbf{(-2,9)}}$.

The original Team D) question was simply "Approximate the larger of these two values <u>to the nearest hundredth</u>." How could you tackle this question without a calculator?
One possibility is outlined at the end of this solution key and is followed by a discussion of an algorithm for computation of square root without a calculator.

Team Round - continued



E) The area of rectangle *LATI* is 15.
The diagram contains a blizzard of similar triangles, namely $\Delta TOP \sim \Delta TIA \sim \Delta LAI \sim \Delta CAL \sim \Delta CLI$ and a pair of congruent triangle, namely $\Delta TIA \cong \Delta LAI$.
For $\Delta CAL \sim \Delta CLI$, the ratio of areas is
$\left(\frac{LA}{LI}\right)^2 = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$. Thus, we know that the area of
ΔTIA may also be represented as 34*k*.
Since $\frac{TO}{TI} = \frac{1}{2}$, the areas of triangles *TOP* and *TIA* are in a 1: 4 ratio.
Multiplying through by 2, the required ratio is #1 : #4 : #3 : #2 = **17 : 18 : 50 : 51**.

F) Consider the chart at the right, where consecutive integers are listed 8 per row.
In the top row, dividing by 8, these entries leave all possible remainders for division by 8, namely 1,2,3,4,5,6,7 and 0.
In each column, division by 8 leaves the same remainder, but division by 5 leaves a different remainder, since 8 and 5 are relatively prime. Only five remainders are possible for division by 5 and each column contains these five remainders, just "shuffled" into a different order.

1	2	3	4	***5***	6	7	*8*
9	***10***	11	12	13	14	***15***	*16*
17	18	19	***20***	21	22	23	*24*
25	26	27	28	29	***30***	31	*32*
33	34	***35***	36	37	38	39	***40***
~~41~~	~~42~~	~~43~~	~~44~~	~~45~~	~~46~~	~~47~~	~~48~~
~~49~~	~~50~~	~~51~~	~~52~~	~~53~~	~~54~~	~~55~~	~~56~~
~~57~~	~~58~~	~~59~~	~~60~~	~~61~~	~~62~~	~~63~~	~~64~~

For example, in column 1, the remainders of division by 5 are 1, 4, 2, 0, 3; in column 2, remainders = 2, 0, 3, 1, 4; in column 3, remainders = 3, 1, 4, 2, 0
As soon as a multiple of 5 is reached in each column, the next and subsequent entries can be written as a (linear) combination of 5 and 8. For example, in column 1, $33 = 5 \cdot 5 + 1 \cdot 8$;
in column 6, $46 = 6 \cdot 5 + 2 \cdot 8$ - etc.
The only chicken nugget purchases possible in the first row of the chart are 5 and 8.
Each entry in the rightmost column is a quantity (*Q*) that may be purchased as each is a multiple of 8-piece nuggets. In the leftmost 7 columns, the numbers above the underlined entry can not be expressed as a (linear) combination of 5 and 8, namely as $Q = 5x + 8y$.
(Notice that all entries after 40 can be written as a linear combination.)
Therefore, the largest possible number of nuggets that may not be purchased is **27**.
Note also that $27 = 8 \cdot 5 - (8 + 5)$.

It is left for you to verify that the maximum value is always $AB - (A + B)$, whenever *A* and *B* are relatively prime.

The original Team D) question:

For exactly two irrational values of the constant *B*, the equation $(2x-3)(Bx-1)=5$ has exactly one real root. **Approximate the larger of these two values <u>to the nearest hundredth</u>.**

A solution starts out the same:
$(2x-3)(Bx-1)=5 \Leftrightarrow 2Bx^2-(2+3B)x-2=0$
To insure exactly one root, we set the discriminant equal to zero.
$b^2-4ac=\left(-(2+3B)\right)^2-4(2B)(-2)=0 \Rightarrow 9B^2+28B+4=0$

$$\Rightarrow B=\frac{-28\pm\sqrt{28^2-4(36)}}{18}=\frac{-28\pm\sqrt{4^2\left(7^2-9\right)}}{18}=\frac{-28\pm 8\sqrt{10}}{18}=\frac{-14\pm 4\sqrt{10}}{9}$$

The larger of the two values is $\dfrac{-14+4\sqrt{10}}{9}$.

We need to approximate $\sqrt{10}$.

Since $3.2^2=10.24$ (an overestimate by 0.24) and $3.1^2=9.61$ (an underestimate by 0.39), we know $\sqrt{10}$ lies between 3.1 and 3.2, closer to 3.2 than 3.1, i.e. $\boxed{3.15<\sqrt{10}<3.20}$.
Since 3.16^2 = 9.9856 (slightly under our target value of 10), we have an outstanding approximation of $\sqrt{10}$ to two decimal places, an error of only 144 (actually 0.0144). For the cautious, since 3.17^2 = 10.0480 (an error of 520), 3.16 is definitely the best two decimal place approximation.
Substituting , $\dfrac{-14+4(3.16)}{9}=\dfrac{-1.36}{9}=-0.15\bar{1}\approx \underline{\mathbf{-0.15}}$.
For comparison, the actual value is approximately –0.150099 to 6 decimal places.

For those who would like to know how to compute square roots directly, READ ON!

Be patient!
Study the two examples worked out in detail and the accompanying dialogue. Then:
Try the four suggested problems.
ENJOY!

Algorithm for Extracting Square Root sans Calculator

An example: *Determine the best two-decimal place approximation of* $\sqrt{8.15}$.

Group digits to the left and to the right of the decimal point into blocks of two.
Since we want accuracy to two decimal places, we write 8.15 as 08.15 00 00
The third decimal place will tell us if we need to round up.

The first digit is the largest N for which $N^2 \le$ leftmost twosome. $N^2 \le 08 \Rightarrow N = 2$
Square N, subtract, and bring down the next twosome. Call this value X. $X = 415$
Double the current approximation (2) and write this value (4) in the space at the left
Let d denote the next digit in the approximation.
We want $(4\,d)\cdot d$ to be less than or equal to X, i.e. forty-something times something $\le$ 415
$(48)\cdot 8 = 384 < 415$, but $(49)9 = 441 > 415$, so the next digit is 8.

This is summarized in the following template:

$$
\begin{array}{ll}
 & \phantom{\sqrt{}}2.d \\
 & \sqrt{08.150000} \\
 & \underline{4} \\
4\underline{d} & 415 \\
 & \underline{384} \\
 & \boxed{d = 8}
\end{array}
$$

Continue repeating these steps until the required number of decimal places have been determined

- Double the current approximation
- Determine the next digit [largest d for which $(\ldots\, d)d \le X$]
- Multiply / Subtract / Bring down the next twosome

The devil is in the details which are shown in the diagrams below:

$$
\begin{array}{ll}
 & \phantom{\sqrt{}}2.\ 8\ \ d \\
 & \sqrt{08.150000} \\
 & \underline{4} \\
48 & 415 \\
 & \underline{384} \\
56\underline{d} & 3100 \\
\multicolumn{2}{l}{(565\cdot 5 = 2825 < 3100)} \\
\multicolumn{2}{l}{(566\cdot 6 = 4396 > 3100)} \\
 & \boxed{d = 5}
\end{array}
$$

$$
\begin{array}{ll}
 & \phantom{\sqrt{}}2.\ 8\ \ 5\ \ d \\
 & \sqrt{08.150000} \\
 & \underline{4} \\
48 & 415 \\
 & \underline{384} \\
565 & 3100 \\
 & \underline{2825} \\
570\underline{d} & 27500 \\
\multicolumn{2}{l}{(5704\cdot 4 = 22816)} \\
\multicolumn{2}{l}{(5705\cdot 5 > 27500)} \\
 & \boxed{d = 4}
\end{array}
$$

In practice, the calculations to determine d are not shown and all the computations are combined into a single template.
Thus, rounded to two decimal places, $\sqrt{8.15} = 2.85$.

Here are the details for $\sqrt{10}$:

$$\begin{array}{ll} & 3.\;1\;\;6\;\;d \\ & \sqrt{10.000000} \\ & \underline{9} \\ 6\underline{1} & 100 \\ & \underline{\;61} \\ 62\underline{6} & \;\;3900 \\ & \;\;\underline{3756} \\ 632\underline{d} & \;\;\;\;14400 \end{array}$$

$(6322 \cdot 2 = 12644)$

$(6323 \cdot 3 > 14400)$

$$\boxed{d = 2}$$

As expected, to two-decimal places, $\sqrt{10} = 3.16$.

<u>Suggested problems</u>

Try approximating $\sqrt{107}$ and $\sqrt{1525}$ to two decimal places.

9.8596 is a perfect square. Evaluate $\sqrt{9.8596}$

An acre has originally defined so that exactly 640 acres was equivalent to 1 square mile.
To the nearest integer, what is the length (in feet) of the side of a square whose area is 1 acre?

The following diagram (2 small squares and 2 rectangles inside a larger square) hints at why this algorithm works. Try explaining why the algorithm works?

$$(a+d)^2 = a^2 + 2ad + d^2$$