MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 – OCTOBER 2013
ROUND 1 VOLUME & SURFACES

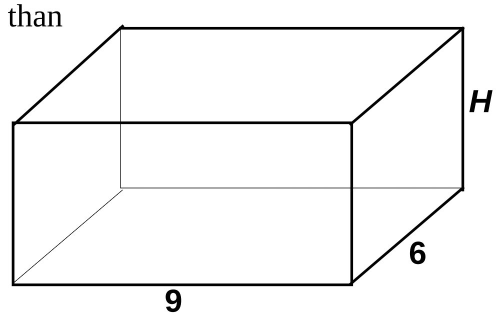
ANSWERS

A) ____________________________ feet

B) __________________________ units ^{3}

C) __________________________ units ^{3}

A) The <u>lateral</u> surface area of a rectangular solid (a box) is 50% more than the sum of the areas of the other two surfaces.
As indicated in the diagram at the right, the dimensions of the base are 6 feet by 9 feet. Compute the height of the box in feet.



B) The total surface area of a cylinder is 484π square units. The circular base of the cylinder has a diameter which is twice the height of the cylinder. Compute the volume of the cylinder.

C) A sphere has a surface area of 96π units^2. A regular hexagon is inscribed in a great circle of this sphere. A pyramid with this hexagonal base has its vertex on the sphere. Compute the <u>maximum</u> possible volume of this pyramid.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013
ROUND 2 PYTHAGOREAN RELATIONS IN RECTILINEAR FIGURES

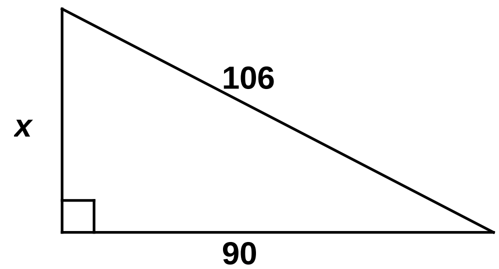
ANSWERS

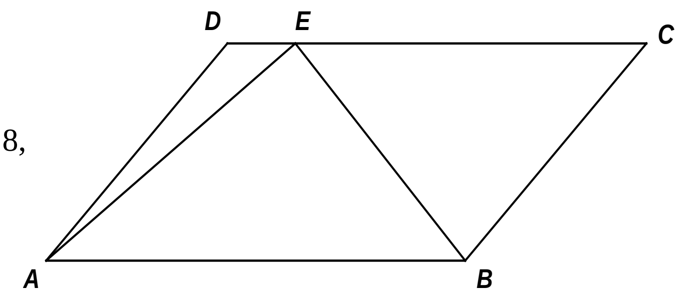
A) ______________________________

B) ______________________________

C) (____________ , ____________)

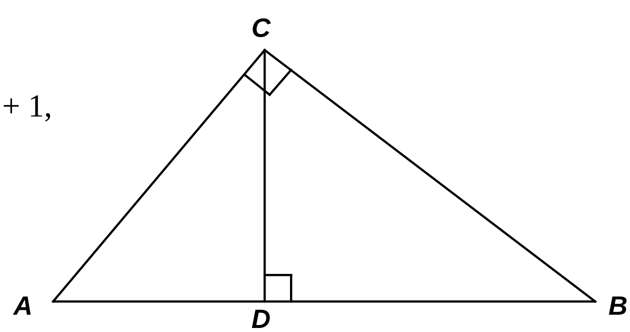
A) Compute x.





B) Given: $ABCD$ is a parallelogram, $AB = 12$, $AD = 8$, ΔAEB is isosceles with $AE = BE$, $m\angle DAB = 60°$

Compute AE.



C) In ΔABC, $m\angle ACB = 90°$, $\overline{CD} \perp \overline{AB}$, $AC = 3x + 2y + 1$, $BC = 6y - 2$, $AB = 7x + 2y + 1$, $AD = 4x - 2$, $DB = 5y - 3$ and $CD = 2(x + y)$.
Compute the ordered pair (x, y).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Given: $\overleftrightarrow{AB}$ has its *x*-intercept at *P*(8, 0) and its *y*-intercept at *Q*(0, –6).
Determine the equation of the line parallel to $\overleftrightarrow{AB}$ that passes through *R*(2, 15) in standard form $Ax + By = C$, where *A* > 0, *A*, *B* and *C* are integers and their greatest common divisor is 1.

.

B) Given: $\begin{cases} x = 2t + 1 \\ y = 6t - 5 \end{cases}$, where *t* denotes any real number
This set of equations is equivalent to a linear function defined by the linear equation $y = mx + k$, where *m* and *k* are constants. For a unique value of *k*, this line passes through the point $(x, y) = (k, 5k)$. Compute *k*.

C) The star field on a flag contains 40 stars arranged in horizontal rows, alternating between long and short rows. A long row contains 3 more stars than a short row. If the top and bottom rows are long rows and the total number of rows is no more than 10, how many stars are there in any two consecutive rows?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013
ROUND 4 ALG 1: FRACTIONS & MIXED NUMBERS

ANSWERS

A) ______________________________ days

B) ______________________________ mph

C) ___________________________________

A) Working alone, A can do a job is 6 days, B can do the same job in x days and C in ($2x$) days. If A works alone for 4 days and stops, B and C can work together and finish the remainder of the job in 3 days. Working alone, in how many days could C do the whole job by himself?

B) The current time is 9:10.
My GPS says I will arrive at my destination at 10:04. (Of course this assumes that I am travelling at an average speed equal to the speed limit for the entire trip. The speed limit is 45mph.)
I am in a hurry and wish to get to my destination by 10:00. Compute my average speed (in mph), if I was able to reach my destination at exactly 10:00.

C) Given: $4AB - 3XA + 4XB - 3AXB = 12AB$
Solve for A in terms of B and X.
Find a <u>simplified</u> expression for X in terms of B which guarantees that A is undefined.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013
ROUND 5 INEQUALITIES & ABSOLUTE VALUE

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Let $y = \begin{cases} \dfrac{|n|}{n} & \text{for } n \neq 0 \\ c & \text{for } n = 0 \end{cases}$, where n denotes an integer and c denotes a real number.

If $\sum_{n=-1}^{n=2013} y = 0$, compute c.

[Fear not! Σ is the summation symbol.
By way of example, $\sum_{n=3}^{n=5}(2n-1) = (2\cdot 3-1)+(2\cdot 4-1)+(2\cdot 5-1) = 5+7+9 = 21$.]

B) Solve for x: $|2x+1| > |x-5|$

C) Determine <u>all</u> real values of x for which each of the fractions $\dfrac{1}{x+5}, \dfrac{1}{13x-60}, \dfrac{1}{5-x}$ are positive and the sequence formed by these three fractions is in strictly increasing order, namely $\dfrac{1}{x+5} < \dfrac{1}{13x-60}$ and $\dfrac{1}{13x-60} < \dfrac{1}{5-x}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013
ROUND 6 ALG 1: EVALUATIONS

ANSWERS

A) _______________________________

B) _______________________________

C) ______________ : ______________

A) Let $\hat{n} = (2n + 1)!$ and $a\#b = \left(a^b + b^a\right)!$

Compute $\dfrac{3\#2}{\hat{7}}$.

Recall:
$n!$ denotes a factorial.
Specifically, $n!$ is defined as the product $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$ and $0! = 1$.

B) The combination padlock for my new laptop is three digits. It is factory preset at 000, but each position can be changed to any digit from 0 through 9 inclusive. How many different combinations are possible, if the sum of the three digits is 10?

C) 6 and 28 are the first two ‘perfect numbers’, i.e. the sum of the proper divisors, excluding the number itself, equals the number. (6 = 1 + 2 + 3 and 28 = 1 + 2 + 4 + 7 + 14)

Numbers for which the sum of the proper divisors is less than the number are termed deficient.
Numbers for which the sum of the proper divisors is more than the number are termed abundant

Compute $D : A$, the ratio of the number of deficient numbers to the number of abundant numbers for integers strictly between the first two perfect numbers.

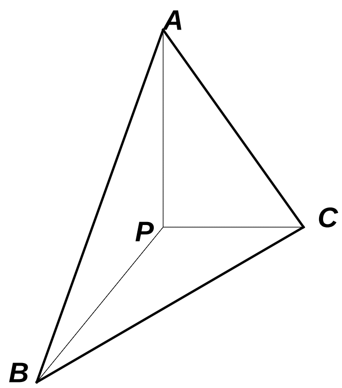
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013
ROUND 7 TEAM QUESTIONS

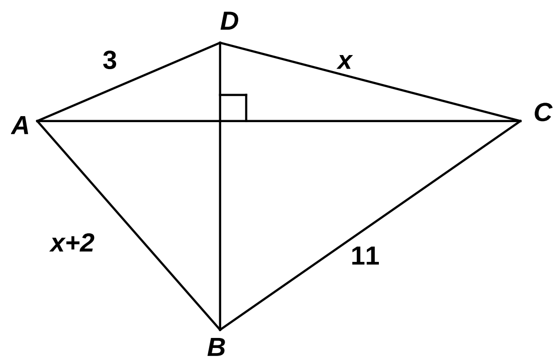
ANSWERS

A) (______ , ______ , ______) D) ____________________________

B) ____________________________ E) ____________________________

C) (__________ , __________) F) (__________ , __________)





A) Each of the angles at vertex P in tetrahedron $PABC$ is a right angle.
$PA^2 = 224$, $PB^2 = 560$ and $PC^2 = 65$. The distance from vertex P to the plane ABC, expressed as a simplified radical is $\frac{A}{B}\sqrt{C}$. Determine the ordered triple of integers (A, B, C).

B) Compute the perimeter of quadrilateral $ABCD$ (See diagram above.)

C) If $y = x + 1$, there are unique <u>integer</u> values of x and y for which the points A, B and C are collinear.

$$A(2x+1,\ 3y),\ B(8y-1,\ 9x),\ C(17y+9x,\ 10(x+y)-3)$$

Compute the coordinates of the point closest to the origin.

D) For constants A, B and C, the following equation is an identity, that is, true for all values of x.

$$\frac{3}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

Of course, both sides are undefined for $x = -1, -2$. Compute $A^3 + B + C$.

E) Let $x \blacklozenge y = \frac{x+1}{2-y}$ and $S = \{(x, y):\ |x| + |y| \le 4$, where x and y are integers$\}$.
For <u>how many</u> ordered pairs (x, y) does $x \blacklozenge y = y \blacklozenge x$?

F) Consider the <u>closed</u> interval $\left[6, (2+a)(3+b)\right]$, where $1 < a \le 10$, $0 < b$ and $ab = 1$.
Let m denote the minimum and M denote the maximum number of integer perfect squares that are included in this interval. Compute the ordered pair (m, M).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013 ANSWERS

Round 1 Geometry Volumes and Surfaces

A) 5.4 or $\frac{27}{5}$ B) 1331π C) $72\sqrt{2}$

Round 2 Pythagorean Relations

A) 56 B) $2\sqrt{21}$ C) (5, 7)

Round 3 Linear Equations

A) $3x-4y=-54$ B) –4 C) 11

Round 4 Fraction & Mixed numbers

A) 27 days B) 48.6 mph or $\frac{243}{5}$ C) $-\frac{8B}{3B+3}$ (or equivalent)

Round 5 Absolute value & Inequalities (Interval notation is acceptable.)

A) –2012 B) $x<-6$ or $x>\frac{4}{3}$ C) $\frac{65}{14}<x<5$

(Comma allowed instead of "or")

Round 6 Evaluations

A) 272 B) 63 C) 17 : 4

(abundant: 12, 18, 20 and 24)

Team Round

A) (4, 3, 26) D) 21

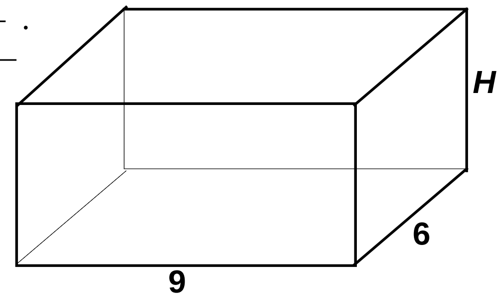
B) 30 E) 6

C) (7, 12) [x = 3, y = 4] F) (1, 4)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013 SOLUTION KEY

Round 1

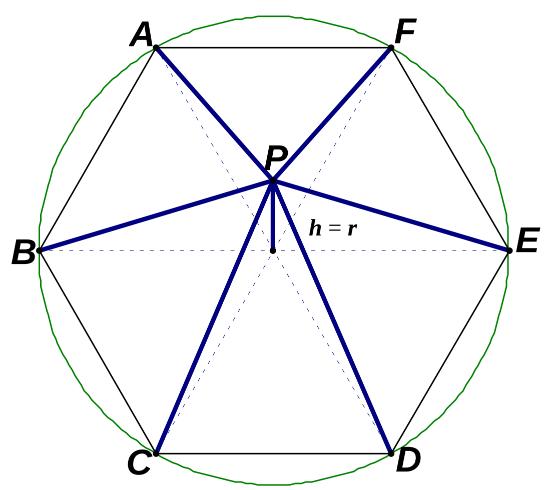
A) $2(6H+9H)=1.5(2\cdot 9\cdot 6) \Rightarrow 30H = 3\cdot 54 \Rightarrow H = \frac{54}{10} = \underline{\mathbf{5.4}}$ or $\underline{\mathbf{\frac{27}{5}}}$.



B) Since $\text{SA} = 2\pi rh + 2\pi r^2$ and we are given $r = h$, we have
$484\pi = 2\left(2\pi r^2\right) \Rightarrow r^2 = 121 \Rightarrow r = 11$.
Thus, $V = Bh = \left(\pi r^2\right)r = \pi r^3 = \pi(11)^3 = \underline{\mathbf{1331\pi}}$.

C) $\text{SA}_{\text{sphere}} = 4\pi r^2 = 96\pi \Rightarrow r^2 = 24 \Rightarrow r = 2\sqrt{6}$
The pyramid with maximum volume will have its vertex P directly above the center of the base. Since the diagonals of the hexagon divide the hexagon into 6 equilateral triangles, the long diagonal of the hexagon is a diameter of the great circle (and of the sphere). The altitude from P to the base will also have length $2\sqrt{6}$. Recall that the area of an equilateral triangle is given by $\frac{s^2\sqrt{3}}{4}$. Thus, the volume of the pyramid is given by

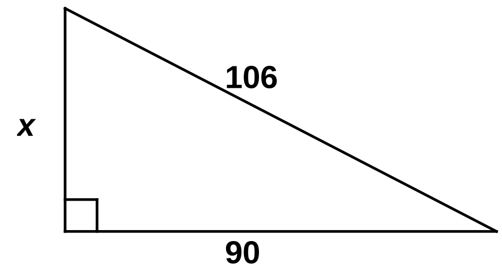
$$\frac{1}{3}Bh = \frac{1}{3}\cdot 6\left(\frac{\left(2\sqrt{6}\right)^2\sqrt{3}}{4}\right)\cdot 2\sqrt{6} = 24\sqrt{18} = \underline{\mathbf{72\sqrt{2}}}.$$



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013 SOLUTION KEY

Round 2

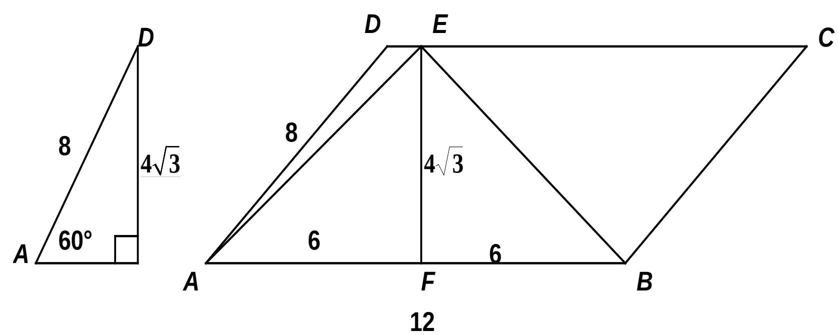
A) Using the Pythagorean Theorem,
$x^2 = 106^2 - 90^2$
Resisting the temptation to play arithmetic,
we factor the right hand side of the equation.
$x^2 = (106 + 90)(106 - 90) = 196(16) = 14^2 4^2$.
Thus, $x = 14(4) =$ **<u>56</u>**.



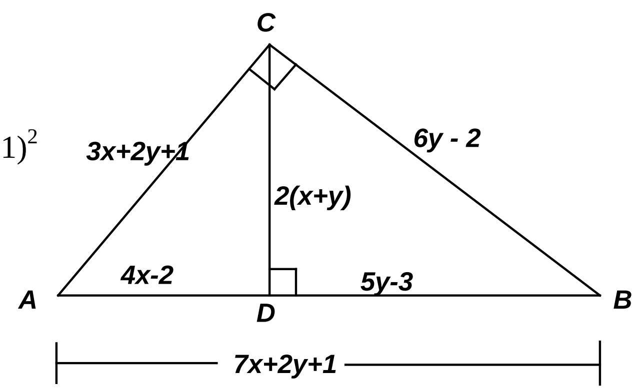
B) $AF = FB = 6$, $EF = 4\sqrt{3}$

$$AE^2 = 6^2 + \left(4\sqrt{3}\right)^2 = 36 + 48 = 84$$

$\Rightarrow AE =$ $\underline{\mathbf{2\sqrt{21}}}$



C) $AD + DB = AB \Rightarrow 4x + 5y - 5 = 7x + 2y + 1$
$\Rightarrow 3x - 3y + 6 = 0 \Rightarrow x = y - 2 \Rightarrow CD = 4(y - 1)$
$CD^2 + DB^2 = BC^2 \Rightarrow 16(y - 1)^2 + (5y - 3)^2 = 4(3y - 1)^2$
$16y^2 - 32y + 16 + 25y^2 - 30y + 9 = 36y^2 - 24y + 4$
$5y^2 - 38y + 21 = (5y - 3)(y - 7) = 0$
$\Rightarrow y = 3/5$ (extraneous) $y = 7 \Rightarrow$ **<u>(5, 7)</u>**



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013 SOLUTION KEY

Round 3

A) The slope of $\overleftrightarrow{PQ}$ is $\dfrac{0+6}{8-0}=\dfrac{3}{4}$. Thus, the equation of the parallel line is $(y-15)=\dfrac{3}{4}(x-2)$.
$\Leftrightarrow 4y-60=3x-6 \Leftrightarrow \underline{\mathbf{3x-4y=-54}}$

B) $x=2t+1 \Rightarrow t=\dfrac{x-1}{2}$ Substituting, $y=6t-5=6\left(\dfrac{x-1}{2}\right)-5=3x-8$
Thus, $3k-8=5k \Rightarrow k=\underline{\mathbf{-4}}$

Alternately, we require that $y = 5x$ for $x = k$. Therefore, $6t-5=5(2t+1) \Rightarrow 4t=-10 \Rightarrow t=-\dfrac{5}{2}$
$\Rightarrow x=k=2\left(-\dfrac{5}{2}\right)+1=\underline{\mathbf{-4}}$

C) If there are S short rows with $x - 3$ stars each, then there are $S + 1$ long rows with x stars each.
$S(x-3)+(S+1)(x)=40 \Leftrightarrow (2S+1)x=40+3S \Rightarrow x=\dfrac{40+3S}{2S+1}$
There must be at least one short row.
$(S, x) = \left(1,\dfrac{43}{3}\right),\left(2,\dfrac{46}{5}\right),\left(3,\dfrac{49}{7}\right),\left(4,\dfrac{52}{9}\right),\left(5,\dfrac{55}{11}\right)$
According to the chart above,
one possibility is 3 short rows of 7 – 3 = 4 stars each and 4 long rows of 7 stars each or 5 short rows of 5 – 3 = 2 stars each and 6 rows of 5 stars each, but the latter exceeds the maximum number of rows. Thus, two consecutive rows contain 4 + 7 = <u>**11**</u> stars.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013 SOLUTION KEY

Round 4

A) $\frac{4}{6}+3\left(\frac{1}{x}+\frac{1}{2x}\right)=1 \Rightarrow \frac{9}{2x}=\frac{1}{3} \Rightarrow 2x = 27 \Rightarrow C\ (2x)$: **<u>27</u>** days

B) At 45 miles per hour, in 54 minutes I would travel $45\cdot\frac{54}{60}=\frac{3\cdot 27}{2}=40.5$ miles.
Since $R\cdot T = D$, to travel 40.5 miles in 50 minutes, I would have to travel at
$\frac{40.5}{\frac{5}{6}}=\frac{81}{2}\cdot\frac{6}{5}=\underline{\mathbf{48.6}}$ mph (or equivalent).

C) $4AB-3XA+4XB-3AXB=12AB \Leftrightarrow 8AB+3XA+3XAB=4XB$
Factoring the left hand side of the equation and solving for *A*, we have $A=\frac{4XB}{8B+3X+3XB}$.
A is undefined, if the denominator is zero.
$8B+3X+3XB=0 \Rightarrow X\left(3B+3\right)=-8B \Rightarrow X = \underline{-\frac{\mathbf{8B}}{\mathbf{3B+3}}}$ (or equivalent)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013 SOLUTION KEY

Round 5

A) $\dfrac{|n|}{n} = \pm 1$ depending on whether n is positive or negative.

$$\sum_{n=-1}^{n=2013} y = -1 + c + 2013(1) = 2012 + c = 0 \Rightarrow c = \underline{\mathbf{-2012}}.$$

B) $|2x+1| > |x-5| \Leftrightarrow \sqrt{(2x+1)^2} > \sqrt{(x-5)^2}$

If $A > B$ and A, $B \geq 0$, then $A^2 > B^2$. Since each radical represents a nonnegative quantity, we can square both sides.

$(2x+1)^2 > (x-5)^2 \Leftrightarrow 4x^2 + 4x + 1 > x^2 - 10x + 25$

$3x^2 + 14x - 24 = (3x-4)(x+6) > 0$

Both factors are positive for $x > \dfrac{4}{3}$ and both factors are negative for $x < -6$.

Therefore, we have $\underline{\boldsymbol{x < -6 \text{ or } x > \frac{4}{3}}}$.

Alternate Solution: $2x + 1 > -x + 5 \Rightarrow 3x > 4 \Rightarrow x > \dfrac{4}{3}$ or $-2x - 1 > -x + 5 \Rightarrow -6 > x$

C) For $\dfrac{60}{13} < x < 5$, each of the fractions is positive.

For this sequence of fractions to be in increasing order the sequence of denominators must be in decreasing order, i.e. $x + 5 > 13x - 60$ and $13x - 60 > 5 - x$

$\Rightarrow 12x < 65$ and $14x > 65 \Rightarrow \dfrac{65}{14} < x < \dfrac{65}{12}$

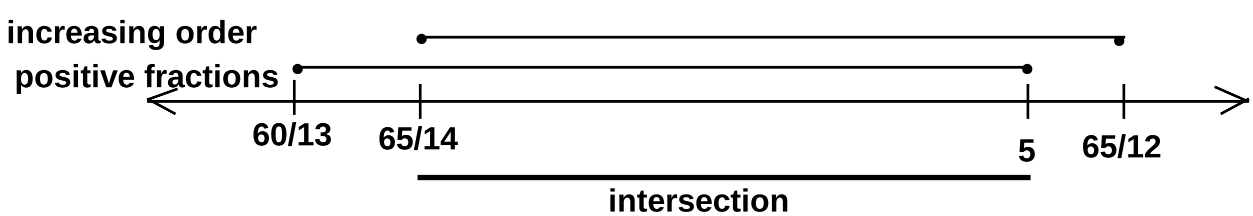
Since both conditions must hold, we must take the intersection of the two intervals.

Clearly, $\dfrac{65}{12} > 5$, but which is larger $\dfrac{60}{13}$ or $\dfrac{65}{14}$.

We can decide by cross multiplying and comparing the products.

[Note: For any positive numbers a, b, c and d, $\dfrac{a}{b} > \dfrac{c}{d} \leftrightarrow ad > bc$.]

$60(14) = 840$ and $65(13) = 5(13)^2 = 5(169) = 845 \Rightarrow \dfrac{65}{14} > \dfrac{60}{13} \Rightarrow \underline{\boldsymbol{\frac{65}{14} < x < 5}}$



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013 SOLUTION KEY

Round 6

A) $\hat{7} = (2 \cdot 7 + 1)! = 15!$

$3\#2 = \left(3^2 + 2^3\right)! = 17!$

$\dfrac{17!}{15!} = 17 \cdot 16 = \underline{\mathbf{272}}$

B) Let's systematically list the possible sets of 3 digits.
Smallest digit 0: 019, 028, 037, 046, <u>*055*</u>
Smallest digit 1: <u>*118*</u>, 127, 136, 145
Smallest digit 2: <u>*226*</u>, 235, <u>*244*</u>
Smallest digit 3: <u>*334*</u>
If the 3 digits are distinct, there are 6 possible combinations.
If two of the digits are the same, there are 3 possible combinations.
8(6) + 5(3) = <u>**63**</u>

C) We must examine the integers from 7 through 27 inclusive.
All primes are deficient since the only proper divisor is 1.
Accordingly, 7, 11, 13, 17, 19 and 23 are deficient numbers.
The only abundant numbers are:

$12 = 2^2 \cdot 3\ [1, 2, 3, 4, 6]$

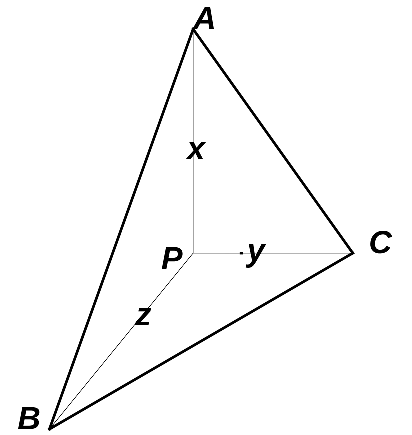
$18 = 2 \cdot 3^2\ [1, 2, 3, 6, 9]$

$20 = 2^2 \cdot 5\ [1, 2, 4, 5, 10]$

$24 = 2^3 \cdot 3\ [1, 2, 3, 4, 6, 8, 12]$

The proper divisors sum to 16, 21, 22 and 36 respectively.
Note that each of the abundant numbers has a repeated prime factor.
Since there are 21 numbers to be classified, the required ratio is <u>**17 : 4**</u>.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013 SOLUTION KEY



Team Round

A) $x^2+y^2=224+65=289 \Rightarrow AC=17$

$x^2+z^2=224+560=784 \Rightarrow AB=28$

$y^2+z^2=65+560=625 \Rightarrow BC=25$

Using Heron's formula, the area of ΔABC can be computed.

The semi-perimeter s is $\frac{28+25+17}{2}=35$. Thus, the area is given by

$\sqrt{35(35-17)(35-25)(35-28)}=\sqrt{35\cdot 18\cdot 10\cdot 7}=\sqrt{5^2\cdot 6^2\cdot 7^2}=210$.

The volume of any pyramid (and this tetrahedron is a pyramid with a triangular base) is given by $\frac{1}{3}Bh$, where B denotes the base and h the height.

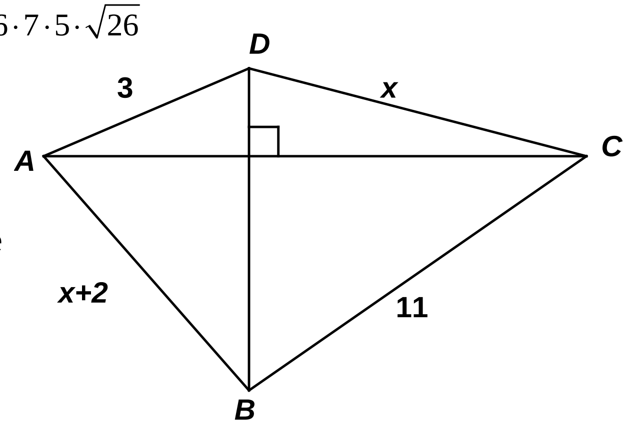
Using PBC as the base, the volume is $\frac{1}{3}\cdot x\cdot\left(\frac{1}{2}yz\right)=\frac{xyz}{6}$

$\sqrt{224}=\sqrt{16\cdot 14}=4\sqrt{14}$, $\sqrt{560}=\sqrt{16\cdot 35}=4\sqrt{35}$

Using ABC as the base, we have

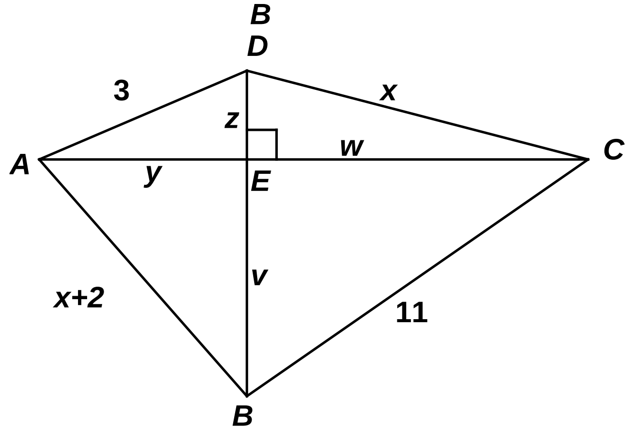
$\frac{1}{3}\cdot h\cdot 210=\frac{xyz}{6} \Rightarrow 420h=xyz=4\sqrt{14}\cdot\sqrt{65}\cdot 4\sqrt{35}=16\cdot 7\cdot 5\cdot\sqrt{26}$

$\Rightarrow h=\frac{560}{420}\sqrt{26}=\frac{4}{3}\sqrt{26}$ $\Rightarrow$ (A, B, C) = **<u>(4, 3, 26)</u>**



B) As noted in a previous contest, since the diagonals of $ABCD$ are perpendicular, the sum of the squares of the lengths of the opposite sides are equal.
See the proof of this fact in the notes included with the solution key for 2009 Round 2 questions.

$3^2 + 11^2 = x^2 + (x + 2)^2$
$\Rightarrow 130 = 2x^2 + 4x + 4$
$\Rightarrow x^2 + 2x - 63 = (x + 9)(x - 7) = 0 \Rightarrow x = 7$
Thus, the perimeter of $ABCD$ is 30.



Alternate Solution:

In ΔADE, (1) $y^2+z^2=9$.

In ΔDCE, (2) $z^2+w^2=x^2$

In ΔBCE, (3) $v^2+w^2=121$

In ΔBAE, (4) $v^2+y^2=x^2+4x+4$

To eliminate z^2, subtract (1) – (2): $y^2-w^2=9-x^2$

Add to (3): $y^2+v^2=130-x^2$

Substitute in (4): $130-x^2=x^2+4x+4$

$\Rightarrow 2x^2+4x-126=0 \Rightarrow x^2+2x-63=(x+9)(x-7)=0 \Rightarrow x=7 \Rightarrow$ Per = 3 + 11 + 7 + 9 = **<u>30</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013 SOLUTION KEY

Team Round

C) The slope of $\overline{AB}$ is $\dfrac{9x-3y}{8y-2x-2}=\dfrac{9x-3x-3}{8x+8-2x-2}=\dfrac{6x-3}{6x+6}=\dfrac{2x-1}{2x+2}$.

The slope of $\overline{AC}$ is $\dfrac{10x+10y-3-3y}{17y+9x-2x-1}=\dfrac{10x+7y-3}{7x+17y-1}=\dfrac{17x+4}{24x+16}$

Since *A*, *B* and *C* are collinear, the slopes of $\overline{AC}$ and $\overline{AB}$ must be equal.
Equating and cross multiplying,

$\dfrac{2x-1}{2x+2}=\dfrac{17x+4}{24x+16}\Rightarrow 48x^2+8x-16=34x^2+42x+8\Rightarrow 14x^2-34x-24=0$

$\Rightarrow\ 7x^2-17x-12=(7x+4)(x-3)=0\Rightarrow x=3$

$y=x+1\ \Rightarrow\ y=4$

$\Rightarrow A(2x+1,3y)\ \Rightarrow A$**<u>(7, 12)</u>**

(*B*(31,27) and *C*(95,67) are clearly further from the origin.)

D) $\dfrac{3}{(x+1)(x+2)^2}=\dfrac{A}{x+1}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$

Multiplying through by $(x+1)(x+2)^2$, we have an equation which is true for **<u>all</u>** values of *x*, namely, $3=A(x+2)^2+B(x+1)(x+2)+C(x+1)$.

If $x=-2$, then two terms on the right hand side disappear and we have $3=C(-1)\Rightarrow C=-3$.
If $x=-1$, we have $3=A(1)^2\Rightarrow A=3$
Picking an arbitrary value of *x*, we can solve for *B*.

$x=0\Rightarrow\ 3=3(2)^2+B(1)(2)+(-3)(1)\Leftrightarrow 3=12+2B-3\Leftrightarrow B=-3$

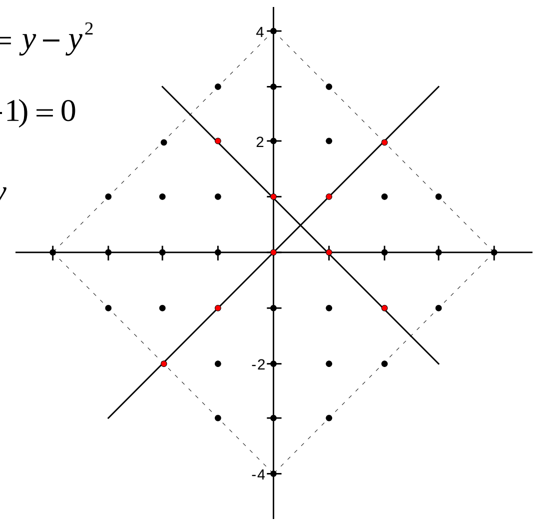
Thus, $A^3+B+C=27-3-3=\underline{\mathbf{21}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2013 SOLUTION KEY

Team Round

E) $x \blacklozenge y = \dfrac{x+1}{2-y}$ and $S = \{(x, y)\text{: } |x| + |y| \le 4$, where x and y are integers$\}$

$\dfrac{x+1}{2-y} = \dfrac{y+1}{2-x} \Rightarrow 2x - x^2 + 2 - x = 2y - y^2 + 2 - y \Rightarrow x - x^2 = y - y^2$
$\Rightarrow x^2 - y^2 = x - y \Rightarrow x^2 - y^2 - (x-y) = 0 \Rightarrow (x-y)(x+y-1) = 0$
$\Rightarrow x = y$ (5 solutions) or $x + y = 1$ (4 solutions)
But 3 of these solutions are extraneous, since neither x nor y can be 2. Thus, there are **<u>6</u>** solutions.



F) $[6, (2+a)(3+b)]$, where $1 < a \le 10$, $0 < b$ and $ab = 1$.

The closed interval is $[6, 6 + 2b + 3a + ab] = [6, 7 + 3a + 2b]$.
Since the coefficient of a is larger than the coefficient of b, if we maximize a (i.e. take $a = 10$) and, correspondingly, take $b = 1/10$, the length of the interval is maximized, namely $6 \le x \le 37.2$.
This interval contains 4 integer perfect squares: 9, 16, 25 and 36 and $M = 4$.
To minimize the interval, we want a as small as possible, but there is no minimum positive value for a.
However, since $ab = 1$ and $a > 1$, $0 < b < 1$. $7 + 3a + 2b > 10$.
Thus, the minimum interval contains 9 and $m = 1$. Therefore, $(m, M) =$ **<u>(1, 4)</u>**.