MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2013
ROUND 1 ALG 2: SIMULTANEOUS EQUATIONS AND DETERMINANTS

ANSWERS

A) ______________________________

B) (_____________ , _____________)

C) ______________________________

******* NO CALCULATORS IN THIS ROUND *******

A) Determine the unique value of k for which the following system has an <u>infinite</u> number of solutions.

$$\begin{cases} y = 2x - 3 \\ \dfrac{x}{2} - \dfrac{y}{4} = k \end{cases}$$

B) The system of equations $\begin{cases} y = 2x + A + B \\ y - A = \dfrac{1}{2}(x - B) \end{cases}$ intersect at $(-2, 1)$.

Compute (A, B).

C) Compute <u>all</u> values of k for which $\begin{vmatrix} 3k-5 & 3 \\ 4 & k-2 \end{vmatrix} = \begin{vmatrix} 1 & k-1 & -2 \\ 1 & 2 & -1 \\ 7-2k & -3 & 0 \end{vmatrix}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2013
ROUND 2 ALG1: EXPONENTS AND RADICALS

ANSWERS

A) ________________________________

B) ________________________________

C) _______________ : _______________

******* NO CALCULATORS IN THIS ROUND *******

A) Given: $\left(A+B\right)^3 = 258$, $(AB)(A+B) = 5$
Compute $A^3 + B^3$.

B) $\sqrt[3]{\dfrac{82-x}{2x+1}}$ is a positive integer.
Compute the <u>largest</u> possible integer value of x for which this is true.

C) The equation $\sqrt{4x+1} + \sqrt{10-x} = 7$ has two real solutions, one an integer and the other a rational fraction. The fractional solution is $\dfrac{A}{B}$. Compute the ratio $A : B$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2013
ROUND 3 TRIGONOMETRY: OPEN

ANSWERS

A) ______________________________ °

B) ______________________________

C) ______________________________ °

******* NO CALCULATORS IN THIS ROUND *******

A) The expression $\tan x - \sin x$ is evaluated for $x = 15°, 30°, 45°, 60°, 75°$.
For which of these values is the value of $\tan x - \sin x$ a <u>minimum</u>?

B) In ΔABC, $\cos A = \dfrac{13}{20}$ and $\cos C = \dfrac{37}{40}$. Compute the measure of B in degrees.
If necessary, write an expression in terms of $Arc\cos n$.

C) Given: $\begin{vmatrix} \cos x & \sin x \\ \sin y & \cos y \end{vmatrix} = \dfrac{1}{2}$ and $\begin{vmatrix} \cos x & -\sin x \\ \sin y & \cos y \end{vmatrix} = \dfrac{\sqrt{2}}{2}$
In <u>degrees</u>, compute the <u>smallest</u> positive value of x for which these conditions hold.

Note: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

FYI: This expression $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2013
ROUND 4 ALG 1: ANYTHING

ANSWERS

A) (_____________ , _____________)

B) _______________________________

C) (_____________ , _____________)

******* NO CALCULATORS IN THIS ROUND *******

A) A mathlete promises his coach that he will participate in all 6 MML meets in 2012-13.
He was disappointed in his performance in the first two meets, despite improving from 0 points in the first contest to 4 points in the second contest. His goal is to average 6 points per meet for the year – a personal best! An average score of N in the last 4 meets would allow him to reach his goal.
He actually scored 6, 10 and 12 in meets 3, 4 and 5.
If he scores k points in meet #6, he reaches his goal, a personal best!
Compute the ordered pair (k, N).

B) The difference $\left(6x^2 - 8x + 11\right) - \left(-5x^2 + kx + 10\right)$ factors over the integers as a unique pair of binomials for exactly one positive value of k. Compute k.

C) Paul started at P and ran towards Q. He ran a mile in 6 minutes.
Ron started at Q and ran towards P. He ran a mile in 9 minutes.
They both started running at the same time and met 1 hour later.

After a month of training, Paul, an avid runner, was able to reduce his mile time by 1 minute.
After a month of "training", Ron, a convicted couch potato, took 1 <u>more</u> minute per mile.

Running over the same course, the elapsed time until they met actually <u>decreased</u> by A minute and B seconds. Compute the ordered pair (A, B).
If necessary, <u>round B to the nearest integer</u>.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2013
ROUND 5 PLANE GEOMETRY: ANYTHING

ANSWERS

A) ________________________ °

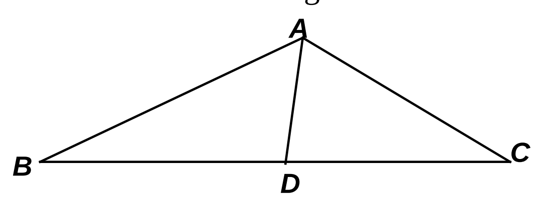
B) ________________________ °

C) (________ , ________ , ________)

******* NO CALCULATORS IN THIS ROUND *******

A) Regular nonagon DECATHLON is inscribed in circle P. Chords $\overline{CO}$ and $\overline{DT}$ intersect at point Q, compute the measure of the obtuse $\angle Q$.

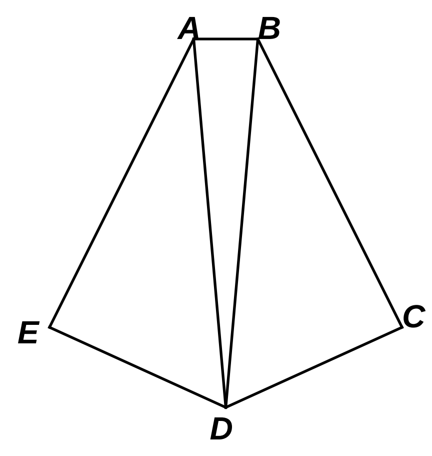
B) In ΔABC, point D is located on $\overline{BC}$ so that $\overline{AD}$ divides ΔABC into 2 similar triangles If $m\angle B = 37°$ and $AB \neq AC$, compute $m\angle BDA + m\angle C$.



C) Given: $\overline{AE} \perp \overline{ED}$, $\overline{BC} \perp \overline{DC}$, $\Delta AED \cong \Delta BCD$

$DE = 6\sqrt{3}$, $AD = 2.5 \cdot AB$ and $BD = 1.25 \cdot BC$

The area of $ABCDE$ can be written in the simplified form as $a\left(b+\sqrt{c}\right)$. Compute the ordered triple (a, b, c).



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2013
ROUND 6 ALG 2: PROBABILITY AND THE BINOMIAL THEOREM

ANSWERS

A) ______________________________

B) ______________________________

C) (_____________ , _____________)

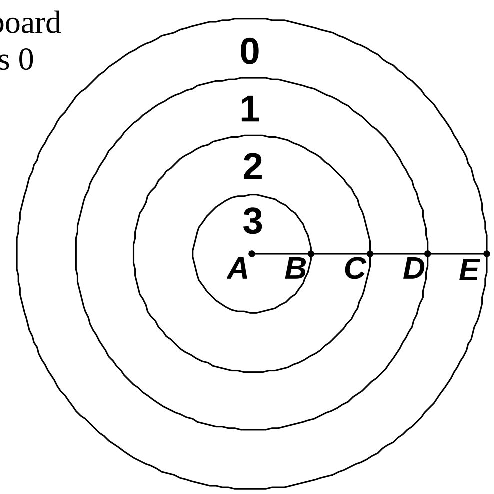
******* NO CALCULATORS IN THIS ROUND *******

A) Let $n(X)$ denote the number of integers in set X.
Let X U Y denote the union of two sets, i.e. the integers in either X or Y or possibly both.

Consider two sets A and B for which
$n(A) = 11$, $n(B) = 10$ and $n(A \text{ U } B) = 15$
Compute the probability <u>as a percentage</u> that a randomly selected integer from one of the sets is in both of the sets.

B) In the expansion of $(a+b)^n$, where n is an integer, the sum of the coefficients is 8^4 .
For $a = 4$ and $b = -\frac{1}{8}$, if n is even, evaluate the middle term in the expansion; if n is odd, evaluate the average of the two middle terms. Express your answer as a reduced ratio of integers.

C) A mathlete determines his score by throwing three darts at the board shown at the right. In this league, the minimum possible score is 0 and the maximum score is 9. The innermost circle scores 3 points, while each of the rings scores the points indicated in the diagram at the right. The mathlete hits the target with each dart. The probability of hitting any region is proportional to the area of the region. The probability of getting a score of 4, as a reduced fraction, is $\frac{P}{Q}$. If $AB = BC = CD = DE$, compute the ordered pair (P, Q). Assume the dart throws are independent.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 - MARCH 2013
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) (____________ , ____________)

B) (_______ , _______ , _______) E) (____________ , ____________)

C) ____________________________ F) ____________________________

******* NO CALCULATORS IN THIS ROUND *******

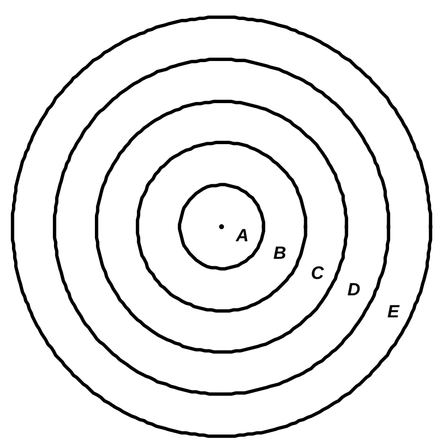
A) Let $A = \begin{bmatrix} 3 & 2 \\ 1 & k \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 \\ 2 & k-3 \end{bmatrix}$. Compute <u>all</u> values of k for which $\det(AB) = 60$.

B) The radical expression $\sqrt{18+\sqrt{35}}$ may be expressed in the form $\dfrac{x+\sqrt{y}}{\sqrt{z}}$, where x, y and z are positive integers. Compute the ordered triple (x, y, z) for which $x + y + z$ is a minimum.

C) Given: $\begin{cases} x = 3\sin(t)+1 \\ y = 2\cos(t)-5 \end{cases}$ for $0 \le t \le \pi$.
Let (X_M, Y_M) denote the maximum values of x and y respectively.
Let (X_m, Y_m) denote the minimum values of x and y respectively.
Compute $X_M Y_M - X_m Y_m$.

D) A record of better than 0.700 is quite an achievement in the modern-era of major league baseball. The best winning percentage of 0.716 belongs to the 2001 Seattle Mariners (116 wins – 46 losses). Suppose our local club's record is currently 72 wins and 38 losses. If we play k more games and lose <u>at least</u> 10 of them, compute the ordered pair (g, W), where g is the <u>minimum</u> value of k for which our winning percentage is <u>over</u> 0.700 and W is our total number of wins for the season.

E) The diagram at the right is a dartboard of concentric circles of radii 1, 2, 3, 4, and 5. I threw a dart and its location is determined by the point of intersection of two perpendicular chords of the largest circle whose lengths are 8 and 9. The lengths of the segments on the shorter chord are in a 25 : 7 ratio. The lengths of the segments on the longer chord are in a 5:1 ratio. Let k denote the region in which the dart landed. Let d denote the exact distance to the <u>nearest</u> circle. Compute the ordered pair (k, d).



F) Compute the coefficient of x^9 in the expansion of $\left(x^2+x-1\right)^6$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2013 ANSWERS

Round 1 Algebra 2: Simultaneous Equations and Determinants

A) $\frac{3}{4}$ B) (3, 2) C) 4, −10

Round 2 Algebra 1: Exponents and Radicals

A) 243 B) 27 C) 234 : 25

Round 3 Trigonometry: Open

A) 15 B) $Arc\cos\left(-\frac{5}{16}\right)$ or $180 - Arc\cos\left(\frac{5}{16}\right)$ [5/16 = 0.3125] C) 7.5

Round 4 Algebra 1: Anything

A) (4, 8) B) 4 C) (4, 27)

Round 5 Plane Geometry: Anything

A) 100 B) 143 C) (24, 6, 6)

Round 6 Algebra 2: Probability and the Binomial Theorem

A) 40% B) $\frac{231}{16}$ C) (39, 256)

Team Round

A) $4, -\frac{5}{6}$ D) (51, 113)

B) (1, 35, 2) E) $\left(D, \frac{1}{4}\right)$

C) −5 F) −10

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2013 SOLUTION KEY

Round 1

A) There will only be an infinite number of solutions if
these equations which look different are actually equivalent equations for the same line.
Multiplying the second equation by 4, we have $2x - y = 4k \Rightarrow y = 2x - 4k$.

Equating the expressions for *y*, $4k = 3 \Rightarrow k = \underline{\mathbf{\frac{3}{4}}}$.

B) Substituting, $\begin{cases} 1 = 2(-2) + A + B \\ 1 - A = \frac{1}{2}(-2 - B) \end{cases} \Leftrightarrow \begin{cases} A + B = 5 \\ 1 - A = \frac{-(2+B)}{2} \end{cases} \Leftrightarrow A - 1 = \frac{(7-A)}{2} \Leftrightarrow A = 3, B = 2 \Rightarrow \underline{\mathbf{(3,\ 2)}}$.

C) $\begin{vmatrix} 3k-5 & 3 \\ 4 & k-2 \end{vmatrix} = (3k-5)(k-2) - 12 = 3k^2 - 11k - 2$

Evaluating the 3 x 3 determinant using the weaving method:
Append copies of the entries in the left and middle columns to the original matrix.
Sum the three diagonal down-products. Call it S_1
Sum the three diagonal up-products. Call it S_2.
Subtract $(S_1 - S_2)$.

$$\begin{vmatrix} 1 & k-1 & -2 \\ 1 & 2 & -1 \\ 7-2k & -3 & 0 \end{vmatrix} \Rightarrow \begin{matrix} 1 & k-1 & -2 & 1 & k-1 \\ 1 & 2 & -1 & 1 & 2 \\ 7-2k & -3 & 0 & 7-2k & -3 \end{matrix}$$

$$\Leftrightarrow \left(1\cdot 2\cdot 0 + (k-1)\cdot -1\cdot(7-2k) + (-2\cdot 1\cdot -3)\right) - \left((7-2k)\cdot 2\cdot -2 + (-3\cdot -1\cdot 1 + 0\cdot 1\cdot(k-1)\right)$$

$$\Leftrightarrow \left(\left(2k^2 - 9k + 7\right) + 6\right) - \left(-28 + 8k + 3\right) = 2k^2 - 17k + 38$$

Equating and re-arranging terms, $k^2 + 6k - 40 = (k-4)(k+10) = 0 \Rightarrow k = \underline{\mathbf{4,\ -10}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2013 SOLUTION KEY

Round 2

A) Expanding $(A + B)^3$, we know $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$ or we expand in stages:

$(A+B)^3 = (A+B)(A+B)^2 = (A+B)\left(A^2 + 2AB + B^2\right)$ and the same result follows.

$A^3 + 3A^2B + 3AB^2 + B^3 = \left(A^3 + B^3\right) + 3AB(A+B) = 258$

Subtracting, using the second condition, we have $A^3 + B^3 = 258 - 3\cdot 5 =$ $\underline{\mathbf{243}}$.

B) To insure that the radicand is itself a positive integer, we require that

$82 - x \ge 2x + 1 \Rightarrow 3x \le 81 \Rightarrow x \le 27$

Thus, the largest possible x-value is 27 and $\sqrt[3]{\dfrac{82-27}{2\cdot 27+1}} = \sqrt[3]{\dfrac{55}{55}} = 1$, so $x =$ $\underline{\mathbf{27}}$ works.

<u>FYI</u>: For $x = 1$, 2 and 5, $\dfrac{82-x}{2x+1}$ evaluates to 27, 16 and 7.

For $x = 7$, 16 and 27, $\dfrac{82-x}{2x+1}$ evaluates to 5, 2 and 1. Did you expect this to happen?

C) Squaring both sides, $(4x+1) + 2\sqrt{4x+1}\sqrt{10-x} + (10-x) = 49$.

Combining like terms and isolating the radicals, $2\sqrt{4x+1}\sqrt{10-x} = 38 - 3x$

Squaring,

$4(4x+1)(10-x) = \left(38-3x\right)^2 \Leftrightarrow -16x^2 + 156x + 40 = 9x^2 - 284x + 1444$

$\Leftrightarrow 25x^2 - 384x + 1404 = 0$

This would be a pain to factor over the integers (if indeed it factored at all), except for the hint that there was an integer solution.

One of the factors must be of the form $(x - c)$, where c is an integer constant.

Thus, we start with $\left(25x - \square\right)\left(x - \square\right)$ and consider possible factorizations of

$1404 = 2^2 \cdot 3^3 \cdot 13$ until we find the one which gives the proper coefficient of the middle term.

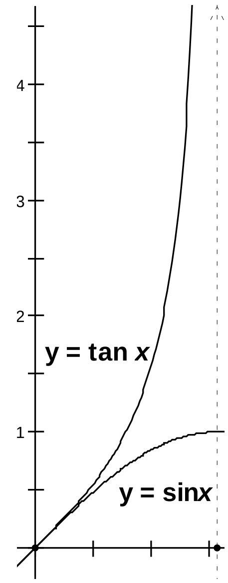
1404
2 702
2 351
3 117
3 39
3 13

$(25x - 234)(x - 6)$ gives the correct middle term (150 + 234 = 384), implying that the fractional solution is $\underline{\mathbf{\dfrac{234}{25}}}$. Note: Both solutions check: $\sqrt{25} + \sqrt{4} = 7$ and, converting $\dfrac{234}{25}$ to the equivalent decimal (9.36), $\sqrt{38.44} + \sqrt{0.64} = 6.2 + 0.8 = 7$

Alternate (**Better!**) Solution (Norm Swanson – Hamilton Wenham)

Guessing the integer solution 6 is a matter of a little trial and error. As above squaring both sides twice, we get $25x^2 - 384x + (\text{We don't care!}) = 0$ and, by inspection, the sum of the roots is $\dfrac{384}{25}$. Since the integer root is 6, the fractional root must be $\dfrac{384-150}{25} = \underline{\mathbf{\dfrac{234}{25}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2013 SOLUTION KEY



Round 3

A) The values of both $\tan x$ and $\sin x$ are 0 for $x = 0$. Over the interval $0 \le x < 90°$, both functions are increasing, but $\tan x$ is increasing faster, as is evidenced by the graphs at the right. Here is the numerical evidence:

At $x = 30°, \dfrac{\sqrt{3}}{3} - \dfrac{1}{2} \approx 0.1$; $x = 45°, 1 - \dfrac{\sqrt{2}}{2} \approx 0.3$; $x = 60°, \sqrt{3} - \dfrac{\sqrt{3}}{2} \approx 0.9$.

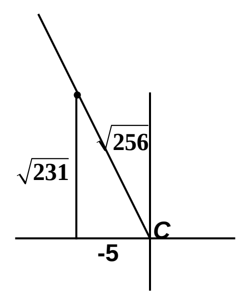
Thus, the smaller the angle, the smaller the difference and the minimum must occur for x = **15**.

Alternately, $\tan x - \sin x = \sin x\left(\dfrac{1}{\cos x} - 1\right)$. Over the -interval $0 \le x < 90°$, as x increases, $\sin x$ increases, $\cos x$ decreases, $\dfrac{1}{\cos x}$ increases and, consequently, $\dfrac{1}{\cos x} - 1$ also increases. Therefore, the product (and equivalently, the given difference) increases over this interval and the minimum occurs for the smallest value of x. For either point of view, number crunching the given values of x was not necessary.

B) Since $B = 180 - (A + C)$, $\cos B = \cos\big(180 - (A + C)\big) = -\cos(A + C)$. Expanding, we have

$$\cos B = -\cos A \cos C + \sin A \sin B = -\frac{13}{20} \cdot \frac{37}{40} + \frac{\sqrt{400 - 169}}{20} \cdot \frac{\sqrt{1600 - 1369}}{40} = \frac{-481 + 231}{800} = -\frac{250}{800} = -\frac{5}{16}$$

Since $\cos B < 0$, B must be an obtuse angle. $\boldsymbol{Arc\cos\left(-\frac{5}{16}\right)}$ denotes an obtuse angle (i.e. in quadrant 2). The corresponding angle in quadrant 1 is $Arc\cos\left(\dfrac{5}{16}\right)$ and the required obtuse angle could also be represented as is $\boldsymbol{180 - Arc\cos\left(\frac{5}{16}\right)}$.



Alternately, using a lesser known identity: (In any ΔABC, $\tan A \cdot \tan B \cdot \tan C = \tan A + \tan B + \tan C$**)**

$$\frac{\sqrt{231}}{13} \cdot \frac{\sqrt{231}}{37} \cdot \tan C = \frac{\sqrt{231}}{13} + \frac{\sqrt{231}}{37} + \tan C \Leftrightarrow \tan C\left(1 - \frac{231}{13 \cdot 37}\right) = -\frac{50\sqrt{231}}{13 \cdot 37} \Leftrightarrow (481 - 231)\tan C = -50\sqrt{231}$$

Since $\tan C = -\dfrac{\sqrt{231}}{5} < 0$, C must be obtuse, so $\cos C = -\dfrac{5}{\sqrt{256}} = -\dfrac{5}{16}$ and the result follows.

C) $\cos x \cos y - \sin x \sin y = \cos(x + y) = \dfrac{1}{2} \Rightarrow x + y = \pm 60° + 360n$ (quadrants 1, 4)

$\cos x \cos y + \sin x \sin y = \cos(x - y) = \dfrac{\sqrt{2}}{2} \Rightarrow x - y = \pm 45 + 360m$ (quadrants 1, 4)

Adding, $2x = \pm 15 + 360k$ or $\pm 105 + 360k$, where $k = n + m$

Thus, $x = \pm 7.5 + 180k$ or $\pm 52.5 + 180k$. The smallest positive value of x is **7.5**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2013 SOLUTION KEY

Round 4

A) Let x denote his average score in the last 4 meets. Then:

$$\frac{0+4+4x}{6}=6 \Leftrightarrow 4x=32 \Rightarrow x=8$$

He needs 36 points for the year to have an average of 6.
Through 5 meets he has scored 32 points, so he needs only 4 points in meet 6.
Thus, (k, N) = <u>**(4, 8)**</u>.

B) $\left(6x^2-8x+11\right)-\left(-5x^2+kx+10\right)=11x^2-\left(8+k\right)x+1$

This only possible factorizations of this trinomial are $\left(11x-1\right)\left(x-1\right)$ or $\left(11x+1\right)\left(x+1\right)$.
Thus, the coefficient of the middle term is ±12.
Equating, $-(8+k)=\pm 12 \Rightarrow 8+k=\mp 12 \Rightarrow k=\cancel{-20}, \underline{\mathbf{4}}$.

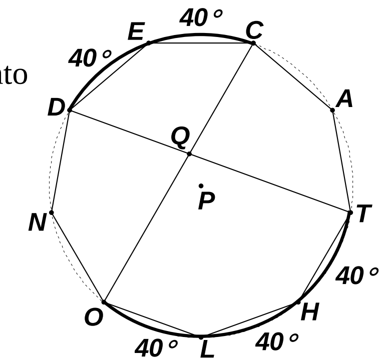
C) At 6 minutes per mile, Paul runs at 10 mph.
At 9 minutes per mile, Ron runs at $\frac{60}{9}=\frac{20}{3}$ or $6\frac{2}{3}$ mph.
Since they meet in 1 hour, PQ = $10\cdot 1+\frac{16}{3}\cdot 1=16\frac{2}{3}$ miles.
Reducing his mile time by 1 minute, increases Paul's rate to $\frac{60}{6-1}=12$ mph
Increasing his mile time by 1 minute decreases Ron's rate to $\frac{60}{9+1}=6$ mph.
Assuming they meet in T hours at these new rates, $12T+6T=16\frac{2}{3} \Rightarrow 54T=50 \Rightarrow T=\frac{25}{27}$ hour

$$\frac{25}{27}\text{ hr}=\frac{25}{27}\cdot 60=\frac{500}{9}=55\frac{5}{9}\text{min}=55\text{min}\,\frac{100}{3}\text{sec}=55\text{min}\,33\frac{1}{3}\text{sec}$$

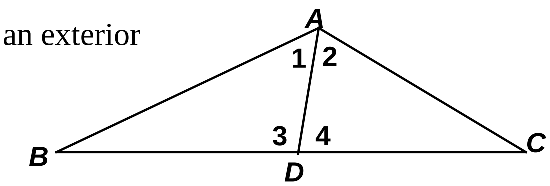
Thus, the elapsed time to cover the distance from P to Q decreased by 4 minutes $26\frac{2}{3}$ seconds, and we have (A, B) = <u>**(4, 27)**</u>.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2013 SOLUTION KEY

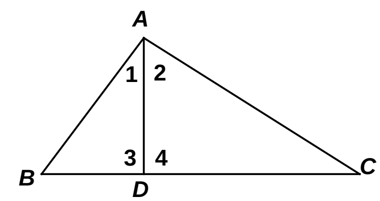
Round 5



A) The nine vertices of regular polygon DECATHLON divide circle P into $\frac{360°}{9} = 40°$ arcs. Since the measure of an angle formed by intersecting chords of a circle equals the average of the intercepted arcs, we have $m\angle DQC = m\angle OQT = \frac{80+120}{2} = \underline{\mathbf{100}}°$.



B) $\angle 1$ in ΔBAD is congruent to $\angle 2$, 4 or C in ΔADC. As an exterior angle of ΔBAD, $\angle 4$ can not be congruent to $\angle 1$ (or $\angle B$).
If $\angle 1$ were congruent to $\angle 2$, then $\angle C$ would have to be congruent to $\angle B$ and $\angle 3$ would have to be congruent to $\angle 4$, forcing $\Delta BAD \cong \Delta CAD$ and $AB = AC$ which is not the case.
Thus, $m\angle 1 \neq m\angle 2$.
The only alternative left is
$m\angle 1 = m\angle C \Rightarrow m\angle 2 = m\angle B = 37°$, $m\angle 3 = m\angle 4 = 90°$ and $m\angle 1 = 53°$
Thus, the required sum is $\underline{\mathbf{143}}°$.



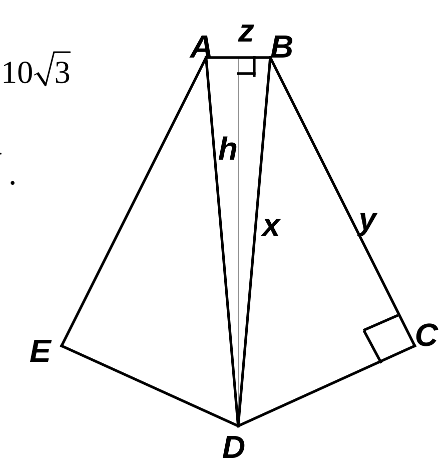
Alternate Solution: $\Delta BAD \sim \Delta ACD$ (Other correspondences lead to contradictions as argued above.) Thus, $m\angle C = m\angle 1$. In ΔBAD, since $m\angle B = 37°$, $m\angle BDA = 143 - m\angle 1 = 143 - m\angle C$,
Transposing, $m\angle BDA + m\angle C = \underline{\mathbf{143}}$. A more accurate diagram is shown above.

C) Let (BD, BC, AB) = (x, y, z) as indicated in the diagram.
Drop a perpendicular (h) from D to $\overline{AB}$. Then: $x = \frac{5}{2}z$, $x = \frac{5}{4}y$

Since $DE = 6\sqrt{3}$, the required area is $2 \cdot \frac{1}{2} \cdot 6\sqrt{3} \cdot y + \frac{1}{2}zh = 6\sqrt{3} \cdot y + \frac{zh}{2}$

$$\begin{cases} y^2 + 108 = x^2 \\ x = \frac{5}{4}y \end{cases} \Rightarrow \frac{9}{16}y^2 = 108 \Rightarrow y^2 = 16(12) \Rightarrow y = 8\sqrt{3} \text{ , } x = 10\sqrt{3}$$

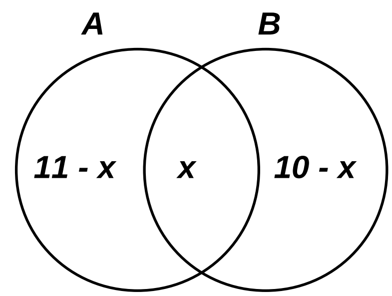
and $z = 4\sqrt{3}$ Thus, $h^2 = 300 - 12 = 288 = 144(2) \Rightarrow h = 12\sqrt{2}$.
The required area is $48(3) + 24\sqrt{6} = 24\left(6 + \sqrt{6}\right) \Rightarrow$ **(24, 6, 6)**.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 6 – MARCH 2013 SOLUTION KEY

Round 6

A) Let $n(X)$ denote the number of integers in set X.
Let X U Y denote the union of two sets, i.e. the integers in either X or Y or possibly both.
Consider two sets A and B.
$n(A) = 11$, $n(B) = 10$ and $n(A \text{ U } B) = 15$



Let x denote the number of integers in the overlap.
Then: $(11 - x) + x + (10 - x) = 21 - x = 15 \Rightarrow x = \underline{\mathbf{6}}$.
Thus, there are 6 out of 15 integers which belong to both sets and the probability is $\frac{6}{15} = \frac{2}{5} = \underline{\mathbf{40\%}}$.

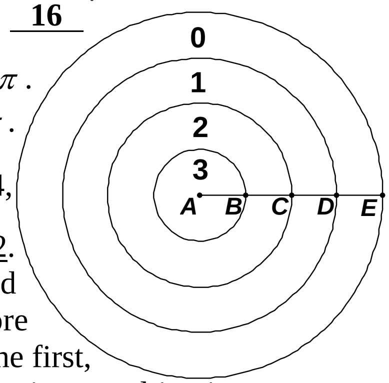
Alternate interpretation: Appeal accepted
The question required choosing an integer from <u>one</u> of the sets. This can be done by first deciding from which set (A or B) to choose my number and then, choosing the number. For example, let A be { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 } and B be { 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 }. Under this interpretation, there are 21 elements that can be chosen (11 from A and 10 from B) Each of the 6 integers, 6 thru 11, could be chosen from A or from B. Thus, there are 12 possible successful draws from 21 possible, resulting in a percentage of $\frac{4}{7} \cdot 100 = \underline{\mathbf{\frac{400}{7} = 57\frac{1}{7} = 57.\overline{142857}}}$ %.

Any of the underlined answers are acceptable under this interpretation. **6/21 is rejected**, since this counts the 6 overlapping integers twice in the denominator and once only in the numerator.

B) Since the sum of the coefficients in the expansion of $(a+b)^n$ is 2^n and $8^4 = 2^{12}$, we know that $n = 12$. The expansion of $(a+b)^{12}$ has 13 terms, the middle term is the 7th term, that is,

$$\binom{12}{6} a^6 b^6 = \frac{\cancel{12} \cdot 11 \cdot \cancel{10} \cdot 9 \cdot 8 \cdot 7}{1 \cdot \cancel{2} \cdot \cancel{3 \cdot 4} \cdot \cancel{5} \cdot 6} \cdot 4^6 \cdot \left(-\frac{1}{8}\right)^6 = 11 \cdot 3 \cdot 4 \cdot 7 \cdot 2^{12} \cdot 2^{-18} = \frac{3 \cdot 7 \cdot 11}{2^4} = \underline{\mathbf{\frac{231}{16}}}.$$

C) If $AB=BC=CD=DE=1$, the areas of the concentric circle are: $\pi, 4\pi, 9\pi, 16\pi$.
The areas of the central circle and the three outer rings are: π, 3π, 5π, 7π.



The probabilities of hitting the regions are: $\frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}$ To score a 4, the three darts must hit (in any order) <u>0, 1 and 3</u> or <u>0, 2 and 2</u> or <u>1, 1 and 2</u>.
In the first case, the first dart could hit any one of the 3 regions, the second dart, any two of the remaining regions, resulting in 6 possible ways to score 4 points. In the last two cases, the singleton score could be the result of the first, second or third dart, but both of the remaining darts must hit the "other" region, resulting in only 3 possible ways to score 4 points. Thus, the probability is

$$6\left(\frac{7}{16} \cdot \frac{5}{16} \cdot \frac{1}{16}\right) + 3\left(\frac{7}{16} \cdot \frac{3}{16} \cdot \frac{3}{16}\right) + 3\left(\frac{5}{16} \cdot \frac{5}{16} \cdot \frac{3}{16}\right) = \frac{210 + 189 + 225}{16^3} = \frac{624}{4096} = \frac{2^4 \cdot 39}{2^{12}} = \frac{39}{256}$$

$\Rightarrow (P, Q) = \underline{\mathbf{(39, 256)}}$.

MASSACHUSETTS MATHEMATICS
CONTEST 6 - MARCH 2013 SOLUTION KEY

Team Round

A) $\begin{bmatrix} 3 & 2 \\ 1 & k \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 2 & k-3 \end{bmatrix} = \begin{bmatrix} 3\cdot 4+2\cdot 2 & 3\cdot -1+2(k-3) \\ 1\cdot 4+2k & 1\cdot -1+k(k-3) \end{bmatrix} = \begin{bmatrix} 16 & 2k-9 \\ 2k+4 & k^2-3k-1 \end{bmatrix}$

Taking the determinant,

$16\left(k^2-3k-1\right)-(2k+4)(2k-9)=16k^2-48k-16-4k^2+10k+36=12k^2-38k+20=60$

Therefore,

$12k^2-38k-40=2(6k^2-19k-20)=2(k-4)(6k+5)=0$

Thus, $k = \underline{\mathbf{4, -\frac{5}{6}}}$.

Alternative solution:

Invoking the theorem $\det(AB)=\det(A)\cdot\det(B)$, we have

$\det\begin{bmatrix} 3 & 2 \\ 1 & k \end{bmatrix} = 3k-2,\ \det\begin{bmatrix} 4 & -1 \\ 2 & k-3 \end{bmatrix} = 4k-12+2=4k-10$

Then: $(3k-2)(4k-10)=60 \Rightarrow 12k^2-38k-40=0$ and the same result follows.

It is left to <u>you</u> as an exercise to prove the theorem that

for all 2 x 2 matrices A and B, $\det(AB)=\det(A)\cdot\det(B)$.

Is this true for any square matrices?

Prove (or disprove) your contention.

Send your proof or counterexamples to olson.re@gmail.com .

The best write-ups will be included in the solution set of the next contest.

Everyone likes to see themselves in print, except certain babies who don't like cash!

Team Round - continued

B) Notice that $\left(1+\sqrt{35}\right)^2 = 36+2\sqrt{35}$. Dividing both sides by 2, we have $\left(\dfrac{1+\sqrt{35}}{\sqrt{2}}\right)^2 = 18+\sqrt{35}$.

Taking the square root, $\sqrt{18+\sqrt{35}} = \dfrac{1+\sqrt{35}}{\sqrt{2}} \Rightarrow$ **<u>(1, 35, 2)</u>** . (Clearly, 1+35+2 = 38 is a minimum.)

<u>An alternative solution</u>, finding an infinite set of solutions, might proceed as follows:

Squaring both sides, $\sqrt{18+\sqrt{35}} = \dfrac{x+\sqrt{y}}{\sqrt{z}}$ becomes $\begin{cases} (1) \quad \dfrac{x^2+y}{z} = 18 \\ (2) \quad \dfrac{2x\sqrt{y}}{z} = \sqrt{35} \Rightarrow 4x^2y = 35z^2 \end{cases}$

If $z > y > x$ and each variable represents a positive integer, then for some positive integers a and b, $z = y + a$ and $y = x + b$.

Substituting in (1), $(y-b)^2 + y = 18(y+a) \Rightarrow (y-b)^2 = 17y + 18a$.

Substituting in (2), $4(y-b)^2 y = 35(y+a)^2 \Rightarrow 4(17y+18a)y = 35(y+a)^2$

$\Rightarrow 68y^2 + 72ay = 35y^2 + 70ay + 35a^2 \Rightarrow 33y^2 + 2ay - 35y^2 = (33y+35a)(y-a) = 0$

$\Rightarrow a = y \Rightarrow z = 2y$.

Eqtn (1) above now becomes $x^2 + y = 36y \Rightarrow x^2 = 35y \Rightarrow x^2 = 35(x+b) \Rightarrow x^2 - 35x - 35b = 0$

Using the quadratic formula, $x = \dfrac{35 \pm \sqrt{35^2 + 35(4)b}}{2} = \dfrac{35 \pm \sqrt{35(35+4b)}}{2}$.

The radicand must be a perfect square.

To force a perfect square radicand, $(35 + 4b)$ must be 35 times a perfect square.

To that end, let $b = 35n$. Then: $35 + 4b = 35(1 + 4n)$. For $n = 0, 2, 6, 12, 20, \ldots$ the radicand satisfies the requirement. Thus, $b = 35n$ and $x = \dfrac{35\left(1 \pm \sqrt{4n+1}\right)}{2}$.

Note: If we use the "–" sign for any value of n, $x \le 0$, violating x's status as a positive integer.

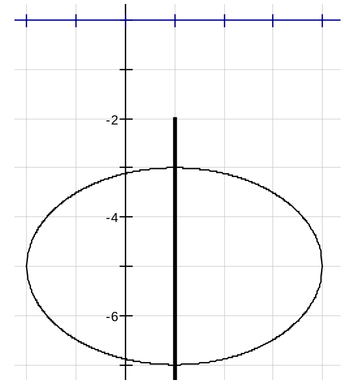
$$n = \begin{cases} 0 \\ 2 \\ 6 \\ 12 \\ 20 \\ \ldots \end{cases} \Rightarrow x = \frac{35}{2} \cdot \begin{cases} 1+1 \\ 1+3 \\ 1+5 \\ 1+7 \\ 1+9 \\ \ldots \end{cases} \Rightarrow x = 35 \cdot \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \ldots \end{cases}$$

The <u>smallest</u> sum is generated by the triple for $n = 0$ (or $b = 35$), namely $(x, y, z) = (35, 35, 70)$.

However, $\dfrac{x+\sqrt{y}}{\sqrt{z}} = \dfrac{35+\sqrt{35}}{\sqrt{70}} = \dfrac{35/\sqrt{35}+\sqrt{35}/\sqrt{35}}{\sqrt{70}/\sqrt{35}} = \dfrac{1+\sqrt{35}}{\sqrt{2}} \Rightarrow (x, y, z) = \underline{\mathbf{(1, 35, 2)}}$.

MASSACHUSETTS MATHEMATICS
CONTEST 6 - MARCH 2013 SOLUTION KEY

Team Round - continued



C) $\begin{cases} x = 3\sin(t)+1 \\ y = 2\cos(t)-5 \end{cases} \Rightarrow \left(\dfrac{x-1}{3}\right)^2 = \sin^2(t)$ and $\left(\dfrac{y+5}{2}\right)^2 = \cos^2(t)$

Adding, we have the equation of a **<u>semi-ellipse</u>**, namely

$\dfrac{(x-1)^2}{9} + \dfrac{(y+5)^2}{4} = 1$, where $1 \le x \le 4$

(since $0 \le t \le \pi$ (not 2π)). Thus, the graph is the right half of the ellipse.
The center is at (1, –5) and the major axis is horizontal, a = 3 and b = 2.
The major axis connects (–2, –5) and (4, –5) , and X_M = 4, but X_m = +1.
The minor axis connects (1, –3) and (1, –7) , Y_M = –3 and X_m = –7.
Thus, $X_M Y_M - X_m Y_m = -12 + 7 =$ **<u>–5</u>**.

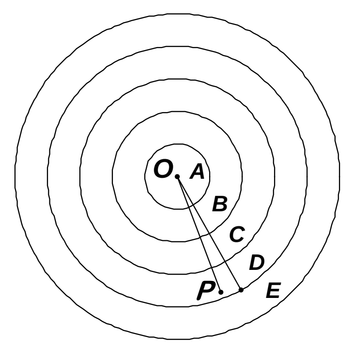
Alternatively, by inspection, the largest and smallest values of x are 1+3(1) = 4 and 1+3(0) = +1.
The largest and smallest values of y are –5±2(1) = –3 and –7. Thus, $X_M Y_M - X_m Y_m =$ **<u>–5</u>**.

D) Assume the club wins N of the remaining k games, i.e. suffers $k - N$ losses. Then:

$\dfrac{72+N}{110+k} > 0.700 = \dfrac{7}{10} \Leftrightarrow 720 + 10N > 770 + 7k \Leftrightarrow N > \dfrac{7k+50}{10}$ and

$k - N \ge 10 \Leftrightarrow N \le k - 10$ By the transitive property, $k - 10 > \dfrac{7k+50}{10} \Leftrightarrow 10k - 100 > 7k + 50 \Leftrightarrow k > 50$

Thus, $k_{\min}$ = 51 and our best record is attained if we lose only 10 of these, i.e. win 41 more games. (g, W) = (51, 72 + 41) = **<u>(51, 113)</u>**.
Check: 113/161 = 0.701$^+$ and 112/161 = 0.695$^+$



E) On $\overline{ST}$, the 25 : 7 ratio on chord of length 8 $\Rightarrow 32x = 8 \Rightarrow 6\frac{1}{4}, 1\frac{3}{4}$

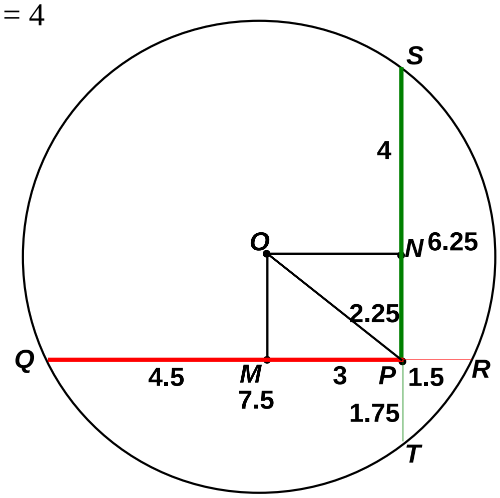
On $\overline{QR}$, the 5 : 1 ratio on chord of length 9 $\Rightarrow 6x = 9 \Rightarrow 7\frac{1}{2}, 1\frac{1}{2}$

Let O be the center of the circle of radius 5, P be the intersection point of the two chords of lengths 8 and 9.

Since a radius drawn perpendicular to any chord bisects the chord, let M and N be the midpoints. $QM = RM = 4.5$ and $SN = NT = 4$
Applying the Pythagorean Theorem in ΔOMP,

$2.25^2 + 3^2 = \left(\dfrac{9}{4}\right)^2 + 9 = \dfrac{81+144}{16} = \dfrac{225}{16} \Rightarrow OP = \dfrac{15}{4} = 3\dfrac{3}{4}$

Thus, the dart fell in region D, $\dfrac{1}{4}$ unit inside the circle of radius 4, implying (k, d) = $\underline{\left(\boldsymbol{D, \frac{1}{4}}\right)}$. **Think we are home free? <u>Read on!</u>**



Thanks to Canton mathletes who, post facto, discovered that Houston we do have a problem, despite the fact that the above solution appears to solve the question nicely.

The segments on the intersecting chords have been color coded to remind us that the product of the lengths of the segments on each chord must be equal.
***For those for whom geometry is a distance memory, the proof is included below.

On chord $\overline{QR}$, the product is $(7.5)\cdot 1.5 = 11.25$, but
on chord $\overline{ST}$, the corresponding product is $(6.25)\cdot 1.75 = 10.9375$. Close, but no cigar!

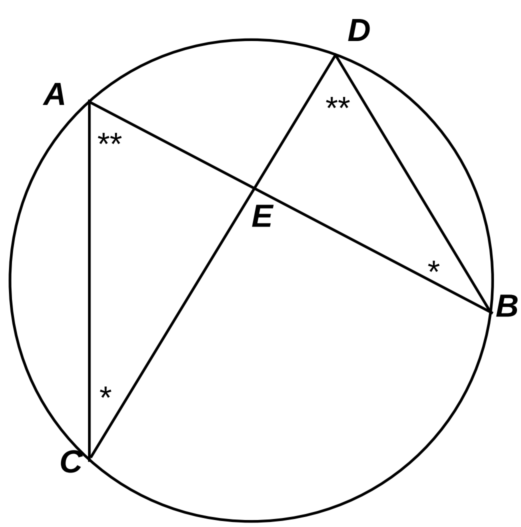
In a circle of radius 10, perpendicular chords of length 8 and 9 can certainly be drawn, but if the short chord divides the long chord into a 5 : 1 ratio, the long chord can <u>not</u> divide the short chord into a 25 : 7 ratio. These conditions are inconsistent and no such circle can be constructed.

The author and the proofreaders missed this inconsistency and at least 4 teams (including Canton) submitted this **best possible wrong answer** to a problem which is not solvable as stated.

Only Canton lodged an appeal to the official answer. (*D*, ¼) or any answer which stated "no answer due to inconsistent conditions" receives credit.

Anyone able to replace these inconsistent conditions with conditions allowing a solution, without drastically changing the problem, should send his ideas to olson.re@gmail.com.

***
Here's a reminder of why the product-chord theorem must be satisfied.
$m\angle A = m\angle D$ and $m\angle B = m\angle C$, since each pair are inscribed angles intercepting the same arc in the circle. Thus, $\Delta ACE \sim \Delta DBE$ by AA.
It follows that the lengths of corresponding sides are proportional, namely $\frac{AE}{DE} = \frac{CE}{BE}$ and, cross multiplying, $AE\cdot BE = DE\cdot CE$.



Team Round - continued

F) Consider the trinomial expansion $\left(x^2+x-1\right)^6$ to be a binomial expansion $\left(x^2+\left(x-1\right)\right)^6$.

The k^{th} term in the expansion is $\binom{6}{k}\left(x^2\right)^{6-k}(x-1)^k = \binom{6}{k}x^{12-2k}\left(x-1\right)^k$

If $k \geq 4$, then the largest exponent of x never exceeds 8.
If $k \leq 1$, then the smallest exponent of x always exceeds 9.
Therefore, we restrict our attention to $k = 2$ and 3.

$k = 2 \Rightarrow \binom{6}{2}x^8\left(x-1\right)^2 = 15(x^8)(x^2-2x+1) \Rightarrow -30x^9$
$k = 3 \Rightarrow 20x^9$
Combining terms, we have <u>**–10**</u>x^9.

<u>An alternative solution</u> applies the <u>Multinomial Theorem</u>. For the expansion of $\left(x^2+x-1\right)^6$, the only terms that will contain x^9 are: $\left(x^2\right)^3\left(x\right)^3\left(-1\right)^0$ or $\left(x^2\right)^4\left(x\right)^1\left(-1\right)^1$

The corresponding multinomial coefficients are $\dfrac{6!}{3!3!0!} = \dfrac{5!}{3!} = 20$ and $-\dfrac{6!}{4!1!1!} = -\dfrac{6\cdot 5}{1} = -30$ and the required coefficient is again <u>**–10**</u>.

<u>Here's the idea behind the multinomial expansion</u> of $\left(x_1+x_2+...+x_m\right)^n$, illustrated for $\left(A+B+C\right)^9$.
Clearly, the first term in the expansion would be A^9 and the last term, C^9.
Will there be terms like A^3B^6, $A^2B^3C^4$ and B^3C^7 ? There are 9 factors of $\left(A+B+C\right)$ to be multiplied and from each trinomial factor, a single monomial multiplier must be selected each time. Thus, the sum of the exponents must be 9 and the last term is <u>not</u> possible. If A is selected as the monomial multiplier 3 times, B 6 times (and C not at all), we will get an A^3B^6 term.
How many are there in the expansion of $\left(A+B+C\right)^9$? 9 objects in groups of 3 identical and 6 identical $\Rightarrow \dfrac{9!}{3!6!} = 84$, which would the coefficient of A^3B^6.

Any *AA BBB CCCC* arrangement will produce $A^2B^3C^4$ and there are $\dfrac{9!}{2!3!4!} = 1260$ such arrangements. Thus, the numerator of the multinomial coefficient will always be n! The sum of the exponents in any term of the expansion must always be n.
The denominator will always be the product of the factorials of the individual exponents.
In general, the multinomial coefficient of the term $x_1^{a_1}x_2^{a_2}\cdot\ldots\cdot x_m^{a_m}$, where $a_1+a_2+...+a_m = n$ in the expansion of $\left(x_1+x_2+...+x_m\right)^n$ is $\dbinom{n}{a_1\ ,a_2\ ,...,a_m} = \dfrac{n!}{a_1!\cdot a_2!\cdot\ \ldots\ \cdot a_m}$.